GEOMETRIC ANALYSIS SECTIONS 3,4

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Section 3. Harmonic maps

Let (M, g) and (N, h) be Riemannian manifolds with M compact, say.

The *energy* of a map $u : M \to N$ is defined to be

$$E(u) = \frac{1}{2} \int_M |Du|^2.$$

Here Du is the derivative of u. At a point $x \in M$:

$$Du_x \in T^*M_x \otimes TN_{u(x)},$$

and the norm $|Du|$ is the standard one computed using the metrics on TM and TN.

An infinitesimal variation in u is given by a section v of the vector bundle bundle $V = u^*(TN)$ over M.
This bundle has a connection, the pull-back of the Levi-Civita connection of N.
So we have a coupled exterior derivative

$$d_\nabla : \Omega^p(V) \to \Omega^{p+1}(V),$$

where Ω^p denotes p-forms on M.

We have formal adjoints

$$d_\nabla^* : \Omega^{p+1}(V) \to \Omega^p(V).$$

We can think of Du as an element of $\Omega^1(V)$.
The "tension" of u is defined to be

$$\tau_u = d_\nabla^* Du.$$

The first variation of E is given by the formula

$$\delta E = -\int_M \langle \tau_u, v \rangle.$$

The map u is called *harmonic* if $\tau_u = 0$. This is the Euler-Lagrange equation associated to the functional E.

If we take local coordinates x^i on M and y^λ on N the equation is

$$\Delta_M y^\lambda + \Gamma^\lambda_{\mu\nu} y^\mu_{,i} y^\nu_{,j} g^{ij} = 0,$$

where $\Gamma^\lambda_{\mu\nu}$ are the Cristoffel symbols on N and g^{ij} is the metric on T^*N.

When $\dim M = 1$ we get the geodesic equations in N and when $\dim N = 1$ we get the equation for a harmonic function on M.

How might one try to prove existence theorems for harmonic maps?
The τ_u lie in *different* spaces, for different maps u, so there is no obvious continuity path.

We can thing of the assignment $u \mapsto \tau_u$ as defining a section of an infinite-dimensional vector bundle over the infinite-dimensional space of maps from M to N. This is the typical situation for differential geometric questions.

In this case the vector bundle in question is the tangent bundle of the space of maps and τ is formally the gradient vector field of the energy functional E.

We could try variational methods, in particular trying to minimise E in a homotopy class of maps.

This works well if $\dim M = 1$ (closed geodesics) and can be used to give some results when $\dim M = 2$ but fails badly for $\dim M > 2$.

The reason is that the Sobolev embedding $L^2_1 \subset C^0$ holds when $\dim M = 1$ and just fails when $\dim M = 2$.

Example. For $p \geq 3$ the infimum of the energy in any homotopy class of maps $S^p \to N$ is zero.

Regard S^p as $\mathbf{R}^p \cup \{\infty\}$ and change the metric on the unit ball $B \subset \mathbf{R}^p \subset S^p$ to the Euclidean one. (It will be clear that this makes no difference.)
Represent a homotopy class of maps $S^p \to N$ by a map U which is constant outside B.
Let $U_\epsilon(x) = U(x/\epsilon)$. Then

$$E(U_\epsilon) = \epsilon^{p-2} E(U).$$

We will see more about this later.

A positive result (Eells and Sampson, 1964).
Suppose that M, N are compact and the sectional curvatures of N are ≤ 0. Then any free homotopy class of maps $M \to N$ contains a harmonic representative.

The results of Eells and Sampson hinge on a differential geometric formula. For $f : M \to N$ set $\mathcal{E} = \frac{1}{2}|Df|^2$.
The we have, schematically:

$$\Delta\mathcal{E} = \langle d_\nabla \tau, Df \rangle + |\nabla Df|^2 + \mathrm{Ricci}_M (Df)^2 - \mathrm{Riem}_N (Df)^4.$$

Explicitly, the curvature terms are, writing $\alpha = Df$:

$$R^M_{ij} \alpha^\lambda_k \alpha^\mu_l g^{ik} g^{jl} + R^N_{\lambda\mu\nu\pi} \alpha^\lambda_i \alpha^\mu_j \alpha^\nu_k \alpha^\pi_l g^{ik} g^{jl}.$$

Here R^M_{ij} is the Ricci curvature of M and $R^N_{\lambda\mu\nu\pi}$ is the Riemann curvature of N, with sign conventions such that the sectional curvature in a bivector ξ, η is

$$-R^N_{\lambda\mu\nu\pi}\xi^\lambda\xi^\nu\eta^\mu\eta^\pi.$$

The key point is that the Ricci term is ≥ 0 if $\mathrm{Ricci}^M \geq$ and the other term is ≥ 0 if the sectional curvatures of N are ≤ 0.
To see this identity, observe first that the Levi-Civita connection on N being torsion-free implies that $d_\nabla \alpha = 0$, where $\alpha = Df \in \Omega^1(V)$. Thus we have

$$\Delta_\nabla \alpha = d_\nabla \tau,$$

where $\Delta_\nabla = d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$.
On the other hand, writing ∇ also for the covariant derivative on $\Omega^1(V)$, we have

$$\Delta \mathcal{E} = \frac{1}{2} \Delta \langle \alpha, \alpha \rangle = \langle \nabla^* \nabla \alpha, \alpha \rangle + |\nabla \alpha|^2.$$

So what we need is a “Weitzenbock formula” comparing $\nabla^*\nabla$ with Δ_∇ on $\Omega^1(V)$. This is

$$\Delta_\nabla = \nabla^*\nabla + \mathrm{Ricci}_M + F_V,$$

where F_V is the curvature tensor of the connection on V. In our case F_V is the pullback of the Riem^N and one gets the formula stated.

Note The identity map of M is harmonic and in that case the two curvature terms cancel.

Assuming the Eells-Sampson result on the existence of harmonic maps, we get a proof of *Preissman's Theorem*:

If N has strictly negative sectional curvature then any abelian subgroup of $\pi_1(N)$ is cyclic.

If we have two commuting elements a, b of $\pi_1(N)$ we can represent them by a map $f : T^2 \to N$. Take the flat metric on T^2. By the Eells-Sampson result we can suppose that f is harmonic. Since the integral of $\Delta\mathcal{E}$ vanishes we see $\nabla Df = 0$ and the curvature term must vanish pointwise, which means that Df has rank 1. It follows that f factors through S^1 and a, b lie in a cyclic subgroup of $\pi_1(N)$.

Eells and Sampson introduced the technique of “nonlinear heat flow” to prove their existence result.
This is the equation for a 1-parameter family of maps $f_t : M \rightarrow N$:

$$\frac{\partial f_t}{\partial t} = \tau(f_t) = d_{\nabla}^{*} Df_t .$$

The strategy is to produce a harmonic map as the limit as $t \rightarrow \infty$ of such a family.
(But the heat flow also has independent interest.)

Digression: a little foundational theory for parabolic PDE.

The ordinary heat equation for functions u_t on M is $\frac{\partial u_t}{\partial t} = \Delta u_t$. With initial condition v there is a solution of the form $u_t = k_t(v)$ for a semigroup of operators k_t which can be understood in various ways.

Spectral description

The operator $(1-\Delta)^{-1} : L^2 \to L^2$ is compact and self-adjoint and so has an orthonormal basis of eigenfunctions. These give eigenfunctions for the Laplacian: $-\Delta\phi_\lambda = \lambda\phi_\lambda$ where λ runs over a sequence tending to ∞. Any function $u \in L^2$ has an expansion $u = \sum u_\lambda \phi_\lambda$. When M is a flat torus this is the usual Fourier series expansion. The L^2_k norm is equivalent to

$$\|u\|^2_{L^2_k} = \sum (1+\lambda^{k/2}) u_\lambda^2.$$

In this description the operator k_t is the multiplication operator acting as $e^{-\lambda t}$ on ϕ_λ.

The behaviour on L^2_k norms is transparent.

Kernel description

For $t > 0$ k_t is an integral operator

$$k_t(v)(x) = \int K_t(x, y)v(y)dy,$$

where k_t is smooth and for small t is well-approximated by the Euclidean heat kernel

$$(2\pi t)^{-n/2} \exp(-r^2/4t).$$

Here $n = \dim M$ and r is the distance from x to y.

Write $L = \frac{\partial}{\partial t} - \Delta$.

We can also study the inhomogeneous equation $LU = \chi$ where χ is a given function on $M \times [0, T]$ for some T, with initial condition $U(x, 0) = 0$.

The solution is

$$U_t = L^{-1}(\chi) = \int_{\tau=0}^{t} k_{t-\tau}(\chi_\tau)\, d\tau.$$

Similar to the elliptic case, we can derive various estimates for the operator L^{-1}.

For example, by integrating $(LU)^2$ over $M \times [0, T]$ we get

$$\|\frac{\partial U}{\partial t}\|^2_{L^2(M\times[0,T])} + \|\Delta_M U\|^2_{L^2(M\times[0,T])} + \|\nabla_M U_T\|^2_{L^2(M)} = \|\chi\|^2_{L^2(M\times[0,T])}.$$

We will use another result. Clearly the integral of $K_t(x,y)$ with respect to y is 1, for all x,t which implies that

$$\|U\|_{C^0} \leq T\|\chi\|_{C^0}. \qquad (***)$$

Set

$$I(x,t)\|\nabla_x K_t(x,y)\|_{L^1} = \int |\nabla_x K_t(x,y)| dy.$$

Then one can one see that $I(x,t) \leq Ct^{-1/2}$ for some C independent of x. It follows that if $LU = \chi$ then

$$\|\nabla_M U\|_{C^0} \leq 2CT^{1/2}\|\chi\|_{C^0}. \qquad (****)$$

Now we can study short time existence for a nonlinear equation of the form

$$\frac{\partial u}{\partial t} = \Delta u + F(u, \nabla_M u) + \rho$$

Here F is a smooth function of its arguments which vanishes when $u = 0$ and ρ is fixed function on M, independent of t. To simplify notation we just write $F(u)$.

We seek a solution on $M \times [0, T]$ with $u = 0$ at $t = 0$. We write $u = L^{-1}(\sigma)$ so the equation is

$$\sigma = F(L^{-1}(\sigma) + \rho.$$

Using the estimates (***), (****) we can solve this for $\sigma \in C^0(M \times [0, T])$ using the contraction mapping theorem, provided that T is small compared to C^{-1}.

This gives a weak solution to our equation, with U in C^1 in the "space" variable.

With some work one can derive further estimates and prove that the solution smooth.

It is clear that the same discussion applies to any similar PDE for a vector-valued function on M.

Now go back to our harmonic map flow.
There are various ways of fitting this into the PDE theory.
One goes as follows. Choose an embedding of N in $\mathbf{R}^p$ for some p. Let Ω be a tubular neighbourhood of $N \subset \mathbf{R}^p$ with an involution$\iota : \Omega \to \Omega$ fixing N. Choose a Riemannian metric on Ω which agrees with the given metric on N and is invariant under ι, so that N is *totally geodesic* in Ω. This means that the flow for maps $f : M \to \Omega$ starting with a map with image in N is the same as the flow for maps $M \to N$.

Writing f^λ for the components of our maps in the Euclidean co-ordinates on $\mathbf{R}^p$ and Γ for the Christoffel symbols of the metric on Ω the PDE is

$$\frac{\partial f^\lambda}{\partial t} = \Delta_M f^\lambda + \Gamma^\lambda_{\mu\nu} f^\mu_{,i} f^\nu_{,j} g^{ij}.$$

If our initial map is h and we set $f = h + u$ this fits into the framework that we discussed.

Another approach is to first extend the PDE theory to sections of vector bundles.

Choose an identification of a neighbourhood of the graph of the initial map h in $M \times N$ with a neighbourhood of the zero section in the bundle $V = h^*(TN)$.

Then small deformations of the map $h : M \to N$ are identified with small sections of $h^*(TN)$ and we can apply the PDE theory.

End of digression: we assume known now that the harmonic map flow has a solution for small time.

The differential geometric formula gives

$$\frac{\partial \mathcal{E}}{\partial t} = \Delta \mathcal{E} - |\nabla (Df)|^2 - \mathrm{Ricci}_M (Df)^2 + \mathrm{Riem}_N (Df)^4.$$

For general M, N this gives

$$\frac{\partial \mathcal{E}}{\partial t} \leq \Delta \mathcal{E} + C_1 \mathcal{E} + C_2 \mathcal{E}^2.$$

The term in e^2 can allow finite-time blow up (compare with $\mathcal{E}(t) = (T - t)^{-1}$).

But under the hypothesis that N has sectional curvature ≤ 0 we get

$$\frac{\partial \mathcal{E}}{\partial t} \leq \Delta \mathcal{E} + C_1 \mathcal{E}.$$

This implies that

$$\frac{\partial e^{C_1 t} \mathcal{E}}{\partial t} \leq \Delta e^{C_1 t} \mathcal{E}.$$

The *maximum principle for the heat equation* tells us that the maximum of $e^{C_1 t}\mathcal{E}$ is nonincreasing with time so

$$\mathcal{E} \leq C_3 e^{C_1 t}.$$

Using this it is not hard to show that the solution exists for all time.

To go further, we have the gradient flow identity

$$\frac{dE}{dt} = -\|\tau\|^2. \qquad (******)$$

By comparison with the solution of the heat equation we get

$$\mathcal{E}_{t+1}(x) \leq e^{C_1} \int K_1(x, y)\mathcal{E}_t(y)dy,$$

and the right hand side is bounded by a multiple of the integral of $\mathcal{E}_t$ which is $E(t)$.
So

$$\max_M \mathcal{E}_{t+1} \leq C_4 E(t)$$

and we see that $\mathcal{E}$ satisfies a fixed bound for all t.

From (******) we can choose a sequence $t_i \to \infty$ such that

$$\|\tau(t_i)\|_{L^2} \to 0.$$

Set $f_i = f_{t_i}$. We know that the $|Df_i\|_{C^0}$ are bounded so by the Ascoli-Arzela Theorem there is a subsequence of the f_i which converge in C^0 to some f_∞.
(Note that this is the first time we use the compactness, as opposed to completeness, of N.)

Starting from the fact that the $\tau(t_i)$ tend to 0 in L^2, it is not hard to show that f_∞ is a smooth harmonic map.

Flat bundles and Hitchin's equations

Examples of manifolds with negative sectional curvature are symmetric spaces G/K where G is a noncompact semisimple Lie group and K is a maximal compact subgroup. We consider a slight variant of the theory discussed above.

Let $P \to M$ be a principle G-bundle with a flat connection.

We may consider reductions of the structure group of the bundle P (but not the flat structure) to $K \subset G$.

These correspond to sections of a bundle $\mathcal{H} \to M$ with fibre G/K (the bundle associated to P via the action of G on G/K.)

The bundle $\mathcal{H}$ has a flat structure so sections are locally given by maps into G/K, well defined up to the action of the isometry group G.

Thus we have a notion of a *harmonic section* of $\mathcal{H}$ and we can ask when do these exist

For simplicity we consider the case $G = SL(2, \mathbf{C})$ when $K = SU(2)$.
Then G/K is the space of hermitian metrics on $\mathbf{C}^2$ of determinant 1.
We have

$$\det \begin{pmatrix} x_1 + x_4 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 - x_4 \end{pmatrix} = x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The space G/K can be identified with a one sheet of a quadric in $\mathbf{R}^{3,1}$ and thence with *hyperbolic* 3-*space* H^3. (This is a consequence of the local isomorphism $SL(2, \mathbf{C}) \sim SO(3, 1)$.)
It has a compactification $\overline{H}^3$, adjoining a 2-sphere at infinity.
$SL(2, \mathbf{C})$ acts on the sphere at infinity by Möbius maps.

Consider a vector bundle $E \to M$ with structure group $SL(2, \mathbf{C})$ and a flat $SL(2, \mathbf{C})$ connection. This is equivalent to a representation

$$\rho : \pi_1(M) \to SL(2, \mathbf{C}).$$

Our bundle $\mathcal{H}$ with fibre H^3 is the bundle of determinant 1 hermitian metrics on E. The harmonic equation for a section h is, in a local flat trivialisation, $d^*(h^{-1}dh) = 0$.
We also have a well-defined heat equation

$$\frac{\partial h}{\partial t} = d^*(h^{-1}dh).$$

Since the local differential geometry is the same the previous discussion carries over *except* for the fact that H^3 is not compact.
We deduce that *if* the heat flow h_t lies in compact subset of $\mathcal{H}$ then there is a harmonic section.

We need a very simple observation in hyperbolic geometry. Let ξ_1, ξ_2 be distinct points in the sphere at infinity in $\overline{H}^3$ and let R be any positive number. Then we can find neighbourhoods U_1, U_2 of ξ_1, ξ_2 in $\overline{H}^3$ such that the hyperbolic distance between $U_1 \cap H^3, U_2 \cap H^3$ exceeds R.

It is elementary to show, using this observation, that if ∇h_t is bounded and h_t do *not* lie in a compact set then there is a point if S^2 fixed by the monodromy $\rho(\pi_1(M))$.
In other words, the representation is *reducible*.
So an irreducible flat $SL(2, \mathbf{C})$ bundle E has a harmonic metric.
One can show that this metric is unique and that conversely if E has a harmonic metric it is either irreducible or decomposes as a sum of flat line line bundles $L \oplus L^*$.

Remark
"Irreducibility", above, is a simple example of a "stability condition" on a geometric object (a flat bundle) under which one can solve a related PDE (for a harmonic metric).

We can take another point of view on the same equation. Let now E be rank 2 vector bundle over M with structure group $SU(2)$ and write $\mathrm{ad}E$ for the bundle of self-adjoint trace-zero endomorphisms of E. Consider pairs (A, ϕ) consisting of an $SU(2)$ connection A on E and $\Phi \in \Omega^1(\mathrm{ad}E)$. Then $A + i\Phi$ is an $SL(2, \mathbf{C})$ connection with curvature

$$(F(A) - \Phi \wedge \Phi) + i(d_A\Phi).$$

One part lies in $\mathrm{ad}E$ and the other in $i\mathrm{ad}\, E$ so this connection is flat if and only if $F(A) = \Phi \wedge \Phi$ and $d_A\Phi = 0$.

The harmonic condition is $d_A^*\Phi = 0$. So the moduli space of (irreducible) solutions of the equations:

- $F(A) = \Phi \wedge \Phi$;
- $d_A\Phi = 0$;
- $d_A^*\Phi = 0$;

is identified with the space of conjugacy classes of irreducible representations $\pi_1(M) \to SL(2, \mathbf{C})$.

Now let M be a compact Riemann surface. Extending what we discussed for line bundles, any connection on a complex vector bundle over M defines a holomorphic structure on the bundle. We can write $\Phi = \phi - \phi^*$ where $\phi \in \Omega^{1,0}(\mathrm{End}_0 E)$ and ϕ^* is defined using the Hermitian structure on E. The last two equations above are equivalent to $\overline{\partial}_A \phi = 0$, i.e. that ϕ is a holomorphic 1-form with values in the holomorphic vector bundle $\mathrm{End}_0(E)$.

We can think of the data as being:

- a holomorphic vector bundle $E \to M$ with structure group $SL(2, \mathbf{C})$;
- a holomorphic section ϕ of $\mathrm{End}_0(E) \otimes K_M$;
- a Hermitian metric h on E whose associated connection has curvature F_h satisfying the equation (Hitchin's equation)

$$F_h + [\phi, \phi_h^*] = 0 \qquad (*********).$$

(Note that ϕ_h^* depends on h.)

Now we have another PDE problem: given the “holomorphic data” (E, ϕ) can we choose a Hermitian metric h to solve (*******)?

This PDE is quite similar to the harmonic section equation we have discussed.

The main result is that the existence of a solution is equivalent to a stability condition on the pair (E, ϕ).

The upshot is that the same moduli space appears as

- equivalence classes of irredicuble representations $\pi_1(M) \to SL(2, \mathbf{C})$;
- equivalence classes of stable “Hitchin pairs” (E, ϕ).

Both descriptions are algebro-geometric in nature but the equivalence depends on solving two PDE.

Section 4. The Seiberg-Witten equations and symplectic 4-manifolds

Review of spin structures in dimensions 3,4.

Dimension 3.

The Lie algebra $\mathbf{su}(2)$ of $SU(2)$ is a 3-dimensional vector space with a natural orientation and Euclidean structure. The adjoint representation gives a 2-1 homomorphism $SU(2) \rightarrow SO(3)$ with kernel $\{\pm 1\}$.

We can define a spin structure on a 3-dimensional oriented Euclidean space Λ to be a 2-dimensional complex vector space S with Hermitian metric and trivialised determinant and an isomorphism $\Lambda = \mathbf{su}(S)$, compatible with the given structures. S is unique "up to sign".

For $\phi \in S$ we have $\phi\phi^* \in \mathrm{End}(S)$. Define $q(\phi)$ to be trace-free part of $\phi\phi^*$. Then $iq(\phi)$ lies in $\mathbf{su}(S)$.

Key observation

$$\langle q(\phi)\phi, \phi\rangle = \frac{1}{2}|\phi|^4.$$

A spin structure on an oriented Riemannian 3-manifold Y is a complex vector bundle $S \to Y$ with structure group $SU(2)$ and an isomorphism $TY = \mathbf{su}(S)$.

Equivalently, it a lift of the frame bundle P of Y to a principal $SU(2)$ bundle $\tilde{P}$. Then S is the vector bundle associated to the fundamental representation of $SU(2)$.

Dimension 4.

Let S^+, S^- be two complex vector spaces as above. They can be regarded as 1-dimensional quaternionic vector spaces. Then $\mathrm{Hom}(S^-, S^+)$ is the complexification of a 4-dimensional real vector space $V(S^-, S^+)$ of quaternion linear maps. The action on $V(S^-, S^+)$ defines a 2-1 homomorphism $SU(2) \times SU(2) \to SO(4)$. A spin structure on an oriented Riemannian 4-manifold M is a pair of bundles $S^+, S^- \to M$ each with structure group $SU(2)$ and an isomorphism $TM = V(S^-, S^+)$.

Equivalently, it is a lift of the frame bundle P of M to a principal $SU(2) \times SU(2)$ bundle $\tilde{P}$. Then $S^{\pm}$ are the vector bundles associated to the fundamental representations of the two $SU(2)$ factors.

Recall that the two-forms on M decompose as $\Lambda^2_+ \oplus \Lambda^2_-$ where the bundles $\Lambda^2_\pm$ have rank 3. Given a spin structure as above we have

$$\Lambda^2_+ = \mathbf{su}(S^+),$$

and for $\phi \in S^+$ we have $iq(\phi) \in \Lambda^2_+$.

The Dirac operator $D : \Gamma(S^-) \to \Gamma(S^+)$ is the composite of the covariant derivative $\nabla : \Gamma(S^-) \to \Gamma(S^- \otimes T^*M)$ with the map $S^- \otimes T^*M \to S^+$ defined by $TM = V(S^-, S^+)$.

Note: We will later give another description of this which may be more familiar.

Let M be a Riemannian 4-manifold with spin structure and $L \to M$ be a Hermitian complex line bundle.
The Seiberg-Witten equations are for a pair (A, ϕ) where A is a connection on L and ϕ is a section of $S^- \otimes L$.

- $D_A \phi = 0$;
- $F^+(A) = -iq(\phi)$.

Here

$$D_A : \Gamma(S^- \otimes L) \to \Gamma(S^+ \otimes L)$$

is the coupled Dirac operator defined by the connection A and $q(\phi)$ is defined using the Hermitian metric on L.

The Lichnerowicz/Weitzenbock formula for coupled Dirac operators gives

$$D_A^* D_A = \nabla_A^* \nabla_A + \frac{R}{4} + iF_A^+. \qquad (*****)$$

Here R is the scalar curvature and $F_A^+ \in \Lambda^2_+$ acts on S^+ via the identification $\Lambda^2 = \mathbf{su}(A^+)$.

So, for a solution of the Seiberg-Witten equations we get

$$0 = \nabla_A^* \nabla_A \phi + \frac{R}{4}\phi + q(\phi)\phi.$$

Taking the L^2-inner product with ϕ and using the key observation this gives

$$\|\nabla_A \phi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{L^4}^4 + \int_M \frac{R}{4}|\phi|^2 = 0.$$

In particular, if $R \geq 0$ the only solution could be with $\phi = 0$ and $F^+(A) = 0$.