On Green functions for Dirichlet sub-Laplacians on H-type groups ☆

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A R T I C L E   I N F O

A B S T R A C T

We construct Green functions of Dirichlet boundary value problems for sub-Laplacians on certain unbounded domains of a prototype Heisenberg-type group (prototype H-type group, in short). We also present solutions in an explicit form of the Dirichlet problem for the sub-Laplacian with non-zero boundary datum on half-space, quadrant-space, etc., as well as in a strip.

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1. Introduction

Prototype H-type groups are an important class of homogeneous stratified Lie groups of step two since any (abstract) H-type group is naturally isomorphic to a prototype H-type group. The abelian group $(\mathbb{R}^d; +)$ and the Heisenberg group $\mathbb{H}^d$ are examples of the prototype H-type groups. We denote a prototype H-type group by $\mathbb{G}$ and by $\mathcal{L}$ a sub-Laplacian on $\mathbb{G}$. We consider a smooth open set $D \subset \mathbb{G}$ with boundary ∂D, and study the Dirichlet problem for its sub-Laplacian

$$\begin{cases} \mathcal{L}u = f & \text{in } D, \\ u = \phi & \text{on } \partial D. \end{cases} \tag{1.1}$$

In Euclidean (elliptic) case the boundary value problem (1.1) for suitable essential class of functions f, ϕ (say, $f \in C^\alpha(D)$, $\alpha > 0$, and $\phi \in C(\partial D)$) has a classical solution, that is, a solution in $C^2(D) \cap C^1(\overline{D})$.

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In general, this fact fails completely for the hypoelliptic boundary value problem (1.1). The example of D. Jerison [16] (see also [2, Section 4]) shows that even if the domain D and the boundary datum ϕ (with $f \equiv 0$) are real analytic in the Heisenberg group $\mathbb{H}^d$, then the solution of the Dirichlet problem (1.1) may be not better than Hölder continuous near a characteristic boundary point, that is, the solution is not classical. We recall that the characteristic set (related to vector fields $\{X_1, ..., X_m\}$) of D is the set

$$\{x \in \partial D \,|\, X_k(x) \in T_x(\partial D),\, k = 1, ..., m\},$$

$T_x(\partial D)$ being the tangent space to ∂D at the point x. Here vector fields $X_1, ..., X_m$ with their commutators span the Lie algebra of $\mathbb{G}$. See Section 2 for more details.

The main aim of this short note is to give an answer to the question: *Is there a class of domains in which the Dirichlet boundary value problem* (1.1) *is explicitly solvable in the classical sense?*

This question is inspired by M. Kac's question: *Is there any boundary value problem for the Laplacian which is explicitly solvable in the classical sense for any smooth domain?*

An answer to M. Kac's question was given in our recent paper [20] for the Heisenberg group and in [23] for general homogeneous stratified Lie groups. The boundary conditions appearing there are, however, nonlocal and the corresponding boundary value problem can be called Kac's boundary value problem. It is interesting to note that the explicit solutions in these papers have been constructed for Kac's boundary value problem for the sub-Laplacian equally well also in the presence of characteristic points on the boundary.

However, the above D. Jerison's example hints that one should seek for an answer to our question concerning the Dirichlet problem (1.1) among domains without characteristic points. Even ball-like bounded domains of non-Abelian H-type groups have non-empty collection (set) of characteristic points. For example, any bounded domain of class C^1 in the Heisenberg group $\mathbb{H}^d$, whose boundary is homeomorphic to the $2d$-dimensional sphere $\mathbb{S}^{2d}$, has non-empty characteristic set (see, for example, [4]). In general, this implies that domains, which we are looking for, should be unbounded. On the other hand, to give an explicit representation of a solution we also need to construct Green functions for these domains, so the domains need to have sufficiently rich symmetry. Thus, in this note we show that the boundary value problem (1.1) is explicitly solvable in the classical sense in such domains as half-spaces, quadrant-spaces and so on. Our analysis is based on the Euclidean ideas combined with further results on H-type groups (and on more general groups) obtained by Kohn–Nirenberg [18], Folland [9] and Kaplan [17]. We also should note that this short paper is partially motivated by the recent work [7] in which the authors construct a Green function for the Neumann sub-Laplacian on the Koranyi ball of the Heisenberg group.

We refer to [12–14,19,21] and [24] as well as to references therein for more general Green function analysis of second order subelliptic (and weighted degenerate) operators.

We also refer to [8] for a general point of view on H-type groups from the perspective of general stratified/graded/homogeneous/nilpotent Lie groups. For functional inequalities on stratified Lie groups and further literature review we can refer to [22].

In Section 2 we very briefly review the main concepts of (prototype) H-type groups and fix the notation. In Section 3 we construct Green functions and give representation formulae for solutions.

2. Preliminaries

Following Bonfiglioli, Lanconelli and Uguzzoni [1] we briefly recall the main notions concerning prototype H-type groups. We adopt the notation from [1] and refer to it for further details.

Definition 2.1. The space $\mathbb{R}^{m+n}$ equipped with the group law

$$(x,t) \circ (y,\tau) = \begin{pmatrix} x_k + y_k, & k = 1, ..., m \\ t_k + \tau_k + \frac{1}{2} \langle A^{(k)} x, y \rangle, & k = 1, ..., n \end{pmatrix} \tag{2.1}$$

and with the dilation $\delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$ is called a *prototype H-type group*. Here $A^{(k)}$ is an $m \times m$ skew-symmetric orthogonal matrix, such that, $A^{(k)} A^{(l)} + A^{(l)} A^{(k)} = 0$ for all $k, l \in \{1, ..., n\}$ with $k \neq l$.

Throughout this paper we use the notation $\mathbb{G}$ for a prototype H-type group $(\mathbb{R}^{m+n}, \delta_\lambda, \circ)$. Note that any (abstract) H-type group is naturally isomorphic to a prototype H-group (see [1, Theorem 18.2.1]). It can be directly checked that the group operation $\circ$ defines a step two nilpotent Lie group in which the inverse of (x,t) is $(-x, -t)$, that is, the identity is the origin. It can be also verified that the vector field in the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ that agrees at the origin with $\frac{\partial}{\partial x_j}$, $j = 1, ..., m$, is given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{n} \left(\sum_{i=1}^{m} a_{j,i}^k x_i \right) \frac{\partial}{\partial t_k}, \tag{2.2}$$

where $a_{j,i}^k$ is the (j,i) element of the matrix $A^{(k)}$. The vector fields $X_1, ..., X_m$ with their commutators span the whole $\mathfrak{g}$. Thus, the sub-Laplacian on $\mathbb{G}$ is given by

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 = \Delta_x + \frac{1}{4} |x|^2 \Delta_t + \sum_{k=1}^{n} \langle A^{(k)} x, \nabla_x \rangle \frac{\partial}{\partial t_k}, \tag{2.3}$$

where Δ and ∇ are the Euclidean Laplacian and the Euclidean gradient, respectively. It is not restrictive to suppose that, if ϱ is the center of $\mathfrak{g}$, $\varrho^\perp$ is the orthogonal complement of ϱ and

$$m = \dim(\varrho^\perp), \quad n = \dim(\varrho).$$

We have that the homogeneous dimension of the group is $Q = m + 2n$. We note that since for H-type groups we have $m \geq 2$ and $n \geq 1$, we actually always have $Q \geq 4$.

Now using a generic coordinate $\xi \equiv (x,t)$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, let us introduce the following functions on $\mathbb{G}$:

$$v : \mathbb{G} \to \varrho^\perp, \quad v(\xi) := \sum_{j=1}^{m} \langle \exp_{\mathbb{G}}^{-1}(\xi), X_j \rangle X_j,$$

where $\{X_1, ..., X_m\}$ is an orthogonal basis of $\varrho^\perp$,

$$z : \mathbb{G} \to \varrho, \quad z(\xi) := \sum_{j=1}^{n} \langle \exp_{\mathbb{G}}^{-1}(\xi), Z_j \rangle Z_j,$$

where $\{Z_1, ..., Z_n\}$ is an orthogonal basis of ϱ. Thus, by the definition of v and z, for any $\xi \in \mathbb{G}$, one has

$$\xi = \exp(v(\xi) + z(\xi)), \quad v(\xi) \in \varrho^\perp, \quad z(\xi) \in \varrho,$$

and a direct calculation shows (see, e.g. [1, Proof of Remark 18.3.3]) that

$$|v(\xi)| = |x|, \quad |z(\xi)| = |t|.$$

It simplifies A. Kaplan's theorem in the following form

Theorem 2.2. *There exists a positive constant c such that*

$$\Gamma(\xi) := c \left(|x|^4 + 16|t|^2 \right)^{(2-Q)/4} \tag{2.4}$$

is the fundamental solution of the sub-Laplacian, that is,

$$\mathcal{L}\Gamma_\zeta = -\delta_\zeta, \tag{2.5}$$

where $\Gamma_\zeta(\xi) = \Gamma(\zeta^{-1} \circ \xi)$ and δ_ζ is the Dirac distribution at $\zeta \equiv (y, \tau) \in \mathbb{G}$.

Note that more general result for abstract H-type groups was established by A. Kaplan in [17]. Now the Green function for the Dirichlet sub-Laplacian in D is defined by the formula

$$G_D(\xi, \zeta) = \Gamma(\zeta^{-1} \circ \xi) - h_\zeta(\xi), \tag{2.6}$$

with

$$G_D(\xi, \zeta) = 0, \quad \xi \in \partial D. \tag{2.7}$$

Here, $h_\zeta(\xi)$ is a harmonic function, that is,

$$\mathcal{L}h_\zeta(\xi) = 0 \quad \text{in } D, \tag{2.8}$$

having as boundary values (in the Perron–Wiener–Brelot sense) the fundamental solution with pole at $\zeta \in D$.

Let ∂D be the boundary of a smooth domain D in $\mathbb{G}$, $d\nu$ the volume element on $\mathbb{G}$, and $\langle X_j, d\nu \rangle$ the natural pairing between vector fields and differential forms. We also recall that the standard Lebesque measure on $\mathbb{R}^{m+n}$ is the Haar measure for $\mathbb{G}$ (see, e.g. [8, Proposition 1.6.6]).

The following version of Green's second formula will be useful for our analysis. It goes back to the integration by parts formula in [2] but the following form (given in [23, Proposition 3.10]) will be useful for us here.

Proposition 2.3 *(Green's second formula). Let $u, v \in C^2(D) \bigcap C^1(\overline{D})$. Then*

$$\int_D (u\mathcal{L}v - v\mathcal{L}u)d\nu = \int_{\partial D} (u\langle \widetilde{\nabla}v, d\nu \rangle - v\langle \widetilde{\nabla}u, d\nu \rangle), \tag{2.9}$$

where $\mathcal{L}$ is the sub-Laplacian on $\mathbb{G}$ and

$$\widetilde{\nabla}u = \sum_{k=1}^m (X_k u) X_k. \tag{2.10}$$

By classical arguments one verifies that the above Green's second formula is valid for the Green function (and the fundamental solution). See [23] for further discussion, but also [2, Section 7]. The relation between the $(n-1)$-form under the integral in the right-hand side of (2.9) and the perimeter and surface measures on ∂D has been discussed in [23].

3. Green functions and representations of solutions

In this section we construct, by using the classical method of reflection, Green functions of Dirichlet boundary value problems for sub-Laplacians on l-wedge like and l-strip like unbounded domains of a

prototype H-type group. We also present solutions in an explicit form of the Dirichlet problem for the sub-Laplacian with non-zero boundary datum on those domains. Of course, the results are well known in the abelian cases.

3.1. Green functions and representations of solutions in l-wedge like spaces

Let $\mathbb{G}^{\ddagger}$ be the l-wedge like space

$$\mathbb{G}^{\ddagger} = \{\xi = (x_1, \ldots, x_m, t_1, \ldots, t_n)|\, x_1, \ldots, x_l > 0\},$$

for some $1 \leq l \leq m$. Let the point $\zeta = (y, \tau) = (y_1, y_2, \ldots, y_m, \tau_1, \ldots, \tau_n)$ lie in this l-wedge like space, $y_1 > 0, \ldots, y_l > 0$. The point

$$\zeta_{x_k} := (y_1, \ldots, -y_k, \ldots, y_m, \tau_1, ..., \tau_n)$$

is said to be symmetric for the point ζ with respect to the hyperplane $x_k = 0$. Similarly, the point

$$\zeta_{x_k x_s} := (y_1, \ldots, -y_k, \ldots, -y_s, \ldots, y_m, \tau_1, ..., \tau_n)$$

is said to be symmetric for the point ζ_{x_k} with respect to the hyperplane $x_s = 0$ and so on. It is clear that the symmetry indices are invariant under permutations. We will also need the notation $\Gamma((\zeta_{(j,l)})^{-1} \circ \xi)$, $j \leq l$, which means sum of the functions $\Gamma((\zeta_{(j,l)})^{-1} \circ \xi)$, $j \leq l$, over all possible (j, l) combination symmetry arguments: here in order to reduce the number of subindices we write $(\zeta_{(j,l)})^{-1} \circ \xi$ for $\zeta^{-1}_{x_{k_1} \ldots x_{k_j}} \circ \xi$. For example, if $l = 3$, $j = 2$, then

$$\Gamma((\zeta_{(2,3)})^{-1} \circ \xi) = \Gamma((\zeta_{x_1 x_2})^{-1} \circ \xi) + \Gamma((\zeta_{x_1 x_3})^{-1} \circ \xi) + \Gamma((\zeta_{x_3 x_2})^{-1} \circ \xi),$$

and if $l = 3$, $j = 3$, then

$$\Gamma((\zeta_{(3,3)})^{-1} \circ \xi) = \Gamma((\zeta_{x_1 x_2 x_3})^{-1} \circ \xi).$$

We have

Proposition 3.1. *The function*

$$G_{\mathbb{G}^{\ddagger}}(\xi, \zeta) = \Gamma(\zeta^{-1} \circ \xi) + \sum_{j=1}^{l}(-1)^j \Gamma((\zeta_{(j,l)})^{-1} \circ \xi) \tag{3.1}$$

is the Green function for the Dirichlet sub-Laplacian in $\mathbb{G}^{\ddagger}$.

Proof of Proposition 3.1. Since by definition $\zeta_{(j,l)} \notin \mathbb{G}^{\ddagger}$, $j = 1, \ldots, l$, it follows from (2.5) that

$$\mathcal{L}\Gamma((\zeta_{(j,l)})^{-1} \circ \xi) = -\delta_{\zeta_{(j,l)}} = 0 \text{ in } \mathbb{G}^{\ddagger},$$

for any $\xi \in \mathbb{G}^{\ddagger}$ and $j = 1, \ldots, l$. Thus, the function $\sum_{j=1}^{l}(-1)^j \Gamma((\zeta_{(j,l)})^{-1} \circ \xi)$ satisfies the condition (2.8), i.e. it is harmonic in $\mathbb{G}^{\ddagger}$. Now it is left to check the boundary condition for the domain $\mathbb{G}^{\ddagger}$, that is, the function $G_{\mathbb{G}^{\ddagger}}$ should become zero at $x_1 = 0$ and at infinity. Recall that

$$d(\xi, \zeta) := \left(\Gamma(\zeta^{-1} \circ \xi)\right)^{\frac{1}{2-Q}} \tag{3.2}$$

is an actual distance on $\mathbb{G}$ (see, e.g. [3]). Now it is easy to see that the d-distance from any point of the hyperplane $x_k = 0$ to the points ζ and ζ_{x_k} is the same, that is, $G_{\mathbb{G}^\ddagger}$ satisfies the Dirichlet condition at the hyperplanes $x_1 = 0, \ldots, x_l = 0$ and it is also clear (by the construction) that the function $G_{\mathbb{G}^\ddagger}$ is zero at the infinity. It proves that

$$G_{\mathbb{G}^\ddagger}(\xi, \zeta) = 0, \quad \xi \in \partial\mathbb{G}^\ddagger. \qquad \square$$

Now we consider a smooth open set $D \subset \mathbb{G}$ with boundary ∂D, and study the Dirichlet problem for the sub-Laplacian $\mathcal{L}$ in D.

For $0 < \alpha < 1$, Folland and Stein (see [10] and see also [9]) defined the anisotropic Hölder spaces $F_\alpha(D)$, $D \subset \mathbb{G}$, by

$$F_\alpha(D) = \{f : D \to \mathbb{C} : \sup_{\substack{\xi, \zeta \in D \\ \xi \neq \zeta}} \frac{|f(\xi) - f(\zeta)|}{[d(\xi, \zeta)]^\alpha} < \infty\},$$

where d is defined by the formula (3.2) in our case. For $k \in \mathbb{N}$ and $0 < \alpha < 1$, one defines $F_{k+\alpha}(D)$ as the space of all $f : D \to \mathbb{C}$ such that all X_j-derivatives of f of order k belong to $F_\alpha(D)$. A bounded function f is called α-Hölder continuous in $D \subset \mathbb{G}$ if $f \in F_\alpha(D)$.

Let $f \in F_\alpha(\mathbb{G}^\ddagger)$, $0 < \alpha < 1$, $\operatorname{supp} f \subset \mathbb{G}^\ddagger$, and $\phi \in C^\infty(\partial\mathbb{G}^\ddagger)$, $\operatorname{supp} \phi \subset \{x_1 = 0\} \bigcup \ldots \bigcup \{x_l = 0\}$. Consider the Dirichlet problem for the sub-Laplacian

$$\begin{cases} \mathcal{L}u = f & \text{in } \mathbb{G}^\ddagger, \\ u = \phi & \text{on } \partial\mathbb{G}^\ddagger. \end{cases} \tag{3.3}$$

Theorem 3.2. *Let* $f \in F_\alpha(\mathbb{G}^\ddagger)$, $0 < \alpha < 1$, $\operatorname{supp} f \subset \mathbb{G}^\ddagger$, *and* $\phi \in C^\infty(\partial\mathbb{G}^\ddagger)$. *Then the boundary value problem* (3.3) *has a unique solution* $u \in C^2(\mathbb{G}^\ddagger) \cap C^1(\overline{\mathbb{G}^\ddagger})$ *and it can be represented by the formula*

$$u(\xi) = \int_{\mathbb{G}^\ddagger} G_{\mathbb{G}^\ddagger}(\xi, \zeta) f(\zeta) d\nu(\zeta) - \int_{\partial\mathbb{G}^\ddagger} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{G}^\ddagger}(\xi, \zeta), d\nu(\zeta) \rangle, \quad \xi \in \mathbb{G}^\ddagger, \tag{3.4}$$

where $\widetilde{\nabla}$ *is defined by* (2.10), *in particular,*

$$\widetilde{\nabla} G_{\mathbb{G}^\ddagger} = \sum_{k=1}^m (X_k G_{\mathbb{G}^\ddagger}) X_k.$$

Proof of Theorem 3.2. Let $u \in C^2(\mathbb{G}^\ddagger) \cap C^1(\overline{\mathbb{G}^\ddagger})$ and assume that u tends to zero at infinity. The Green's second formula (2.9) is in bounded domains, but it is still applicable for functions, with necessary decay rates at infinity, in unbounded domains. It can be shown by the standard argument using quasi-balls with radii $R \longrightarrow \infty$. Thus, if we apply Green's second formula (2.9) to the function u with $v(\zeta) = G_{\mathbb{G}^\ddagger}(\xi, \zeta)$, we shall obtain

$$u(\xi) = \int_{\mathbb{G}^\ddagger} G_{\mathbb{G}^\ddagger}(\xi, \zeta) f(\zeta) d\nu(\zeta) - \int_{\partial\mathbb{G}^\ddagger} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{G}^\ddagger}(\xi, \zeta), d\nu(\zeta) \rangle.$$

Here we have used the properties of the Green function

$$G_{\mathbb{G}^\ddagger}(\xi, \zeta) = 0, \quad \zeta \in \partial\mathbb{G}^\ddagger,$$

and, by construction the function $G_{\mathbb{G}^\ddagger}$ is symmetric, that is, $G_{\mathbb{G}^\ddagger}(\xi, \zeta) = G_{\mathbb{G}^\ddagger}(\zeta, \xi)$ in $\mathbb{G}^\ddagger$, so

$$\mathcal{L}_\zeta G_{\mathbb{G}^\ddagger}(\xi, \zeta) = -\delta_\xi,$$

where δ_ξ is the Dirac distribution at $\xi \in \mathbb{G}^\ddagger$. Now we need to show that the function defined by (3.4) belongs to $C^2(\mathbb{G}^\ddagger) \cap C^1(\overline{\mathbb{G}^\ddagger})$. Since $f \in F_\alpha(\mathbb{G}^\ddagger)$, supp $f \subset \mathbb{G}^\ddagger$, the volume potential (the first term of the right hand side in (3.4)) belongs to $C^2(\overline{\mathbb{G}^\ddagger})$ by Folland's theorem (see [9, Theorem 6.1], see also [10]). Hörmander's hypoellipticity theorem (see [15]) guarantees that every harmonic function is C^∞, hence the Dirichlet double layer potential (the second term of the right hand side in (3.4)) is in $C^2(\mathbb{G}^\ddagger)$. On the other hand, since $\phi \in C^\infty(\partial\mathbb{G}^\ddagger)$, supp $\phi \subset \{x_1 = 0, \ldots, x_l = 0\}$ and the boundary hyperplanes $\{x_1 = 0\}, \ldots, \{x_l = 0\}$ have no characteristic points (see [11, Section 8] for more discussions on the non-characteristic hyperplanes in $\mathbb{G}$) the Dirichlet double layer potential is continuous on the boundary by the Kohn–Nirenberg theorem (see [2, Theorem 3.12], which is a consequence of [18, Theorem 4], see also [5,6]). □

Remark 3.3. One may consider $\mathbb{G}^\ddagger_a$, $a = (a_1, \ldots, a_l) \in \mathbb{R}^l$, l-wedge like space $\{\xi = (x_1, \ldots, x_m, t_1, \ldots, t_n) | x_1 > a_1, \ldots, x_l > a_l\}$, but in this l-wedge like space the Green function $G_{\mathbb{G}^\ddagger_a}$ has the same formula as the formula (3.1) in which the symmetry points are chosen, in this case, with respect to the hyperplanes $\{x_1 = a_1\}, \ldots, \{x_l = a_l\}$. Of course, an analogue of Theorem 3.2 will be obtained by the same argument.

Let us demonstrate some simple cases of Theorem 3.2 and Proposition 3.1 with different (simpler) notations. First, as above we construct a Green function for the Dirichlet sub-Laplacian in a half-space on $\mathbb{G}$. Let $\mathbb{G}^+$ be the half-space

$$\mathbb{G}^+ = \{\xi = (x_1, \ldots, x_m, t_1, \ldots, t_n) | x_1 > 0\}.$$

Let the point $\zeta = (y, \tau) = (y_1, y_2, \ldots, y_m, \tau_1, \ldots, \tau_n)$ lie in this half-space, $y_1 > 0$. The point

$$\zeta^* = (y^*, \tau) := (-y_1, y_2, \ldots, y_m, \tau_1, \ldots, \tau_n)$$

is said to be symmetric for the point ζ with respect to the hyperplane $x_1 = 0$. We have the following direct consequence of Proposition 3.1.

Corollary 3.4. *The function*

$$G_{\mathbb{G}^+}(\xi, \zeta) = \Gamma(\zeta^{-1} \circ \xi) - \Gamma((\zeta^*)^{-1} \circ \xi) \tag{3.5}$$

is the Green function for the Dirichlet sub-Laplacian in $\mathbb{G}^+$.

Let $f \in F_\alpha(\mathbb{G}^+)$, supp $f \subset \mathbb{G}^+$, and $\phi \in C^\infty(\partial\mathbb{G}^+)$, supp $\phi \subset \{x_1 = 0\}$. Consider the Dirichlet problem for the sub-Laplacian

$$\begin{cases} \mathcal{L}u = f & \text{in } \mathbb{G}^+, \\ u = \phi & \text{on } \partial\mathbb{G}^+. \end{cases} \tag{3.6}$$

In this case Theorem 3.2 can be restated in the following form.

Corollary 3.5. *The boundary value problem (3.6) has a unique solution $u \in C^2(\mathbb{G}^+) \cap C^1(\overline{\mathbb{G}^+})$ and it can be represented by the formula*

$$u(\xi) = \int_{\mathbb{G}^+} G_{\mathbb{G}^+}(\xi, \zeta) f(\zeta) d\nu(\zeta) - \int_{\partial \mathbb{G}^+} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{G}^+}(\xi, \zeta), \, d\nu(\zeta) \rangle, \quad \xi \in \mathbb{G}^+, \tag{3.7}$$

where

$$G_{\mathbb{G}^+}(\xi, \zeta) = \Gamma(\zeta^{-1} \circ \xi) - \Gamma((\zeta^*)^{-1} \circ \xi).$$

Now we construct a Green function for the Dirichlet sub-Laplacian in a quadrant-space on $\mathbb{G}$. Let $\mathbb{G}^{\oplus}$ be the quadrant-space

$$\mathbb{G}^{\oplus} = \{\xi = (x_1, x_2, ..., x_m, t_1, ..., t_n) | \, x_1 > 0, \, x_2 > 0\}.$$

Let the point $\zeta = (y, \tau) = (y_1, y_2, ..., y_m, \tau_1, ..., \tau_n)$ lie in this quadrant-space, $y_1 > 0$, $y_2 > 0$. Denote by

$$\zeta^* = (y^*, \tau) := (-y_1, y_2, ..., y_m, \tau_1, ..., \tau_n)$$

and

$$\overline{\zeta} = (\overline{y}, \tau) := (y_1, -y_2, ..., y_m, \tau_1, ..., \tau_n)$$

the symmetric points for ζ with respect to the hyperplanes $x_1 = 0$ and $x_2 = 0$, respectively. The point

$$\overline{\zeta}^* = (\overline{y}^*, \tau) = (-y_1, -y_2, ..., y_m, \tau_1, ..., \tau_n)$$

is the symmetric point for ζ^* with respect to the hyperplane $x_2 = 0$ and the symmetric point for $\overline{\zeta}$ with respect to the hyperplane $x_1 = 0$.

We have the following another direct consequence of Proposition 3.1.

Corollary 3.6. *The function*

$$G_{\mathbb{G}^{\oplus}}(\xi, \zeta) = \Gamma(\zeta^{-1} \circ \xi) + \Gamma((\overline{\zeta}^*)^{-1} \circ \xi) - \Gamma((\zeta^*)^{-1} \circ \xi) - \Gamma((\overline{\zeta})^{-1} \circ \xi) \tag{3.8}$$

is the Green function for the Dirichlet sub-Laplacian in $\mathbb{G}^{\oplus}$.

3.2. Green functions and representations of solutions in l-strip spaces

Let $\mathbb{G}^{\models}$ be the *l*-strip like space

$$\mathbb{G}^{\models} = \{\xi = (x_1, \ldots, x_m, t_1, \ldots, t_n) | \, a > x_l > 0\},$$

for some $1 \leq l \leq m$. Let the point $\zeta = (y, \tau) = (y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)$ lie in this *l*-strip space, $a > y_l > 0$. We will use the notations

$$\zeta_{+,j} := (y_1, \ldots, y_l - 2aj, \ldots, y_m, \tau_1, ..., \tau_n),$$

and

$$\zeta_{-,j} := (y_1, \ldots, -y_l + 2aj, \ldots, y_m, \tau_1, ..., \tau_n),$$

for all $j = 0, 1, 2, \ldots$. Following familiar pattern as above we obtain

Proposition 3.7. *The function*

$$G_{\mathbb{G}^{\vDash}}(\xi, \zeta) = \sum_{j=-\infty}^{\infty} \left(\Gamma(\zeta_{+,j}^{-1} \circ \xi) - \Gamma(\zeta_{-,j}^{-1} \circ \xi) \right) \tag{3.9}$$

is the Green function for the Dirichlet sub-Laplacian in $\mathbb{G}^{\vDash}$.

Proof of Proposition 3.7. It is evident that the first additive component in the $j = 0$ term of (3.9), i.e. the term $\Gamma(\zeta_{+,0}^{-1} \circ \xi)$ represents the fundamental solution and all the other terms are subharmonic functions in $\mathbb{G}^{\vDash}$. Let us check that traces of (3.9) vanish on hyperplanes $x_l = 0$ and $x_l = a$. If $x_l = 0$, then (3.9) gives

$$G_{\mathbb{G}^{\vDash}}(\xi, \zeta)|_{x_l=0} =$$

$$c \sum_{j=-\infty}^{\infty} \Big(\big(((x_1 - y_1)^2 + \ldots + (-y_l + 2aj)^2 + \ldots + (x_m - y_m)^2)^2 + 16|t - \tau|^2 \big)^{(2-Q)/4}$$

$$- \big(((x_1 - y_1)^2 + \ldots + (y_l - 2aj)^2 + \ldots + (x_m - y_m)^2)^2 + 16|t - \tau|^2 \big)^{(2-Q)/4} \Big) = 0. \tag{3.10}$$

If $x_l = a$, then (3.9) gives

$$G_{\mathbb{G}^{\vDash}}(\xi, \zeta)|_{x_l=a} =$$

$$c \sum_{j=-\infty}^{\infty} \Big(\big(((x_1 - y_1)^2 + \ldots + (a - y_l + 2aj)^2 + \ldots + (x_m - y_m)^2)^2 + 16|t - \tau|^2 \big)^{(2-Q)/4}$$

$$- \big(((x_1 - y_1)^2 + \ldots + (a + y_l - 2aj)^2 + \ldots + (x_m - y_m)^2)^2 + 16|t - \tau|^2 \big)^{(2-Q)/4} \Big) =$$

$$c \sum_{j=0}^{\infty} \big(((x_1 - y_1)^2 + \ldots + (a - y_l + 2aj)^2 + \ldots + (x_m - y_m)^2)^2 + 16|t - \tau|^2 \big)^{(2-Q)/4}$$

$$- c \sum_{j=1}^{\infty} \big(((x_1 - y_1)^2 + \ldots + (a + y_l - 2aj)^2 + \ldots + (x_m - y_m)^2)^2 + 16|t - \tau|^2 \big)^{(2-Q)/4}$$

$$+ c \sum_{j=-1}^{-\infty} \big(((x_1 - y_1)^2 + \ldots + (a - y_l + 2aj)^2 + \ldots + (x_m - y_m)^2)^2 + 16|t - \tau|^2 \big)^{(2-Q)/4}$$

$$- c \sum_{j=0}^{-\infty} \big(((x_1 - y_1)^2 + \ldots + (a + y_l - 2aj)^2 + \ldots + (x_m - y_m)^2)^2 + 16|t - \tau|^2 \big)^{(2-Q)/4} = 0. \tag{3.11}$$

Here the first term ($j = 0$ term) of the first sum is canceled with the first term ($j = 1$ term) of the second sum and the second terms of the first sum is canceled with the second term of the second sum and so on, that is, the first two sums give zero. Similarly, the first term of the third sum is canceled with the first term of the last sum and the second term of the third sum is canceled with the second term of the last sum and so on, that is, the last two sums also give zero. As a result, the trace vanishes at $x_l = a$. $\quad\square$

Let $f \in F_\alpha(\mathbb{G}^{\vDash})$, $0 < \alpha < 1$, $\mathrm{supp}\, f \subset \mathbb{G}^{\vDash}$, and $\phi \in C^\infty(\partial\mathbb{G}^{\vDash})$, $\mathrm{supp}\, \phi \subset \{x_l = 0\} \bigcup \{x_l = a\}$. Consider the Dirichlet problem for the sub-Laplacian

$$\begin{cases} \mathcal{L}u = f & \text{in } \mathbb{G}^{\vDash}, \\ u = \phi & \text{on } \partial\mathbb{G}^{\vDash}. \end{cases} \tag{3.12}$$

Theorem 3.8. *Let $f \in \mathcal{F}_\alpha(\mathbb{G}^\vDash)$, $0 < \alpha < 1$, supp $f \subset \mathbb{G}^\vDash$, and $\phi \in C^\infty(\partial\mathbb{G}^\vDash)$. Then the boundary value problem* (3.12) *has a unique solution $u \in C^2(\mathbb{G}^\vDash) \cap C^1(\overline{\mathbb{G}^\vDash})$ and it can be represented by the formula*

$$u(\xi) = \int\limits_{\mathbb{G}^\vDash} G_{\mathbb{G}^\vDash}(\xi, \zeta) f(\zeta) d\nu(\zeta) - \int\limits_{\partial\mathbb{G}^\vDash} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{G}^\vDash}(\xi, \zeta), d\nu(\zeta) \rangle, \quad \xi \in \mathbb{G}^\vDash, \tag{3.13}$$

where $\widetilde{\nabla}$ is defined by (2.10)*, in particular,*

$$\widetilde{\nabla} G_{\mathbb{G}^\vDash} = \sum_{k=1}^m \left(X_k G_{\mathbb{G}^\vDash} \right) X_k.$$

Proof of Theorem 3.8. Similar to proof of Theorem 3.2. $\quad\square$

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