GEOMETRIC ANALYSIS SECTIONS 1,2

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"GEOMETRIC ANALYSIS":

Study of partial differential equations (usually elliptic or parabolic) related to differential geometry on manifolds.

The course will focus on *examples*, mainly from *Riemannian Geometry* and *Gauge theory*.

Outline plan:

1. Short review of background theory.
2. The equation of constant Gauss curvature on surfaces and the vortex equation.
3. The theorem of Eells and Sampson on harmonic maps to spaces of negative curvature and applications to Hitchin's equation.
4. The Seiberg-Witten equation on symplectic 4-manifolds.
5. The Yamabe problem for constant scalar curvature metrics.
6. Yau's solution of the Calabi conjecture in complex differential geometry.
7. Perelman's monotonicity formula for Ricci flow.
8. Adiabatic approximations, such as the result of Dostoglou and Salamon relating Yang-Mills instantons to holomorphic maps

We may not get to all of these topics, and the level of detail will be variable.

Some books:

- T. Aubin *Nonlinear Analysis on manifolds: Monge-Ampère equations*; *Nonlinear problems in Riemannian geometry*.
- J. Jost *Geometric analysis and Riemannian geometry*.
- R. Schoen and S-T Yau *Lectures on differential geometry*.

SECTION 1. REVIEW OF SOME BACKGROUND.
Linear elliptic equations.

Let $\mathcal{D}$ be a linear elliptic operator of order r over a compact manifold M so

$$\mathcal{D} : \Gamma(E) \to \Gamma(F)$$

for vector bundles E, F over M.

Fundamental example: the Laplace operator on functions on a Riemannian manifold.

$$\Delta u = \mathrm{div}\ \mathrm{grad} u$$

$$\Delta u = -d^* du = \pm * d * du,$$

$$\Delta u = g^{-1/2} \sum \frac{\partial}{\partial x^i} \left(g^{1/2}\, g^{ij} \frac{\partial u}{\partial x^j} \right).$$

Usually there will be metrics on M, E, F.

Basic fact: $\mathrm{Ker}\ \mathcal{D}$ is finite dimensional and $\mathrm{Im}\ \mathcal{D}$ consists of sections of F which are L^2-orthogonal to $\mathrm{ker}\mathcal{D}^*$.

Here $\mathcal{D}^*$ is the formal adjoint operator defined by the condition

$$\langle \sigma, \mathcal{D} f \rangle = \langle \mathcal{D}^* \sigma, f \rangle.$$

where

$$\langle p, q \rangle = \int_M (p(x), q(x))\, d\mu_x = \int_M (p(x), q(x))\, dx.$$

For example, for any ρ we can solve the equation $-\Delta f + f = \rho$ since $-\Delta + 1$ is a strictly positive self-adjoint operator:

$$\langle(-\Delta + 1)f, f\rangle = \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2.$$

The solution is given by an integral operator

$$f(x) = \int_M G(x, y)\rho(y)dy.$$

There are multitudes of *function spaces*, with associated norms.

- L^p_k for $p \geq 1$—functions with k derivatives in L^p.
- $C^{k,\alpha}$ for $0 < \alpha < 1$—functions whose first k derivatives are Hölder continuous with exponent α.

Elliptic operators over compact manifolds behave well on these function spaces (taking $p > 1$ in the first case).

$$\mathcal{D} : L^2_{k+r} \to L^2_k$$

is *Fredholm*, with finite dimensional kernel and closed image of finite codimension. Similarly for the $C^{k,\alpha}$. There are *elliptic estimates* of the shape:

$$\|f\|_{L^2_{k+r}} \leq C \left(\|\mathcal{D}f\|_{L^2_k} + \|f\|_{L^2} \right).$$

Also *regularity*: if f is only *a priori* in L^2 (or even just a distribution) and $\mathcal{D}f = \rho$ in the weak sense for $\rho \in L^2_k$ then in fact $f \in L^2_{k+r}$.

Results of “Ascoli-Arzela type”: inclusions $L^2_{k+1} \to L^2_k$ and $C^{k+1,\alpha} \to C^{k,\alpha}$ are *compact*.

Sobolev embedding theorems

The quantity $k - n/p$ where $n = \mathrm{dim} M$ is the *scaling weight* of the L^p_k norm.

If $1 - n/p > 0$ then functions in L^p_1 are *continuous*.

Outline proof.

If f is a smooth function on $\mathbf{R}^n$ supported in the unit ball then we can write $f(0)$ as the integral of the radial derivative $\frac{\partial f}{\partial r}$ along any ray through the origin. Now average over these rays to get a formula of the shape

$$f(0) = \int k(x).(\nabla f)(x) dx, \qquad (*)$$

where $|k(x)| = O(|x|^{1-n})$ and estimate the integral (*) using Hölders inequality.

In fact for $1 - n/p > 0$ the functions in L^p_1 are in $C^{,\alpha}$ for $\alpha = 1 - n/p$.

If $1 - n/p \geq -n/q$ then $L^p_1 \subset L^q$.

This can be reduced to the case $p = 1, q = n/(n-1)$ for nonnegative functions of compact support on $\mathbf{R}^n$. That is, the Sobolev inequality

$$\left(\int f^{n/n-1} \right)^{(n-1)/n} \leq C_n \int |\nabla f|, \qquad (**)$$

which can be shown to be equivalent to the *isoperimetric inequality*.

Outline proof.

Start with case $n = 2$ and write

$$g_1(x_2) = \int |\frac{\partial f}{\partial x_1}| \ dx_1.$$

Then $|f(x_1, x_2)| \leq g_1(x_2)$ and

$$\int g_1 x_2) dx_2 \leq \|\nabla f\|_{L^1}.$$

Define $g_2(x_1)$ similarly. So

$$f(x_1, x_2)^2 \leq g_1(x_2) g_2(x_1)$$

and

$$\int f^2 \leq \int g_1 dx_2 \times \int g_2 dx_1 \leq \|\nabla f\|^2_{L^1_2}.$$

For $n = 3$ we get in the same way functions $g_1(x_2, x_3)$ etc. and

$$f^3 \leq g_1(x_2, x_3) g_2(x_1, x_3) g_3(x_1, x_2).$$

Now use the Cauchy-Schwartz inequality twice to estimate the integral of $f^{3/2}$.

Similarly for general n.

The corresponding result for *sets* $K \subset \mathbf{R}^n$ is

$$\mathrm{Vol}(K)^{n-1} \leq \Pi_1 \times \cdots \times \Pi_n,$$

where Π_i is the volume of the projection of K to the ith. coordinate hyperplane.

Inverse and Implicit function theorems in Banach spaces

For example, suppose that H_1, H_2 are Banach spaces and

$$\mathcal{F} : U \to H_2$$

is a continuously differentiable map from an open set $U \subset H_1$ containing 0 and $\mathcal{F}(0) = 0$.

Suppose that $d\mathcal{F}$ at 0 is an isomorphism from H_1 to H_2.

Then for all small y in H_2 there is a unique small solution x to the equation $\mathcal{F}(x) = y$.

Outline proof

Write $\mathcal{F}(x) = x + E(x)$, so the equation to be solved is $x = T(x)$ where $T(x) = y - E(x)$. Then

$$T(x_1) - T(x_2) = E(x_1) - E(x_2),$$

and the hypotheses imply that T is a *contraction* for x_i sufficiently small.

We find a solution $x = \lim_{k\to\infty} T^k(0)$.

For example, consider the PDE $-\Delta f + f^2 = 1 + \rho$ on a compact manifold of dimension $n \leq 8$. We can apply the above to show that for sufficiently small $\rho \in L^2$ there is a solution $1 + \eta$ where $\eta \in L^2_2$ is small.

That is, we define

$$\mathcal{F}(\eta) = -\Delta\eta + (1 + \eta)^2 - 1.$$

Sobolev embedding shows that if $\eta \in L^2_2$ then $\eta \in L^4$ so $\mathcal{F}$ is defined as a map from L^2_2 to L^2.

The derivative at $\eta = 0$ is the linear map $\mathcal{D}(\xi) = -\Delta\xi + 2\xi$ which is positive, self-adjoint hence invertible from L^2_2 to L^2.

Explicitly, if G is the integral operator $(-\Delta + 2)^{-1}$ we define a sequence

$$\sigma_{k+1} = \sigma_k + \left(\rho - (\sigma_k + (G\sigma_k)^2)\right).$$

Then $\sigma_k \to \sigma_\infty$ and the solution η is $G\sigma_\infty$.

Suppose, more generally, that the derivative of $\mathcal{F}$ is surjective, with finite dimensional kernel of dimension p. Then for small y the small solutions of the equation $\mathcal{F}(x) = y$ are parametrised by a manifold of dimension p.

SECTION 2. **Constant Gauss curvature and vortices**

Let M be a compact 2-manifold. A Riemannian metric g on M has a Gauss curvature K_g. We are interested in finding a metric of constant Gauss curvature in a given conformal class.

The basic differential geometric formula we need is that if $g = e^{2u} g_0$ then

$$K_g = e^{-2u} (K_0 - \Delta_0 u)$$

One way of seeing this is through complex differential geometry. If L is a holomorphic line bundle over a complex manifold with a hermitian metric h on the fibres then there is a unique connection on L compatible with the holomorphic and metric structures.

If s is a local holomorphic section of L the curvature Θ of this connection is the 2-form $\overline{\partial}\partial \left(\log |s|^2\right)$.

Suppose that our manifold M is oriented, so it becomes a Riemann surface with area form ω. Then $\overline{\partial}\partial f = \frac{i}{2}\Delta f\ \omega$.

Applying the discussion above to the tangent bundle we get a curvature form Θ which (from the definitions) is $\Theta = -iK\omega$.

We treat the case of *negative Euler characteristic*.

By the Gauss-Bonnet Theorem this means that the integral of the curvature of any metric is negative.

As a first step we choose u so that $K_0 - \Delta_0 u$ is a negative constant. Then $K_g < 0$ so without loss of generality we may suppose that the original metric has $K_0 = -\rho$ with $\rho > 0$.

The equation to solve to get $K = -1$ is

$$-\Delta u + e^{2u} = \rho. \qquad (***)$$

The main result is that there is a unique solution u.
This gives a proof of the Uniformisation Theorem (for compact Riemann surfaces of negative Euler characteristic).

We prove the existence of a solution to this equation (***) using the *continuity method*.

Let $\rho_t = (1-t) + t\rho$ for $t \in [0,1]$.
We have a family of equations $-\Delta u_t + e^{2u_t} = \rho_t$. Let $S \subset [0,1]$ be the set of parameter values for which a solution exists.
The strategy is to prove

1. S is nonempty;
2. S is open;
3. S is closed.

(1) is easy since $u_0 = 0$ is a solution.

To prove (2) we use the inverse function theorem.
Let $\mathcal{F}(u) = -\Delta u + e^{2u}$. In this dimension$L^2_2$ functions are continuous, so $\mathcal{F}$ is defined as a map from L^2_2 to L^2. The derivative of $\mathcal{F}$ at u is the linear map $\mathcal{D}(\xi) = -\Delta\xi + 2e^{2u}\xi$ which is positive self-adjoint hence invertible.
So if $t \in S$ and σ is sufficiently small in L^2 there is an L^2_2 solution v to the equation $\mathcal{F}(v) = \rho_t + \sigma$.
In particular this is true for $\rho_t + \sigma = \rho_{t'}$ for t' close to t.

To complete the proof of openness we need a *regularity* result.

If ρ is C^∞ and u is an L^2_2 solution to $-\Delta u + e^{2u} = \rho$ then u is also smooth.

This follows by "bootstrapping".

For smooth f we have

$$\Delta(e^{2f}) = e^{2f}(2\Delta f + 4|\nabla f|^2).$$

We have inclusions $L^2_2 \subset C^0$ and $L^2_2 \subset L^4_1$. This means that

$$\tau = e^{2u}(2\Delta u + 4|\nabla u|^2)$$

is defined in L^2.
If $f_i \in C^\infty$ converge in L^2_2 to u then $\Delta(e^{2f_i})$ converge in L^2 to τ which implies that the equation $\tau = \Delta e^{2u}$ is true in the weak sense.
So $\Delta^2 u = \Delta\rho - \tau$ in the weak sense and the right hand side is in L^2. Elliptic regularity implies that $u \in L^2_4$

. . . and so on.

Now we want to prove that S is closed. This is done using *a priori* estimates from the maximum principle.

Let $-\Delta u + e^{2u} = \rho$ where ρ is strictly positive. At a point x_0 where u attains is maximum we have $\Delta u \leq 0$. So

$$2u(x_0) \leq \log \rho(x_0).$$

Similarly at a point x_1 where u attains its minimum we get

$$2u(x_1) \geq \log \rho(x_1).$$

This implies that there is some fixed C so that for all $t \in S$

$$\|u_t\|_{C^0} \leq C$$

Then we get a bound on the L^2 norm of Δu_t, hence on the L^2_2 norm of u_t.

Differentiating the equation repeatedly we get bounds on all L^2_k norms of u_t.

Suppose $t_i \in S$ converge to $t_\infty \in [0,1]$.

Taking a subsequence, we can suppose that u_{t_i} converge in L^2_k for all k.

The limit shows that $t_\infty \in S$.

Uniqueness of the solution follows easily from the maximum principle. (Exercise)

Variational method

This gives another approach to the problem. Define

$$E(u) = \int_M |\nabla u|^2 + e^{2u} - \rho u.$$

Then our equation $-\Delta u + e^{2u} - \rho = 0$ is the Euler-Lagrange equation associated to this functional.

We will find a solution by minimising E.

For $x \in M$ write

$$V_x(y) = e^{2y} - \rho(x)y.$$

Our functional is

$$\int_M |\nabla u|^2 + V_x(u) \quad dx$$

The key point is that $V_x(y) \to +\infty$ as $y \to \pm\infty$.

Clearly E is bounded below so we have a number θ defined as the infimum of $E(u)$ as u runs over all smooth functions.

We can choose a “minimising sequence”: a sequence u_i so that $E(u_i) \to \theta$.

We first argue that we can choose such a sequence which satisfies a fixed bound

$$\|u_i\|_{L^\infty} \leq C \qquad (*****)$$

For each $x \in M$ the function V_x has a unique minimum achieved at some $\nu(x) \in \mathbf{R}$. Let $\overline{\nu}, \underline{\nu}$ be the maximum and minimum values of $\nu(x)$.

Suppose we have a u with $\max_x u(x) > \overline{\nu}$. Let $\Omega \subset M$ be the set where $u < \overline{\nu}$. Define a new function u^* by $u^*(x) = u(x)$ for $x \in \Omega$ and $u^*(x) = \overline{\nu}$ if $x \notin \Omega$.

Ignoring for the moment the fact that u^* need not be smooth, we see that $E(u^*) \leq E(u)$.

So we can change any minimising sequence to get a new one which is bounded above. Similarly for the lower bound.

The lack of smoothness of u^* is handled by a straightforward approximation argument.

(Note that changing $\overline{\nu}$ by an arbitrarily small amount we can assume, by Sard's Theorem, that Ω is a domain in M with smooth boundary.)

Next we can prove that our bounded minimising sequence converges in L^2_1.

This is the same idea as in the standard proof of the Riesz representation Theorem, in Hilbert space theory.

We see that, for u_1, u_2 satisfying the bound (*****), we have

$$E(\frac{u_1+u_2}{2}) \leq \frac{E(u_1)+E(u_2)}{2} - \delta \|u_1-u_2\|^2_{L^2_1},$$

for some (computable) $\delta > 0$ depending on C.

This uses the *convexity* of the functions V_x.

So suppose that $u_i \to u_\infty$ in L^2_1.

The bound (*****) means that there is no problem defining e^{2u_∞}.

We claim that u_∞ satisfies the Euler-Lagrange equation in a weak sense. That is, for all smooth ψ we have

$$\langle \nabla u_\infty, \nabla \psi \rangle + \langle e^{2u_\infty} - \rho, \psi \rangle = 0.$$

Write the left hand side of the above as $D(u_\infty, \psi)$. It is the derivative of the functional E at the point u_∞ in the direction ψ.

Arguing for a contradiction, suppose that $D(u_\infty, \psi) < 0$.

We have

$$E(u_i + t\psi) = E(u_i) + tD(u_i, \psi) + \alpha(u_i, t\psi),$$

say, where it is straightforward to show that $|\alpha(u_i, t\psi)| \leq ct^2$ for small t.
It is also straightforward to see that, for this fixed ψ, we have

$$D(u_i, \psi) \to D(u_\infty, \psi)$$

as $i \to \infty$.
Then we can choose a fixed $t > 0$ so that
$E(u_i + t\psi) \leq E(u_i - \eta)$ for some $\eta > 0$ independent of i.
This contradicts the minimising property.

To finish the proof we just have to establish regularity: that u_∞ is smooth (Exercise).

Remark

Although we have avoided needing it; the functional E can be defined on L^2_1.

The Sobolev inequalities in dimension say that $L^2_1 \subset L^p$ for all finite p, but not for $p = \infty$.

If f is a function supported on the unit disc in $\mathbf{R}^2$ with $\|\nabla f\|_{L^2} \leq 1$, a careful study of the Sobolev embedding constants shows that there is an $\epsilon > 0$ such that $\exp(\epsilon f^2)$ is integrable, which implies that $\exp(Kf)$ is integrable for all K.

See Gilbarg and Trudinger, Chapter 7.

The Vortex equation

Let M be a Riemann surface with a compatible metric and area form ω.

Let L be a Hermitian complex line bundle over M. Fix a positive real number τ.

The vortex equation is a system of non-linear equations for a pair (A, ϕ) where A is a unitary connection on L and ϕ is a section of L.

$$\overline{\partial}_A \phi = 0 \quad , \left(\tau - |\phi|^2\right) \omega = iF_A.$$

Here F_A is the curvature of A (a purely imaginary 2-form) and $\overline{\partial}_A$ is the $\overline{\partial}$-operator on sections of L defined by A.

In a local complex co-ordinate $x + iy$ on M we have

$$\overline{\partial}_A s = \frac{1}{2} \left(\nabla_x s + i \nabla_y s \right),$$

where ∇_x, ∇_y are the covariant derivatives in the x, y directions defined by A.

We review the fact that for any connection A the operator $\overline{\partial}_A$ defines a *holomorphic structure* on the line bundle L.

We define a sheaf on M by the local solutions of the equation $\overline{\partial}_A s = 0$. It follows from the definition that this is a sheaf of modules over the structure sheaf of M. We just have to see that there is a non-vanishing solution of the equation in the neighbourhood of any point in M.

In some local trivialisation of L we write
$\nabla_x = \partial_x + A_x, \nabla_y = \partial_y + A_y$.
We need to find a non-trivial solution f of the equation

$$\left(\partial_x + i\partial_y\right) f = (A_x + iA_y)f.$$

Writing $A_x + iA_y = \alpha$ and $f = e^g$ this becomes the equation $\overline{\partial} g = \alpha$, which we can solve locally using the Cauchy kernel. (The first case of the "$\overline{\partial}$-Poincaré lemma)

So the first part of the vortex equation says that ϕ is a holomorphic section of line bundle, for the holomorphic structure defined by A. This implies that the degree d of L is ≥ 0. The Chern-Weil theory says that

$$d = \frac{i}{2\pi} \int_M F_A,$$

and the second equation implies that

$$2\pi d = \int_M (\tau - |\phi|^2)\omega.$$

So $\tau \geq 2\pi d / \mathrm{Area}(M)$ and if equality holds we have $\phi = 0$.

Fix $d > 0$ and $\tau > 2\pi d/\mathrm{Area})$.

The main result is that there is a 1-1 correspondence between solutions of the vortex equation, up to natural equivalence, and positive divisors of degree d on M.

More precisely, the group of “gauge transformations”

$$\mathrm{Aut}(L) = Maps(M, S^1)$$

acts on the space of solutions (A, ϕ) and we regard solutions in the same orbit as equivalent.

Suppose that we start with a positive divisor D.

From complex geometry theory this defines a holomorphic line bundle $\mathcal{L}$ with a holomorphic section Φ having zero divisor D. Also, a Hermitian metric h on $\mathcal{L}$ defines a unitary connection. Now the second vortex equation becomes an equation for the metric h. If we take some initial metric h_0 and set $h = e^{2u} h_0$ the equation is

$$-\Delta u + \Gamma e^{2u} = \rho,$$

where $\Gamma = |\Phi|^2_{h_0}$ and $\rho\, \omega = \tau - iF_0$.

Our hypotheses imply that the integral of ρ is strictly positive.

We can choose the initial metric h_0 so that $c\Gamma \leq \rho \leq C\Gamma$ for positive constants c, C.

Consider the continuity path with equations

$$-\Delta u_t + \Gamma e^{2u_t} = t\rho + (1-t)\Gamma,$$

so $u_0 = 0$ is a solution for $t = 0$.

The same argument as before gives openness.

The proof of closedness is a little more involved. If the maximum and minimum of u_t occur outside D the same argument as before gives upper and lower bounds. We just have to see that the maximum or minimum cannot occur only at points of D.

This follows from a stronger version of the maximum principle, based on the identity

$$-\int_\Omega f\,\Delta f = \int_\Omega |\nabla f|^2$$

if f vanishes on $\partial\Omega$.

We apply this to $f = u_t + \mathrm{constant}$ for a suitable constant.

These vortex equations have their origins in real-world physics.

Solutions are absolute minimisers of the Landau-Ginzburg functional

$$\int |\nabla_A \phi|^2 + |F_A|^2 + (\tau - |\phi|^2)^2.$$

The vortex equations are a prototype for a large family of important equations involving pairs (A, ϕ) comprising a connection A and an additional field ϕ.

The set of positive divisors of degree d is the symmetric product $s^d M$: a complex manifold of complex dimension d
It is the *moduli space* $\mathcal{M}_\tau$ of solutions to the vortex equation.

One consequence is that we get natural metrics on $s^d M$.

Let A_0 be some fixed connection on L. We consider connections $A_0 + ia$ where a is a 1-form on M with $d^* a = 0$. This fixes the gauge freedom $\mathrm{Aut} L$ except for the constant maps from $M \to S^1$.

A tangent vector to $\mathcal{M}_\tau$ at a point $[A, \phi]$ is given by a pair (a, ψ) satisfying the linear equations

- $d^* a = 0$,
- $da = -2\mathrm{Re}\langle \psi, \phi \rangle$,
- $\overline{\partial}_A \psi + i a^{0,1} \phi = 0$,
- $\int_M \mathrm{Im}\langle \psi, \phi \rangle = 0$.

To define the metric on $\mathcal{M}$ we use the L^2 norm of (a, ψ).