IGR Report on Distorted elastic waveguides: theory and application

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I. BACKGROUND/ CONTEXT

The project arose from a requirement for fundamental results pertaining to propagating waves along distorted and bent elastic waveguides. This was a collaborative project between an applied mathematician (Imperial Mathematics Department) and the Non-Destructive Testing group (Imperial Mechanical Engineering Department), the aim being to develop numerical, asymptotic and experimental techniques in parallel, drawing upon the strengths of each group.

The concept of guided wave NDT is straightforward. A guided wave signal which propagates along a structure is modified in some way by the presence of a defect; the defect is then detected by observing the change in the signal. In practice the inspection is normally achieved in a pulse-echo configuration: a short-time signal is generated by a transducer at a chosen location on the structure; any reflection of the guided wave from a defect is then detected at the same transducer location. The amplitude of the reflection indicates the severity of the defect and the arrival time indicates its distance from the transducer.

However, despite the conceptual simplicity, the development of guided wave NDT techniques is extremely complex, for a number of reasons. Even in the simplest of structures, there are many different modes which can propagate. These have different speeds which are also, in general, frequency-dependent. These need to be understood in order to select the mode with the most useful properties: good sensitivity to the type of defect to be detected, good long-distance propagation characteristics, easy to excite and receive. Ideally also, the practical set-up should enable the chosen mode to be excited and received in isolation from the other modes in order to avoid confusion in the interpretation. In more complicated structures there can be additional constraints. If a structure is immersed in a fluid or embedded in another solid then much of the energy of the modes can be lost by radiation into the surrounding medium; therefore knowledge of the extent to which each mode "leaks" is also required. Similarly, material damping can differentially affect the propagation distance of the modes and needs to be understood if present.

The NDT group is the world leader in the exploitation of guided waves for NDT, and already have highly developed models and experimental techniques for their research in this topic. They have also been extremely successful in bringing their findings to market, having initiated two spin-out companies and being engaged for many years in applied research projects and consultancy in this field. From their experience it has become clear that the development of new NDT techniques is only possible when there is thorough knowledge of the wave mode properties. In practical terms this involves finding dispersion curves for the propagating modes or interpreting the wave structures in terms of modes. Sadly, only special geometries such as flat plates and straight cylinders are "separable" and allow for an obvious interpretation in terms of modes. Before this project began there was a reasonable fundamental knowledge of guided waves in circular elastic geometries, say, circumferential modes around a single-layer annulus. But, there were very few results on waves guided in plates of arbitrary curvature, and even less was known about wave propagation along bent elastic bars.

We pursued a parallel strategy of asymptotic modelling and numerical simulations in the Mathematics Department with numerical validation, implementation and experimental work in the NDT group, with the purpose of developing knowledge and procedures for modelling guided waves in these structural forms where hitherto it had not been possible.

II. RESEARCH ACHIEVEMENTS

The grant was used to fund a Post-Doctoral RA, Dr Dmitri Gridin, in the Mathematics Department and an RA, Mr Jimmy Fong, in the Mechanical Engineering Department.

Our stated aim in the original proposal was: **<u>Aims</u>**: The aim of the proposed work is to deliver a theoretical model for predicting high frequency elastic wave propagation in curved elastic bars and pipes. This model will also be validated by comparison with experiments and Finite Element simulations.

A. Mathematical developments

We have developed an asymptotic theory, based upon a refined ray theory, for waves in guided media. This theory as outlined in our articles [1, 8–11, 13] is an advance over those theories that were being used in related areas, mainly ocean acoustics. The theories that were being used were based upon a so-called adiabatic approach: here the local wavenumber and mode-shape of the lowest order approximation coincide with that of a usual, normal, mode in a waveguide with the same local cross-section; higher order terms then correct for the transverse structure of the wavefield. Our original hope had been that we could, more or less directly, adapt these theories to elasticity. However, it is unfortunately the case that these theories are slightly flawed and do not provide good agreement with test cases. However, by modifying the ray ansatz appropriately one can develop a consistent asymptotic theory and we have done so [10, 11]. Moreover, there are different regimes depending upon whether only a few modes or many modes propagate, and this too now falls within our new theory [9, 11]. The theory has been validated numerically upon test cases, and since guided waves arise in many contexts throughout physics and mathematics we were also able to provide useful results in quantum mechanics [8].

The published papers are all for elastic or acoustic (SH waves) plates, but the methods we have developed carry across to bars too and publications for this are now in submission/preparation [1, 9].

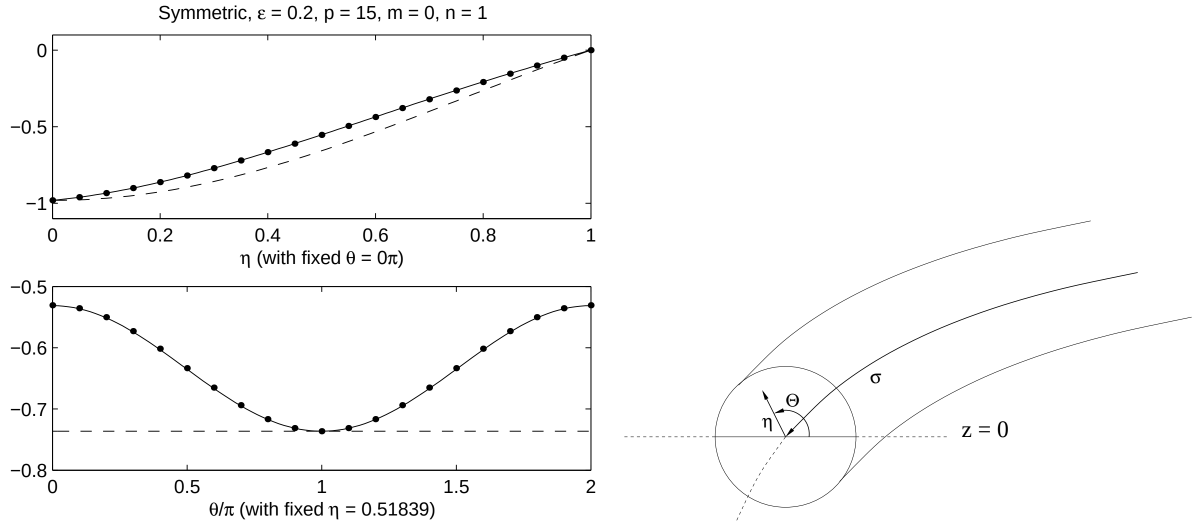

FIG. 1: On the left: Typical modeshapes, these are taken from the asymptotic theory for wave propagation in bars developed in [1]. This figure shows results from numerical (solid line) computations for a test case which is a propagation around a circular torus that has constant radii, with, for comparison, the adiabatic (dashed line), and first-order corrected (dots) wave amplitudes for the $m = 0$, $n = 1$ symmetric mode (m, n are mode numbers, ϵ and p are properties of the guide). The top graph shows the variation of wave amplitude with (radial distance) η for a fixed value of (angle) θ and the bottom graph shows the variation with θ for fixed η; clearly the newly developed asymptotic theory (dots) is highly accurate whereas the earlier adiabatic theory (dashed) is poor. The coordinates η and θ are shown in the sketch on the right.

Dispersion curves: Even for cylindrical geometries, dispersion curves are often hard and time consuming to find numerically: it is not just a simple root-finding exercise. However, they provide a cornerstone upon which much guided wave theory and interpretation depends. In many cases the roots emerge from determinants of large matrices whose elements contain, say, Bessel functions. We developed an asymptotic approach based directly upon ray theory, with a nice physical explanation, to ease the existing numerical approaches [12]: this has now been implemented with the existing codes of the NDT group [3]. Experimental work was also initiated to validate the predictions [7]. Motivated by several issues that arose: the existing theories and approaches cannot easily deal with anisotropy, viscoelasticity, inhomogeneous media and cannot guarantee that all modes have been found, we moved on and developed a general approach for finding dispersion curves in curved geometries. This does not involve root-finding at all - instead one solves the underlying ODE (for a separable geometry) or PDE using spectral methods to turn the differential eigenvalue problem into a matrix eigenvalue problem. This turns out to be remarkably effective as demonstrated in [2]; this too will be implemented in the NDT codes making them very versatile and efficient.

Trapped modes: An interesting issue arose during the research work, could trapped modes, that is, finite energy, non-trivial, eigensolutions, exist in curved elastic structures? Such trapped modes have been of considerable interest in the theory of water waves, quantum mechanics and elsewhere, but little has been done in elasticity. It turns out that the theory for trapped modes is very similar to that for long wave propagation and near cut-off behaviour in bent waveguides. Hence we developed a theory capable of describing modes near cut-off and used it to demonstrate both theoretically and using full numerical simulations that trapped modes exist [8, 13] (see also figure 1). The theory also translates to elastic bars and that work is currently in preparation [9].

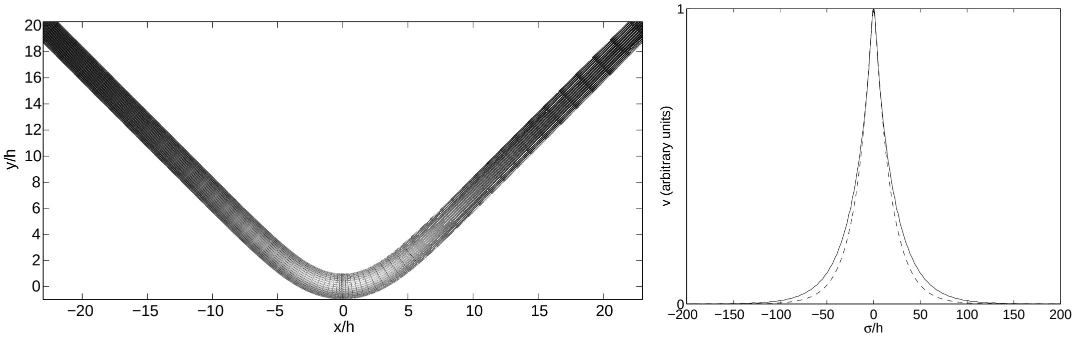

FIG. 2: The left panel shows the elastic energy density of the lowest trapped mode in a bent elastic plate from a numerical simulation; the light shading corresponds to high energy. The right panel shows the normal displacment evaluated along the lower edge with solid as the numerical and dashed as the asymptotic solutions. Thus demonstrating that not only do trapped modes exist, but also that the asymptotic method predicts and represents them accurately.

B. Numerical and experimental developments and validations

As stated in the proposal, one case of guided waves in curved structures had already been derived: the waves which propagate around a single-layer cylindrical annulus [16]. This was a small step but nevertheless it formed a good starting point for validation studies. The results of Liu and Qu's [16] derivation were implemented and used to predict reference "exact" results. This is a non-trivial task because of poor numerical conditioning of the Bessel functions, and so it was not possible to solve throughout the desired solution space. However, as far as it was possible to find results, practically perfect agreement was found between the asymptotic solutions and the "exact" solutions. Furthermore, the asymptotic solutions were shown to be substantially faster and to cover the parts of the solution space which had not been possible with the exact solution [3, 12].

The successful numerical study of the waves in a circular annulus was followed by an experimental validation. This was achieved using a thin plate which was progressively curved so that results could be found for different degrees of curvature. A very precise apparatus was developed using a permanently attached transducer, so that the guided wave velocity could be measured. This study validated the modelling work and also revealed very useful insights into the nature of the waves in curved structures [4, 6].

The single layer "exact" numerical model was developed to allow the modelling of multiple cylindrical layers, and also to include the possibility of a fluid or solid within a pipe or surrounding a pipe. These cases are of great practical importance for lined or coated pipes, pipes which transport fluids, and pipes which are buried. The case of a pipe with a fluid or solid surrounding it is particularly important. It is also fundamentally significant from the theoretical point of view, because in this case the energy of the waves can "leak" away from the pipe into the surrounding material: thus the analysis becomes complex, needing an imaginary wavenumber component to represent the attenuation of the leaking wave. This presents a serious numerical challenge. This problem was solved numerically and was also validated experimentally, revealing how the degree of curvature has a strong influence on the leakage attenuation [5, 7].

The other important application area is bars or pipes which are curved along their axes. The relevance here is to the NDT of pipe bends and bent bars. A breakthrough was made in numerical modelling, by the development of a procedure for finding the dispersion curves in a curved structure of any (arbitrary) shape of cross-section using a regular commercial Finite Element programme (ANSYS for example). Previous research by others had shown that FE could be used to find the dispersion curves in waveguides of arbitrary cross sections, but the new technique enabled this to be done without developing specific FE codes, instead just using a regular commercial code. As a further novelty, it allowed any desired curvature of the waveguide to be modelled too [15]. This technique was used to study the modes in a pipe bend, and led to an explanation of the observed behaviour when a wave in a straight pipe meets a bend [14]. This technique has also been used to great effect to calculate the guided wave behaviour in railway lines [17], the results of which are now being exploited for NDT of railway lines by a spin-out company, Guided Ultrasonics (Rail) Ltd.

The prediction of trapped modes was an exciting discovery in the mathematical programme, and it would be interesting to demonstrate these experimentally. An experimental set-up to do this has been designed and the experiment will be attempted shortly. If successful this will result in another joint publication.

C. Implementation

All of the mathematical developments have initially been implemented in stand-alone code, using the MATLAB programming environment. This enables platform portability and considerable upgrade durability. However in the longer term is is desirable to implement the key findings into larger programs which are documented and supported and available to other researchers. The obvious choice for this implementation is the Disperse program, developed by the NDT group, which is licensed to about 60 organisations worldwide.

The "exact" model for waves travelling around a circular annulus, of any number of layers, has been implemented in Disperse and is now operational. The coding for the case of leaky waves (when a pipe is embedded or immersed in another material) has been developed and is being implemented in the program. It is also planned that the asymptotic models and spectral methods will be implemented for cases where there is strong demand and their speed will be of great benefit. The spectral method will also offer the advantage of ensuring that all modes are found, this being a current concern in complex analyses. It is planned that these implementations will be done during 2005, using a specialist programmer and other funds.

The Finite Element method for waveguides of arbitrary cross section has already been implemented, in that it employs a regular commercial Finite Element code, and has already been used by others for a variety of studies. The procedure to perform this kind of analysis has been reported in the literature and so is available to all [15].

III. RESEARCH IMPACT

Before this project one could argue that there was not a coherent viable theory for propagating modes along arbitrarily bent waveguides, nor were the numerical methods available to rapidly, accurately and efficiently various dispersion curves. Additionally, it was not known whether trapped modes could exist in elastic bars and plates and what criteria were required for them to exist.

Substantial progress has been made upon all fronts: a coherent, accurate theory now exists. Highly accurate, novel and versatile numerical schemes have been developed and implemented. Trapped modes have been discovered.

This research has increased the essential modelling foundation, opening the way for NDT developments in curved structures. This will have a serious impact upon the area of guided waves as anyone interested in propagating waves along curved waveguides, not just in elasticity, will have to use these results.

Besides achieving the objective goals of the project, the collaboration between the mathematicians and the engineers has proved to be extremely fruitful. Mathematicians and engineers approach problems in different ways and it is profitable to learn insights from each other and deepen understanding of the topic. This is not a tangible outcome but it is very encouraging and valuable for the long term development of the research. Furthermore, the existence of the collaborative arrangement has enabled the mathematicians to gain access to application areas and the potential for further applications, and the engineers to receive mathematical support in a variety of topics, extending beyond the present project.

IV. DISSEMINATION

Results of the research programme have been disseminated by presentations at the following conferences:

1 Review of Progress in Quantitative NDE (QNDE), Bellingham, Washington, USA, 2002. QNDE is the most important international conference in NDT.

2 QNDE, Green Bay, Wisconsin, USA, 2003 (× 2)

3 QNDE, Golden, Colorado, USA, 2004

4 Anglo-French Physical Acoustics Conference, Wissant, France, 2002

5 Anglo-French Physical Acoustics Conference, Wye, UK, 2004 (× 2)

6 International Congress on Acoustics, Kyoto, Japan, 4-9 April 2004.

7 British Applied Mathematics Colloquium, UEA, Norwich, UK, 19-22 April 2004.

The published results of the programme will amount to 10 refereed journal papers and 3 conference papers; these are references 1-13 in the references section below. 7 of these articles are already in print.

V. EXPENDITURE

The expenditure was very close to that outlined in the original proposal. It paid the salaries of Dr Gridin and, partly, Mr Fong. Some minor changes occurred in the expenditure on travel versus consumables: conference trips to Japan and the US were more expensive that originally budgeted for. Additionally we paid travel expenses for visiting speakers for a joint research seminar series run on Elastic Waves.

VI. FURTHER RESEARCH

The collaborative relationship has been fruitful, and both parties are keen to take on further joint projects. With this in mind, they have prepared a proposal for a project which aims to make a step change improvement in the way Finite Elements are used to model wave propagation and its interaction with defects. They have recently gained full funding from an industrial source to undertake this and have started work on it; this is funding the continuing employment of Dr Gridin.

The development of the Finite Element technique for calculating the dispersion curves in arbitrary section waveguides offers promise for further research. Currently the technique is limited to waves which do not attenuate. Development of the ideas to include the possibility of attenuating waves would enable the model to be applied to study waveguides which are immersed in fluids or embedded in solids. Funding is currently being sought to research this idea.

Trapped modes are of considerable interest in a variety of other subjects involving waves. Oceanographic waves are sometimes trapped due to density or topographic changes and our work on trapped modes turns out to be relevant; this has led to us beginning a collaboration with a colleague in the USA on this topic.

The development of accurate and versatile numerical schemes for finding dispersion curves and the asymptotic schemes valid for arbitrary geometries have been picked up as being of interest by the Ministry of Defence (DSTL) and they are supporting a CASE studentship to develop this work further.

VII. SUMMARY

A very considerable amount of progress has been made during this project in understanding how waves propagate along bent elastic bars and plates in a variety of different contexts. An important aspect has been that the collaboration between the Mathematical and Engineering pieces of the project have been truly beneficial to both parties: the applications fueling the mathematical progress, and mathematical ideas being implemented directly where they are needed.

The main outcomes are:

- An efficient, accurate, and versatile numerical technique for calculating dispersion curves in a (possibly multi-layer, inhomogeneous or anisotropic) cylindrical annulus. The numerical work is complemented by experimental validation and an asymptotic theory. This means that the understanding of guided waves around a cylindrical elastic annulus (i.e. pipe walls) is now complete.
- Numerical and experimental studies of "leaky" waves, and how curvature affects the leakage attenuation. This provides valuable and practical information on how, say, fluid inside or outside a pipe affects the guiding of elastic waves around the pipe walls.
- Finite Element modelling of dispersion curves in arbitrary cross-section bars - vital for the modelling of guided waves along real structures, this is timely for, say, railway lines.
- The development of refined asymptotic methods, this is a mathematical advance that allows us to sidestep lengthy numerical calculations. This has led to us discovering, both numerically and asymptotically, trapped modes (resonances) caused by curvature, and to predict when they will occur.

This project has enabled us to move the subject forward, and some of the ideas and knowledge gained have already been of practical benefit to NDT: the most visible application being to the NDT inspection of possible damage to railway lines. Curved and deformed geometries are so commonplace in real applications that the methods developed here will be of use and interest for evermore.

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