Wave equation for sums of squares on compact Lie groups

Claudia Garetto [a], Michael Ruzhansky [b,*,1]

[a] *Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU, United Kingdom*
[b] *Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom*

Abstract

In this paper we investigate the well-posedness of the Cauchy problem for the wave equation for sums of squares of vector fields on compact Lie groups. We obtain the loss of regularity for solutions to the Cauchy problem in local Sobolev spaces depending on the order to which the Hörmander condition is satisfied, but no loss in globally defined spaces. We also establish Gevrey well-posedness for equations with irregular coefficients and/or multiple characteristics. As in the Sobolev spaces, if formulated in local coordinates, we observe well-posedness with the loss of local Gevrey order depending on the order to which the Hörmander condition is satisfied.
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1. Introduction

In this paper we investigate the well-posedness of the Cauchy problem for time-dependent wave equations associated to sums of squares of invariant vector fields on compact Lie groups.

Such analysis is motivated, in particular, by general investigations of the well-posedness and wave propagation governed by subelliptic operators and problems with multiplicities. An often encountered example of subelliptic behaviour is a sum of squares of vector fields, extensively analysed by Hörmander [17,18], Oleinik and Radkevich [27], Rothschild and Stein [28], and by many others. For invariant operators on compact Lie groups, the sum of squares becomes formally self-adjoint, making the corresponding wave equation hyperbolic, a necessary condition for the analysis of the corresponding Cauchy problem. Already in this setting, we discover a new phenomenon of the loss of the local Gevrey regularity for its solutions. Moreover, this loss is linked to the order to which the Hörmander condition is satisfied.

Thus, let G be a compact Lie group of dimension n with Lie algebra $\mathfrak{g}$, and let $X_1, \dots, X_k$ be a family of left-invariant vector fields in $\mathfrak{g}$. Let

$$\mathcal{L} := X_1^2 + \cdots + X_k^2 \tag{1.1}$$

be the sum of squares of derivatives defined by the vector fields. If the iterated commutators of $X_1, \dots, X_k$ span the Lie algebra of G, the operator $\mathcal{L}$ is a sub-Laplacian on G, hypoelliptic in view of Hörmander's sum of the squares theorem.

With or without the Hörmander condition, it can be shown that the operator $\partial_t^2 - \mathcal{L}$ is (weakly) hyperbolic (see Remark 3.2). For a continuous function $a = a(t) \geq 0$, we will be concerned with the Cauchy problem

$$\begin{cases} \partial_t^2 u(t,x) - a(t)\mathcal{L}u(t,x) = 0, & (t,x) \in [0,T] \times G, \\ u(0,x) = u_0(x), & x \in G, \\ \partial_t u(0,x) = u_1(x), & x \in G. \end{cases} \tag{1.2}$$

When localised, the Cauchy problem (1.2) is a weakly hyperbolic equation with both time and space dependent coefficients, and the available results and techniques are rather limited compared to, for example, the situation when the coefficients depend only on time. For example, general Gevrey well-posedness results of Bronshtein [3] or Nishitani [26] may apply for some a and $\mathcal{L}$, but in general they do not take into account the geometry of the problem and of the operator $\mathcal{L}$.

In the case of the Euclidean space $\mathbb{R}^n$, the Cauchy problem for the operator $\partial_t^2 - a(t)\Delta$ with the Laplacian Δ has been extensively studied. It is known that the Cauchy problem for this operator may be not well-posed in $C^\infty(\mathbb{R}^n)$ and in $\mathcal{D}'(\mathbb{R}^n)$ if the function $a(t)$ becomes zero or is irregular, see, respectively, Colombini and Spagnolo [7], and Colombini, Jannelli and Spagnolo [6]. Thus, Gevrey spaces appear naturally in such well-posedness problems already on $\mathbb{R}^n$, and for the latter equation, a number of sharp well-posedness results in Gevrey spaces have been established by Colombini, de Giorgi and Spagnolo [5]. We note that problems with lower (e.g. distributional) regularity of coefficients require different methods, see e.g. the authors' paper [14]. At the same time, even for analytic principal part, inclusion of lower order terms may require suitable Levi conditions, see e.g. [13].

Our analysis will cover the case of the Laplacian Δ on the compact Lie group G since we can write it as $\mathcal{L} = X_1^2 + \cdots + X_n^2$ for a basis $X_1, \dots, X_n$ of the Lie algebra of G. For different ways of representing Laplacians on compact Lie groups we refer to an extensive discussion in Stein [33]. In the case of the Laplacian we recover the orders that can be obtained from the work of Nishitani [26] since in this case we can write $\mathcal{L}$ in local coordinates in the divergence form. For sub-Laplacian $\mathcal{L}$ this no longer applies (neither are the results of Jannelli [19] because of the lack of divergence form and appearing lower order terms).

Also in the case of the Laplacian, the strictly hyperbolic wave and Schrödinger equations on compact Lie groups have been recently analysed in the framework of the KAM theory by Berti and Procesi [1], with further additions of nonlinear terms. We can refer to their paper and references therein, as well as to Helgason [16], for a thorough explanation of the appearance of these partial differential equations on compact Lie groups, and the need to study them contributed greatly to the development of the modern theory of compact Lie groups, starting with Weyl [35]. L^p-estimates for wave equations have been considered on Lie groups as well. Here, in the case of the Laplacian on compact Lie groups, the loss of regularity in L^p has been obtained by Chen, Fan and Sun [4]; however, for $p > 1$, this loss can be also deduced from the localised L^p-estimates for Fourier integral operators by Seeger, Sogge and Stein [32]. In turn, different techniques are required for the wave equation with sub-Laplacians, see e.g. the case of the standard sub-Laplacian on the Heisenberg group by Müller and Stein [24]. The finite propagation speed results for wave equations for subelliptic operators are also known in different related settings: analysis for abstract operators was developed by Melrose [23], with explicit formulae for the wave kernels on the Heisenberg group obtained by Taylor [34] and Nachman [25] and, more recently, by Greiner, Holcman and Kannai [15].

The point of this paper is that by applying the global Fourier analysis on G to the Cauchy problem (1.2) we can view it as an equation with coefficients depending only on t, leading to a range of sharp results depending on further properties of the function $a(t)$. However, since the global Fourier coefficients of functions on G become matrix valued, on the Fourier transform side the scalar equation (1.2) becomes a system, with the size of the system going to infinity with the dimension of representations, unless G is a torus. An important observation enabling our analysis is that we can explore the sum of squares structure of the operator $\mathcal{L}$ using a notion of a matrix symbol for operators on compact Lie groups. Thus, we will show that the system for Fourier coefficients decouples completely into independent scalar equations for the matrix components of Fourier coefficients. The equations are determined by the entries of the matrix symbol of $\mathcal{L}$ which we also study for this purpose, in particular establishing lower bounds for its eigenvalues in terms of the order to which the Hörmander condition is satisfied (in the case when it is indeed satisfied).

Our results will apply to general operators $\mathcal{L}$ of the form (1.1) without $X_1, \ldots, X_k$ necessarily satisfying the Hörmander condition. However, if the Hörmander condition is satisfied, the well-posedness of (1.2) in $C^\infty(G)$, $\mathcal{D}'(G)$, or usual Gevrey spaces on G viewed as a manifold will follow. Moreover, such well-posedness statements can be refined with respect to the loss of regularity and the orders of appearing Sobolev or Gevrey spaces if we know also the order to which the Hörmander condition is satisfied. To our knowledge, this phenomenon of local loss of Gevrey regularity appears to be new in the study of weakly hyperbolic equations.

Let us give an example of such an equation (1.2) on the 3-sphere $G = \mathbb{S}^3$. Here, if X, Y, Z are an orthonormal basis (with respect to the Killing form) of left-invariant vector fields on $\mathbb{S}^3$, then we can set $\mathcal{L}$ to be the sub-Laplacian

$$\mathcal{L} := \mathcal{L}_{\mathbb{S}^3, sub} := X^2 + Y^2. \tag{1.3}$$

Here, we can view the 3-sphere $\mathbb{S}^3$ as a Lie group with respect to the quaternionic product of $\mathbb{R}^4$, and note that it is globally diffeomorphic and isomorphic to the group SU(2) of unitary 2×2 matrices of determinant one, with the usual matrix product. We also note that in Euler's angles

(ϕ, ψ, θ) the sub-Laplacian $\mathcal{L}_{\mathbb{S}^3, sub}$ has the form

$$\mathcal{L}_{\mathbb{S}^3, sub} = \frac{1}{\sin^2 \theta} \partial_\phi^2 - 2 \frac{\cos \theta}{\sin^2 \theta} \partial_\phi \partial_\psi + \left(\frac{1}{\sin^2 \theta} - 1 \right) \partial_\psi^2 + \partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta,$$

see e.g. [29, Section 11.9], where we can take the almost injective range for Euler angles $0 \leq \phi < 2\pi$, $0 < \theta < \pi$, $-2\pi \leq \psi < 2\pi$ (see [29, Section 11.3]). Denoting by η the dual variables to Euler's angles, we can see that the principal symbol of $\mathcal{L}_{\mathbb{S}^3, sub}$ in these coordinates is

$$\frac{1}{\sin^2 \theta} (\eta_1 - \cos \theta \eta_2)^2 + \eta_3^2$$

so that Eq. (1.2) is weakly hyperbolic, with multiplicities on the set $\eta_1 = \cos \theta \eta_2$, $\eta_3 = 0$, even if $a(t) \equiv 1$.

Other examples of sub-Laplacians of different steps can be constructed from the lists of roots systems (see e.g. Fegan [9, Chapter 8]), although there are certain limitations on possible root strings, see e.g. Knapp [22, Section II.5].

The paper is organised as follows. In Section 2 we will formulate our results. In Section 3 we will set the notation for our approach and will establish properties of matrix symbols and Sobolev spaces associated to sub-Laplacians. In Section 4 we will give proofs of our results.

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2. Main results

Thus, for this paper, as in the introduction, we let G be a compact Lie group and let $X_1, \ldots, X_k$ be a family of left-invariant vector fields in $\mathfrak{g}$. Then we fix the operator $\mathcal{L}$ as in (1.1). In what follows, we restrict our considerations to non-negative functions $a \geq 0$ to ensure that the Cauchy problems we consider are hyperbolic.

In our results below, concerning the Cauchy problem (1.2), we will aim at carrying out comprehensive analysis and distinguish between the following cases:

Case 1: $a(t) \geq a_0 > 0$, $a \in C^1([0, T])$;
Case 2: $a(t) \geq a_0 > 0$, $a \in C^\alpha([0, T])$, with $0 < \alpha < 1$;
Case 3: $a(t) \geq 0$, $a \in C^\ell([0, T])$ with $\ell \geq 2$;
Case 4: $a(t) \geq 0$, $a \in C^\alpha([0, T])$, with $0 < \alpha < 2$.

Thus, Case 1 is the regular time-non-degenerate case when we obtain the well-posedness in Sobolev spaces associated to the operator $\mathcal{L}$. If $\mathcal{L}$ is hypoelliptic with Hörmander condition satisfied to order r we show the loss in local regularity depending on r. Case 2 is devoted to non-zero $a(t)$ but allowing it to be of Hölder regularity α only. Cases 3 and 4 are devoted to the situation when there may be also degeneracies with respect to t, in both situations when $a(t)$ is regular and not. The threshold $\alpha = 2$ is natural from the point of view that in general, if $a \in C^\alpha([0, T])$ with $0 < \alpha < 2$, the characteristic roots are in Hölder spaces $C^{\frac{\alpha}{2}}([0, T])$ with $0 < \frac{\alpha}{2} < 1$, thus providing a similar setting to that in Case 2. Thus, the proofs in Cases 2 and 4 will be similar and based on the regularisation and separation of characteristic roots.

Case 3 will rely on construction of a quasi-symmetriser, while in Case 1 a symmetriser will suffice.

For any $s \in \mathbb{R}$, we define Sobolev spaces $H^s_{\mathcal{L}}(G)$ associated to $\mathcal{L}$ by

$$H^s_{\mathcal{L}}(G) := \{ f \in \mathcal{D}'(G) : (I - \mathcal{L})^{s/2} f \in L^2(G) \}, \tag{2.1}$$

with the norm

$$\| f \|_{H^s_{\mathcal{L}}} := \| (I - \mathcal{L})^{s/2} f \|_{L^2}.$$

At this moment, we note that the formal self-adjointness makes these Sobolev spaces well defined in our setting, e.g. through the Plancherel formula on the Fourier transform side as in (3.5).

If $\mathcal{L}$ is a Laplacian, e.g. if $\mathcal{L} = X_1^2 + \cdots + X_n^2$ for a basis of vector fields in $\mathfrak{g}$, we will omit the subscript and simply write $H^s(G)$ in this case. Since the Laplacian is elliptic, the spaces H^s coincide with the usual Sobolev space on G considered as a smooth manifold. In Section 3 we will analyse the main relevant properties of these spaces.

Let us now formulate the corresponding results. The first result deals with strictly positive and regular propagation speed $a(t)$.

Theorem 2.1 (*Case 1*). *Assume that $a \in C^1([0, T])$ and that $a(t) \geq a_0 > 0$. For any $s \in \mathbb{R}$, if the Cauchy data satisfy $(u_0, u_1) \in H^{1+s}_{\mathcal{L}} \times H^s_{\mathcal{L}}$, then the Cauchy problem* (1.2) *has a unique solution $u \in C([0, T], H^{1+s}_{\mathcal{L}}) \cap C^1([0, T], H^s_{\mathcal{L}})$ which satisfies the estimate*

$$\| u(t, \cdot) \|^2_{H^{1+s}_{\mathcal{L}}} + \| \partial_t u(t, \cdot) \|^2_{H^s_{\mathcal{L}}} \leq C \big(\| u_0 \|^2_{H^{1+s}_{\mathcal{L}}} + \| u_1 \|^2_{H^s_{\mathcal{L}}} \big). \tag{2.2}$$

Furthermore, suppose that the vector fields $X_1, \ldots, X_k$ satisfy Hörmander condition of order r, i.e. that their iterated commutators of length $\leq r$ span the Lie algebra of G. Then the Cauchy problem (1.2) *is well-posed in $C^\infty(G)$ and in $\mathcal{D}'(G)$. Moreover, for any $s \geq 0$, we have the estimate in the usual Sobolev spaces:*

$$\| u(t, \cdot) \|^2_{H^{(1+s)/r}} + \| \partial_t u(t, \cdot) \|^2_{H^{s/r}} \leq C \big(\| u_0 \|^2_{H^{1+s}} + \| u_1 \|^2_{H^s} \big). \tag{2.3}$$

We then deal with the situations when the function $a(t)$ may become zero or when it is less regular than C^1. In this case, already for elliptic $\mathcal{L}$ (for example, $\mathcal{L}$ being the Laplacian), we can not expect the well-posedness in C^∞ on in $\mathcal{D}'$, by an adaptation of results in [7] and [6]. However, it would hold in Gevrey spaces but the appearing Gevrey space may depend on the operator $\mathcal{L}$.

Thus, for $0 < s < \infty$, we define the $\mathcal{L}$-Gevrey space $\gamma^s_{\mathcal{L}}(G) \subset C^\infty(G)$ by

$$f \in \gamma^s_{\mathcal{L}}(G) \Longleftrightarrow \exists A > 0 : \big\| e^{A(-\mathcal{L})^{\frac{1}{2s}}} f \big\|_{L^2(G)} < \infty. \tag{2.4}$$

The expression on the right is well-defined, for example in the sense of semi-groups, since the operator $-\mathcal{L}$ is formally self-adjoint and positive. It can be also easily understood on the Fourier transform side, see (3.6). The first part of the following proposition justifies the terminology.

Proposition 2.2. *We have the following properties.*

(i) *If $\mathcal{L} = X_1^2 + \cdots + X_n^2$ is the Laplacian on G then for $1 \leq s < \infty$, the space $\gamma_{\mathcal{L}}^s(G)$ in local coordinates coincides with the usual Gevrey space $\gamma^s(\mathbb{R}^n)$, i.e. the space of smooth functions $\psi \in C^\infty(\mathbb{R}^n)$ for which there exist constants $C > 0$ and $A > 0$ such that*

$$\left| \partial^\alpha \psi(x) \right| \leq C A^{|\alpha|} (\alpha!)^s.$$

In the case of the Laplacian $\mathcal{L}$, we denote the space $\gamma_{\mathcal{L}}^s(G)$ simply by $\gamma^s(G)$ dropping the subscript $\mathcal{L}$.

(ii) *If $\mathcal{L} = X_1^2 + \cdots + X_k^2$ with $X_1, \ldots, X_k$ satisfying the Hörmander condition of order r, i.e. their iterated commutations of length $\leq r$ span the Lie algebra of G, then we have the continuous inclusions*

$$\gamma^s(G) \subset \gamma_{\mathcal{L}}^s(G) \subset \gamma^{rs}(G).$$

We note that Part (i) of Proposition 2.2 has been proved in [8]. The continuous embeddings in Part (ii) follow from the formula (3.6) and estimates (3.7) in Proposition 3.1. We also note that especially for $s \geq 1$, dropping the subscript $\mathcal{L}$ in the notation in the case of Laplacians should not cause problems since in this case the space coincides with the usual Gevrey space on manifolds, as stated in Part (i) of Proposition 2.2.

We formulate the well-posedness of the Cauchy problem (1.2) in the Gevrey spaces $\gamma_{\mathcal{L}}^s$. If the Hörmander condition is satisfied, the embeddings in Part (ii) of Proposition 2.2, in view of Part (i) of Proposition 2.2, yield a well-posedness results in the Gevrey spaces γ^s, or in the usual $\gamma^s(\mathbb{R}^n)$ in local coordinates, provided all the Gevrey indices are ≥ 1. However, the well-posedness formulated in $\gamma_{\mathcal{L}}^s$ is a more refined statement since the space $\gamma_{\mathcal{L}}^s$ is in general bigger than γ^s, or maybe unrelated to it if the Hörmander condition is not satisfied.

Theorem 2.3 (*Case 2*). *Assume that $a(t) \geq a_0 > 0$ and that $a \in C^\alpha([0, T])$ with $0 < \alpha < 1$. Then for initial data $u_0, u_1 \in \gamma_{\mathcal{L}}^s(G)$, the Cauchy problem (1.2) has a unique solution $u \in C^2([0, T], \gamma_{\mathcal{L}}^s(G))$, provided that*

$$1 \leq s < 1 + \frac{\alpha}{1 - \alpha}. \tag{2.5}$$

Furthermore, suppose that the vector fields $X_1, \ldots, X_k$ satisfy Hörmander condition of order r, i.e. that their iterated commutators of length $\leq r$ span the Lie algebra of G. Then, in particular, for initial data $u_0, u_1 \in \gamma^s(G)$ with s satisfying (2.5), the Cauchy problem (1.2) has a unique solution $u \in C^2([0, T], \gamma^{rs}(G))$.

The second part of Theorem 2.3 follows from the first one in view of the embeddings in Proposition 2.2, Part (ii). By Proposition 2.2, Part (i), for $s \geq 1$, the spaces $\gamma^s(G)$ and $\gamma^{rs}(G)$ can be identified with Gevrey spaces $\gamma^s(\mathbb{R}^n)$ and $\gamma^{rs}(\mathbb{R}^n)$ in local coordinates, respectively. Consequently, we obtain the local version of Theorem 2.3 with loss of Gevrey regularity in local coordinates:

Corollary 2.4. *Assume that the vector fields $X_1, \ldots, X_k$ satisfy Hörmander condition of order r. Assume further that $a(t) \geq a_0 > 0$ and that $a \in C^\alpha([0, T])$ with*

$$0 < \alpha < 1.$$

Let the initial data u_0, u_1 belong to $\gamma^s(\mathbb{R}^n)$ in any local coordinate chart, for

$$1 \leq s < 1 + \frac{\alpha}{1 - \alpha}.$$

Then the Cauchy problem (1.2) *has a unique solution u such that $u(t, \cdot)$ belongs to $\gamma^{rs}(\mathbb{R}^n)$ in every local coordinate chart.*

We now consider the situation when the propagation speed $a(t)$ may become zero but is regular, i.e. $a \in C^\ell$ for $\ell \geq 2$.

Theorem 2.5 *(Case 3). Assume that $a(t) \geq 0$ and that $a \in C^\ell([0, T])$ with $\ell \geq 2$. Then for initial data $u_0, u_1 \in \gamma_{\mathcal{L}}^s(G)$, the Cauchy problem* (1.2) *has a unique solution $u \in C^2([0, T], \gamma_{\mathcal{L}}^s(G))$, provided that*

$$1 \leq s < 1 + \frac{\ell}{2}. \tag{2.6}$$

If $a(t) \geq 0$ belongs to $C^\infty([0, T])$ then the Cauchy problem (1.2) *is well-posed in every Gevrey class $\gamma_{\mathcal{L}}^s(G)$, $s \geq 1$.*

We now consider the case which is complementary to that in Theorem 2.5, namely, when the propagation speed $a(t)$ may become zero and is less regular, i.e. $a \in C^\alpha$ for $0 < \alpha < 2$.

Theorem 2.6 *(Case 4). Assume that $a(t) \geq 0$ and that $a \in C^\alpha([0, T])$ with $0 < \alpha < 2$. Then, for initial data $u_0, u_1 \in \gamma_{\mathcal{L}}^s(G)$ the Cauchy problem* (1.2) *has a unique solution $u \in C^2([0, T], \gamma_{\mathcal{L}}^s(G))$, provided that*

$$1 \leq s < 1 + \frac{\alpha}{2}. \tag{2.7}$$

Theorems 2.5 and 2.6 have obvious consequences, similar to those in the second part of Theorem 2.3 and in Corollary 2.4. Namely, for initial data $u_0, u_1 \in \gamma^s(G)$, $s \geq 1$, the Cauchy problem (1.2) has a unique solution $u \in C^2([0, T], \gamma^{rs}(G))$, provided that s also satisfies conditions (2.6) or (2.7), respectively.

We note that we could have united formulations of Theorems 2.5 and 2.6 in a single statement but we decided to separate them since our proofs of these two theorems are in fact very different, based on quasi-symmetrisers and regularisation and separation of characteristic roots, respectively.

Finally, we note that using a characterisation of ultradistributions on compact Lie groups from [8], one can obtain counterparts of the Gevrey results also in the corresponding spaces of ultradistributions (see [11] for an example of such an argument in $\mathbb{R}^n$).

3. Fourier analysis and symbolic properties of sub-Laplacians

In this section we recall the necessary elements of the global Fourier analysis that we will be using, and establish properties of the matrix symbols of sub-Laplacians, leading to embedding properties of the associated Sobolev spaces. The matrix symbols for operators on compact Lie groups have been developed in [29,31] to which we refer also for the details of the Fourier analysis reviewed below.

Let $\widehat{G}$ be the unitary dual of G, consisting of the equivalence classes $[\xi]$ of the continuous irreducible unitary representations $\xi : G \to \mathbb{C}^{d_\xi \times d_\xi}$, of matrix-valued functions satisfying $\xi(xy) = \xi(x)\xi(y)$ and $\xi(x)^* = \xi(x)^{-1}$ for all $x, y \in G$. For a function $f \in C^\infty(G)$ we can define its Fourier coefficient at $[\xi] \in \widehat{G}$ by

$$\widehat{f}(\xi) := \int_G f(x)\xi(x)^* dx \in \mathbb{C}^{d_\xi \times d_\xi},$$

where the integral is (always) taken with respect to the Haar measure on G, and with a natural extension to distributions. The Fourier series becomes

$$f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}\big(\xi(x)\widehat{f}(\xi)\big),$$

with Plancherel's identity taking the form

$$\|f\|_{L^2(G)} = \left(\sum_{[\xi] \in \widehat{G}} d_\xi \big\|\widehat{f}(\xi)\big\|_{\mathrm{HS}}^2 \right)^{1/2} =: \|\widehat{f}\|_{\ell^2(\widehat{G})}, \tag{3.1}$$

which we take as the definition of the norm on the Hilbert space $\ell^2(\widehat{G})$, and where

$$\big\|\widehat{f}(\xi)\big\|_{\mathrm{HS}}^2 = \operatorname{Tr}\big(\widehat{f}(\xi)\widehat{f}(\xi)^*\big)$$

is the Hilbert–Schmidt norm of the matrix $\widehat{f}(\xi)$. For a Laplacian Δ on G, we have that for a fixed $[\xi] \in \widehat{G}$, all $\xi_{ij}(x)$, $1 \leq i, j \leq d_\xi$, are eigenfunctions of $-\Delta$ with the same eigenvalue, which we denote by $|\xi|^2$, so that we have

$$-\Delta \xi_{ij}(x) = |\xi|^2 \xi_{ij}(x) \quad \text{for all } 1 \leq i, j \leq d_\xi.$$

We denote

$$\langle \xi \rangle := \big(1 + |\xi|^2\big)^{1/2},$$

which are the eigenvalues of the first order elliptic operator $(I - \Delta)^{1/2}$.

Smooth functions and distributions on G can be characterised in terms of their Fourier coefficients. Thus, we have

$$f \in C^\infty(G) \Longleftrightarrow \forall N \exists C_N \quad \text{such that } \big\|\widehat{f}(\xi)\big\|_{\mathrm{HS}} \leq C_N \langle \xi \rangle^{-N} \text{ for all } [\xi] \in \widehat{G}.$$

Also, for distributions, we have

$$u \in \mathcal{D}'(G) \Longleftrightarrow \exists M \exists C \quad \text{such that } \left\| \widehat{u}(\xi) \right\|_{\mathrm{HS}} \leq C \langle \xi \rangle^M \text{ for all } [\xi] \in \widehat{G}.$$

Furthermore, importantly for our results, it was established in [8] that the Gevrey ultradifferentiable functions and ultradistributions on compact Lie groups, initially defined in local coordinates, can be also characterised in terms of their Fourier coefficients. Thus, for $s \geq 1$,

$$f \in \gamma^s(G) \Longleftrightarrow \exists A > 0, \quad C > 0 \text{ such that } \left\| \widehat{f}(\xi) \right\|_{\mathrm{HS}} \leq C \mathrm{e}^{-A \langle \xi \rangle^{1/s}} \text{ for all } [\xi] \in \widehat{G}.$$

Here the space $\gamma^s(G)$ is the usual Gevrey space $\gamma^s(\mathbb{R}^n)$ extended to G viewed as an analytic manifold, as in Proposition 2.2, Part (i).

Given a linear continuous operator $T : C^\infty(G) \to C^\infty(G)$ (or even $T : C^\infty(G) \to \mathcal{D}'(G)$), we define its matrix symbol by

$$\sigma_T(x, \xi) := \xi(x)^*(T\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi},$$

where $T\xi$ means that we apply T to the matrix components of $\xi(x)$. In this case we may prove that

$$Tf(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}\big(\xi(x) \sigma_T(x, \xi) \widehat{f}(\xi)\big). \tag{3.2}$$

The correspondence between operators and symbols is one-to-one, and we will write T_σ for the operator given by (3.2) corresponding to the symbol $\sigma(x, \xi)$. The quantization (3.2) has been extensively studied in [29,31], to which we refer for its properties and for the corresponding symbolic calculus.

In particular, if $X_1, \ldots, X_n$ is an orthonormal basis of the Lie algebra of G, then the symbol of the Laplacian $\Delta = X_1^2 + \cdots + X_n^2$ is

$$\sigma_\Delta(\xi) = -|\xi|^2 I_{d_\xi},$$

where $I_{d_\xi} \in \mathbb{C}^{d_\xi \times d_\xi}$ is the identity matrix.

We now turn to analysing properties of the matrix symbol of the operator (1.1), namely, of the operator

$$\mathcal{L} = X_1^2 + \cdots + X_k^2.$$

The operator $\mathcal{L}$ is formally self-adjoint, therefore its symbol $\sigma_\mathcal{L}$ can be diagonalised by a choice of the basis in the representation spaces. Moreover, the operator $-\mathcal{L}$ is positive definite as sum of squares of vector fields. Therefore, without loss of generality, we can always write

$$\sigma_{-\mathcal{L}}(\xi) = \begin{pmatrix} \nu_1^2(\xi) & 0 & \ldots & 0 \\ 0 & \nu_2^2(\xi) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \nu_{d_\xi}^2(\xi) \end{pmatrix}, \tag{3.3}$$

for some $\nu_j(\xi) \geq 0$.

Consequently, we can also define Sobolev spaces $H^s_{\mathcal{L}}(G)$ associated to sums of squares. Thus, for any $s \in \mathbb{R}$, we set

$$H^s_{\mathcal{L}}(G) := \left\{ f \in \mathcal{D}'(G) : (I - \mathcal{L})^{s/2} f \in L^2(G) \right\}, \tag{3.4}$$

with the norm

$$\|f\|_{H^s_{\mathcal{L}}} := \left\| (I - \mathcal{L})^{s/2} f \right\|_{L^2}.$$

Using Plancherel's identity (3.1), we can write

$$\|f\|_{H^s_{\mathcal{L}}} = \left\| (I - \mathcal{L})^{s/2} f \right\|_{L^2} = \left(\sum_{[\xi] \in \widehat{G}} d_\xi \left\| \left(I_{d_\xi} - \sigma_{\mathcal{L}}(\xi) \right)^{s/2} \widehat{f}(\xi) \right\|^2_{\mathrm{HS}} \right)^{1/2}$$

$$= \left(\sum_{[\xi] \in \widehat{G}} d_\xi \sum_{j=1}^{d_\xi} \left(1 + v_j^2(\xi) \right)^s \sum_{m=1}^{d_\xi} \left| \widehat{f}(\xi)_{jm} \right|^2 \right)^{1/2}. \tag{3.5}$$

There are different characterisations of such Sobolev spaces, also in more generality: for example, see [10] for a heat kernel description, etc. However, for our purposes, the Fourier description (3.5) will suffice.

We note that using the Plancherel identity, the Gevrey space $\gamma^s_{\mathcal{L}}(G)$ defined in (2.4) can be characterised by the condition

$$f \in \gamma^s_{\mathcal{L}} \Longleftrightarrow \exists A > 0 : \left\| e^{A(-\mathcal{L})^{\frac{1}{2s}}} f \right\|^2_{L^2} = \sum_{[\xi] \in \widehat{G}} d_\xi \left\| e^{A\sigma_{-\mathcal{L}}(\xi)^{\frac{1}{2s}}} \widehat{f}(\xi) \right\|^2_{\mathrm{HS}}$$

$$= \sum_{[\xi] \in \widehat{G}} d_\xi \sum_{j=1}^{d_\xi} e^{A v_j(\xi)^{1/s}} \sum_{m=1}^{d_\xi} \left| \widehat{f}(\xi)_{jm} \right|^2 < \infty, \tag{3.6}$$

where now the matrix $e^{A\sigma_{-\mathcal{L}}(\xi)^{\frac{1}{2s}}}$ is well-defined in view of the diagonal form of $\sigma_{-\mathcal{L}}(\xi)$ in (3.3).

We now assume that $X_1, \ldots, X_k$ is a family of left-invariant vector fields such that their iterated commutators of length $\leq r$ span the Lie algebra of G, and establish a relation between $v_j(\xi)$ and the eigenvalues of the Laplacian, yielding also the embedding properties between Sobolev spaces $H^s_{\mathcal{L}}(G)$ and the usual Sobolev spaces $H^s(G)$ on G viewed as a smooth manifold. Here we note that the usual Sobolev spaces $H^s = H^s(G)$ can be also characterised as the set of $f \in \mathcal{D}'(G)$ such that $(I - \Delta)^{s/2} f \in L^2(G)$, with the corresponding equivalence of norms. For integers $s \in \mathbb{N}$, the embedding $H^s_{\mathcal{L}} \subset H^{s/r}$ in (3.8) is, in fact, Theorem 13 in [28].

Proposition 3.1. *Let $X_1, \ldots, X_k$ is a family of left-invariant vector fields such that their iterated commutators of length $\leq r$ span the Lie algebra of G. Let*

$$\mathcal{L} := X_1^2 + \cdots + X_k^2$$

be the corresponding sub-Laplacian with symbol (3.3). *Then there exists a constant $c > 0$ such that*

$$c\langle\xi\rangle^{1/r} \le \nu_j(\xi) + 1 \le \sqrt{2}\langle\xi\rangle \quad \text{for all } [\xi] \in \widehat{G} \text{ and } 1 \le j \le d_\xi. \tag{3.7}$$

Consequently, for $s \ge 0$ we have the continuous embeddings

$$H^s \subset H^s_{\mathcal{L}} \subset H^{s/r} \quad \text{and} \quad H^{-s/r} \subset H^{-s}_{\mathcal{L}} \subset H^{-s}. \tag{3.8}$$

More precisely, for any $s \ge 0$ there exist constants $C_1, C_2 > 0$ such that we have

$$C_1\|f\|_{H^{s/r}} \le \|f\|_{H^s_{\mathcal{L}}} \le C_2\|f\|_{H^s} \quad \text{and} \quad C_1\|f\|_{H^{-s}} \le \|f\|_{H^{-s}_{\mathcal{L}}} \le C_2\|f\|_{H^{-s/r}}. \tag{3.9}$$

Proof. The proof of (3.7) is easy if we use the following result by Rothschild and Stein [28]. In Theorem 18 in [28] it was shown, in particular, that for a sub-Laplacian $\mathcal{L}$ satisfying Hörmander condition of order $\le r$, we have the estimate

$$\|f\|^2_{H^{2/r}} \le C\big(\|\mathcal{L}f\|^2_{L^2} + \|f\|^2_{L^2}\big).$$

Using the Fourier series and Plancherel's theorem, this is equivalent to the estimate

$$\sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{4/r} \big\|\widehat{f}(\xi)\big\|^2_{\mathrm{HS}} \le C \sum_{[\xi]\in\widehat{G}} d_\xi \big(\big\|\sigma_{\mathcal{L}}(\xi)\widehat{f}(\xi)\big\|^2_{\mathrm{HS}} + \big\|\widehat{f}(\xi)\big\|^2_{\mathrm{HS}}\big),$$

holding for all $f \in H^2_{\mathcal{L}}$. In particular, applying this to f such that $\widehat{f}(\xi) = A$ for some $[\xi] \in \widehat{G}$ and zero for all other $[\tilde{\xi}]$, it follows that we have the estimate

$$\langle\xi\rangle^{2/r}\|A\|_{\mathrm{HS}} \le C\big(\big\|\sigma_{\mathcal{L}}(\xi)A\big\|_{\mathrm{HS}} + \|A\|_{\mathrm{HS}}\big)$$

for all $A \in \mathbb{C}^{d_\xi \times d_\xi}$. Now, recalling that the symbol $\sigma_{\mathcal{L}}$ is diagonal of the form (3.3), we obtain that

$$\langle\xi\rangle^{2/r} \le C\big(\nu_j^2(\xi) + 1\big)$$

for all $1 \le j \le d_\xi$, proving the first (left) inequality in (3.7). The second estimate in (3.7) follows from the relation $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ and the estimate $\nu_j^2(\xi) \le |\xi|^2$, which is a consequence of the fact that the operator $\Delta - \mathcal{L} = X^2_{k+1} + \cdots + X^2_n$ is formally self-adjoint and negative definite.

To obtain (3.9), we observe that the second estimate in (3.9) follows from the first one by duality. In turn, the first part of (3.9) follows from (3.5) using estimate (3.7). □

Remark 3.2. We note that the Cauchy problem (1.2) in local coordinates is always hyperbolic. In fact, the positivity of the matrix symbol $\sigma_{-\mathcal{L}}$ implies that the operator $\mathcal{L}$ satisfies the sharp Gårding inequality, see [30]. Consequently, the principal symbol of $-\mathcal{L}$ in any local coordinate system is non-negative, implying that the operator $\partial_t^2 - \mathcal{L}$ is hyperbolic.

4. Reduction to first order system and energy estimates

The operator $\mathcal{L}$ has the symbol (3.3), which we can write in matrix components as

$$\sigma_{-\mathcal{L}}(\xi)_{mk} = v_m^2(\xi)\delta_{mk}, \quad 1 \le m, k \le d_\xi,$$

with δ_{mk} standing for Kronecker's delta. Taking the Fourier transform of (1.2), we obtain the collection of Cauchy problems for matrix-valued Fourier coefficients:

$$\partial_t^2 \widehat{u}(t,\xi) - a(t)\sigma_{\mathcal{L}}(\xi)\widehat{u}(t,\xi) = 0, \quad [\xi] \in \widehat{G}. \tag{4.1}$$

Writing this in the matrix form, we see that this is equivalent to the system

$$\partial_t^2 \widehat{u}(t,\xi) + a(t) \begin{pmatrix} v_1^2(\xi) & 0 & \dots & 0 \\ 0 & v_2^2(\xi) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_{d_\xi}^2(\xi) \end{pmatrix} \widehat{u}(t,\xi) = 0,$$

where we put explicitly the diagonal symbol $\sigma_{\mathcal{L}}(\xi)$. Rewriting (4.1) in terms of matrix coefficients

$$\widehat{u}(t,\xi) = \big(\widehat{u}(t,\xi)_{mk}\big)_{1 \le m, k \le d_\xi},$$

we get the equations

$$\partial_t^2 \widehat{u}(t,\xi)_{mk} + a(t)v_m^2(\xi)\widehat{u}(t,\xi)_{mk} = 0, \quad [\xi] \in \widehat{G}, \ 1 \le m, k \le d_\xi. \tag{4.2}$$

The main point of our further analysis is that we can make an individual treatment of the equations in (4.2). Thus, let us fix $[\xi] \in \widehat{G}$ and m, k with $1 \le m, k \le d_\xi$, and let us denote

$$\widehat{v}(t,\xi) := \widehat{u}(t,\xi)_{mk}.$$

We then study the Cauchy problem

$$\partial_t^2 \widehat{v}(t,\xi) + a(t)v_m^2(\xi)\widehat{v}(t,\xi) = 0, \qquad \widehat{v}(t,\xi) = \widehat{v}_0(\xi), \qquad \partial_t \widehat{v}(t,\xi) = \widehat{v}_1(\xi), \tag{4.3}$$

with ξ, m being parameters, and want to derive estimates for $\widehat{v}(t,\xi)$. Combined with characterisations of Sobolev, smooth and Gevrey functions, this will yield the well-posedness results for the original Cauchy problem (1.2).

In the sequel, for fixed m, we set

$$|\xi|_v := v_m^2(\xi). \tag{4.4}$$

Hence, the equation in (4.3) can be written as

$$\partial_t^2 \widehat{v}(t,\xi) + a(t)|\xi|_v^2 \widehat{v}(t,\xi) = 0. \tag{4.5}$$

Note that if $|\xi|_v \neq 0$, Eq. (4.5) is of strictly hyperbolic type in Cases 1 and 2 and weakly hyperbolic in Cases 3 and 4. We now proceed with a standard reduction to a first order system of this equation and define the corresponding energy. The energy estimates will be given in terms of t and $|\xi|_v$ and we then go back to t, ξ and m by using (4.4).

We now use the transformation

$$V_1 := i|\xi|_v \widehat{v},$$
$$V_2 := \partial_t \widehat{v}.$$

It follows that Eq. (4.5) can be written as the first order system

$$\partial_t V(t, \xi) = i|\xi|_v A(t) V(t, \xi), \tag{4.6}$$

where V is the column vector with entries V_1 and V_2 and

$$A(t) = \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}.$$

The initial conditions $\widehat{v}(0, \xi) = \widehat{v}_0(\xi)$, $\partial_t \widehat{v}(0, \xi) = \widehat{v}_1(\xi)$ are transformed into

$$V(0, \xi) = \begin{pmatrix} i|\xi|_v \widehat{v}_0(\xi) \\ \widehat{v}_1(\xi) \end{pmatrix}.$$

Note that the matrix A has eigenvalues $\pm\sqrt{a(t)}$ and symmetriser

$$S(t) = \begin{pmatrix} 2a(t) & 0 \\ 0 & 2 \end{pmatrix}.$$

By definition of the symmetriser we have that

$$SA - A^*S = 0.$$

It is immediate to prove that

$$2 \min_{t \in [0,T]} \big(a(t), 1\big)|V|^2 \leq (SV, V) \leq 2 \max_{t \in [0,T]} \big(a(t), 1\big)|V|^2, \tag{4.7}$$

where $(\cdot, \cdot)$ and $|\cdot|$ denote the inner product and the norm in $\mathbb{C}^2$, respectively.

4.1. Case 1: proof of Theorem 2.1

In Case 1 ($a(t) > 0$, $a \in C^1([0, T])$) it is clear that there exist constants $a_0 > 0$ and $a_1 > 0$ such that

$$a_0 = \min_{t \in [0,T]} a(t) \quad \text{and} \quad a_1 = \max_{t \in [0,T]} a(t).$$

Hence (4.7) implies,

$$c_0|V|^2 = 2\min(a_0, 1)|V|^2 \leq (SV, V) \leq 2\max(a_1, 1)|V|^2 = c_1|V|^2, \tag{4.8}$$

with $c_0, c_1 > 0$. We then define the energy

$$E(t, \xi) := \big(S(t)V(t, \xi), V(t, \xi)\big).$$

We get, from (4.8), that

$$\begin{aligned}
\partial_t E(t, \xi) &= \big(\partial_t S(t)V(t, \xi), V(t, \xi)\big) + \big(S(t)\partial_t V(t, \xi), V(t, \xi)\big) + \big(S(t)V(t, \xi), \partial_t V(t, \xi)\big) \\
&= \big(\partial_t S(t)V(t, \xi), V(t, \xi)\big) + i|\xi|_v \big(S(t)A(t)V(t, \xi), V(t, \xi)\big) \\
&\quad - i|\xi|_v \big(S(t)V(t, \xi), A(t)V(t, \xi)\big) \\
&= \big(\partial_t S(t)V(t, \xi), V(t, \xi)\big) + i|\xi|_v \big((SA - A^*S)(t)V(t, \xi), V(t, \xi)\big) \\
&= \big(\partial_t S(t)V(t, \xi), V(t, \xi)\big) \leq \|\partial_t S\| |V(t, \xi)|^2 \leq c' E(t, \xi)
\end{aligned}$$

i.e. we obtain

$$\partial_t E(t, \xi) \leq c' E(t, \xi), \tag{4.9}$$

for some constant $c' > 0$. By Gronwall's lemma applied to inequality (4.9) we conclude that for all $T > 0$ there exists $c > 0$ such that

$$E(t, \xi) \leq c E(0, \xi).$$

Hence, inequalities (4.8) yield

$$c_0 |V(t, \xi)|^2 \leq E(t, \xi) \leq c E(0, \xi) \leq c c_1 |V(0, \xi)|^2,$$

for constants independent of $t \in [0, T]$ and ξ. This allows us to write the following statement: there exists a constant $C_1 > 0$ such that

$$|V(t, \xi)| \leq C_1 |V(0, \xi)|, \tag{4.10}$$

for all $t \in [0, T]$ and ξ. Hence

$$|\xi|_v^2 |\widehat{v}(t, \xi)|^2 + |\partial_t \widehat{v}(t, \xi)|^2 \leq C_1' \big(|\xi|_v^2 |\widehat{v_0}(\xi)|^2 + |\widehat{v_1}(\xi)|^2\big).$$

Recalling the notations $\widehat{v}(t, \xi) = \widehat{u}(t, \xi)_{mk}$ and $|\xi|_v = v_m(\xi)$, this means

$$v_m^2(\xi)|\widehat{u}(t, \xi)_{mk}|^2 + |\partial_t \widehat{u}(t, \xi)_{mk}|^2 \leq C_1' \big(v_m^2(\xi)|\widehat{u_0}(\xi)_{mk}|^2 + |\widehat{u_1}(\xi)_{mk}|^2\big) \tag{4.11}$$

for all $t \in [0, T]$, $[\xi] \in \widehat{G}$ and $1 \leq m, k \leq d_\xi$, with the constant C_1' independent of ξ, m, k. Now we recall that by Plancherel's equality, we have

$$\left\|\partial_t u(t,\cdot)\right\|_{L^2}^2 = \sum_{[\xi]\in\widehat{G}} d_\xi \left\|\partial_t \widehat{u}(t,\xi)\right\|_{\mathrm{HS}}^2 = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{m,k=1}^{d_\xi} \left|\partial_t \widehat{u}(t,\xi)_{mk}\right|^2$$

and

$$\left\|\mathcal{L}u(t,\cdot)\right\|_{L^2}^2 = \sum_{[\xi]\in\widehat{G}} d_\xi \left\|\sigma_{\mathcal{L}}(\xi)\widehat{u}(t,\xi)\right\|_{\mathrm{HS}}^2 = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{m,k=1}^{d_\xi} v_m^2(\xi)\left|\widehat{u}(t,\xi)_{mk}\right|^2.$$

Hence, the estimate (4.11) implies that

$$\left\|\mathcal{L}u(t,\cdot)\right\|_{L^2}^2 + \left\|\partial_t u(t,\cdot)\right\|_{L^2}^2 \le C\big(\|\mathcal{L}u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2\big), \tag{4.12}$$

where the constant $C > 0$ does not depend on $t \in [0, T]$. More generally, modulo analytic functions corresponding to trivial representations, multiplying (4.11) by powers of $v_m(\xi)$, for any s, we get

$$v_m^{2+2s}(\xi)\left|\widehat{u}(t,\xi)_{mk}\right|^2 + v_m^{2s}(\xi)\left|\partial_t \widehat{u}(t,\xi)_{mk}\right|^2$$
$$\le C_1'\big(v_m^{2+2s}(\xi)\left|\widehat{u}_0(\xi)_{mk}\right|^2 + v_m^{2s}(\xi)\left|\widehat{u}_1(\xi)_{mk}\right|^2\big). \tag{4.13}$$

Taking the sum over ξ, m and k as above, this yields the estimate (2.2).

If the vector fields $X_1, \ldots, X_k$ satisfy Hörmander's condition of order r, the estimate (2.3) follows from (2.2) and Proposition 3.1. Consequently, taking α arbitrarily large, we can also conclude that the solution u belongs to $C^\infty(G)$ and by duality to $\mathcal{D}'(G)$ in x if the initial data belong to $C^\infty(G)$ and $\mathcal{D}'(G)$, respectively. This completes the proof of Theorem 2.1.

Before proceeding to proving Cases 1–3, we note that in $\mathbb{R}^n$, due to the necessity to introduce compactly supported cut-offs to explore the finite propagation speed of the equation, one has to distinguish between the analytic case $s = 1$ and Gevrey cases $s > 1$. The case $s = 1$ can be then handled by using, e.g. Kajitani [20], see also earlier results by Bony and Shapira [2]. However, with our method of proof, it will not be necessary to make such a distinction since the group G is already compact.

4.2. Case 2: proof of Theorem 2.3

We assume now still $a(t) \ge a_0 > 0$ but this time the regularity of a is reduced, i.e., $a \in C^\alpha([0, T])$, with $0 < \alpha < 1$. As above $a(t) \ge a_0 > 0$ for all $t \in [0, T]$. Keeping the notation (4.4) and inspired by [11] we look for a solution of the system (4.6), i.e. of

$$\partial_t V(t,\xi) = \mathrm{i}|\xi|_\nu A(t) V(t,\xi), \tag{4.14}$$

of the following form

$$V(t,\xi) = \mathrm{e}^{-\rho(t)|\xi|_\nu^{1/s}} (\det H)^{-1} H W,$$

where $\rho \in C^1([0, T])$ is a real-valued function which will be suitably chosen in the sequel, $W = W(t, \xi)$ is to be determined,

$$H(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1(t) & \lambda_2(t) \end{pmatrix},$$

and, for $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi \geq 0$ with integral 1,

$$\lambda_1(t) = (-\sqrt{a} * \varphi_\epsilon)(t),$$
$$\lambda_2(t) = (+\sqrt{a} * \varphi_\epsilon)(t), \tag{4.15}$$

$\varphi_\epsilon(t) = \frac{1}{\epsilon}\varphi(t/\epsilon)$. By construction, λ_1 and λ_2 (where the dependence on ϵ is omitted for the sake of simplicity) are smooth in $t \in [0, T]$. Moreover,

$$\lambda_2(t) - \lambda_1(t) \geq 2\sqrt{a_0},$$

for all $t \in [0, T]$ and $\varepsilon \in (0, 1]$,

$$\left|\lambda_1(t) + \sqrt{a(t)}\right| \leq c_1\varepsilon^\alpha$$

and

$$\left|\lambda_2(t) - \sqrt{a(t)}\right| \leq c_2\varepsilon^\alpha,$$

uniformly in t and ε. By substitution in (4.14) we get

$$e^{-\rho(t)|\xi|_\nu^{\frac{1}{s}}}(\det H)^{-1}H\partial_t W + e^{-\rho(t)|\xi|_\nu^{\frac{1}{s}}}\left(-\rho'(t)|\xi|_\nu^{\frac{1}{s}}\right)(\det H)^{-1}HW - e^{-\rho(t)|\xi|_\nu^{\frac{1}{s}}}\frac{\partial_t \det H}{(\det H)^2}HW$$

$$+ e^{-\rho(t)|\xi|_\nu^{\frac{1}{s}}}(\det H)^{-1}(\partial_t H)W = i|\xi|_\nu e^{-\rho(t)|\xi|_\nu^{\frac{1}{s}}}(\det H)^{-1}AHW.$$

Multiplying both sides of the previous equation by $e^{\rho(t)|\xi|_\nu^{\frac{1}{s}}}(\det H)H^{-1}$ we get

$$\partial_t W - \rho'(t)|\xi|_\nu^{\frac{1}{s}}W - \frac{\partial_t \det H}{\det H}W + H^{-1}(\partial_t H)W = i|\xi|_\nu H^{-1}AHW.$$

Hence,

$$\partial_t |W(t,\xi)|^2 = 2\operatorname{Re}\big(\partial_t W(t,\xi), W(t,\xi)\big)$$
$$= 2\rho'(t)|\xi|_\nu^{\frac{1}{s}}|W(t,\xi)|^2 + 2\frac{\partial_t \det H}{\det H}|W(t,\xi)|^2 - 2\operatorname{Re}\big(H^{-1}\partial_t HW, W\big)$$
$$- 2|\xi|_\nu \operatorname{Im}\big(H^{-1}AHW, W\big). \tag{4.16}$$

It follows that

$$\partial_t |W(t,\xi)|^2 \leq 2\operatorname{Re}\big(\partial_t W(t,\xi), W(t,\xi)\big)$$
$$\leq 2\rho'(t)|\xi|_\nu^{\frac{1}{s}}|W(t,\xi)|^2 + 2\left|\frac{\partial_t \det H}{\det H}\right||W(t,\xi)|^2$$
$$+ 2\|H^{-1}\partial_t H\||W(t,\xi)^2| + |\xi|_\nu\|H^{-1}AH - (H^{-1}AH)^*\||W(t,\xi)|^2. \tag{4.17}$$

We proceed by estimating

(1) $\frac{\partial_t \det H}{\det H}$,
(2) $\|H^{-1}\partial_t H\|$,
(3) $\|H^{-1}AH - (H^{-1}AH)^*\|$.

Note that the matrices H and A depend only on t here. Hence, in complete analogy with [11] (Remark 21) we get that for all $T > 0$ there exist constants $c_1, c_2, c_3 > 0$ such that

$$\left|\frac{\partial_t \det H}{\det H}\right| \leq c_1 \varepsilon^{\alpha-1}, \tag{4.18}$$

$$\left\|H^{-1}\partial_t H\right\| \leq c_2 \varepsilon^{\alpha-1}, \tag{4.19}$$

$$\left\|H^{-1}AH - \left(H^{-1}AH\right)^*\right\| \leq c_3 \varepsilon^{\alpha}, \tag{4.20}$$

for all $t \in [0, T]$ and $\varepsilon \in (0, 1]$. Hence, combining (4.18), (4.19) and (4.20) with the energy (4.17) we obtain

$$\partial_t \left|W(t, \xi)\right|^2 \leq \left(2\rho'(t)|\xi|_\nu^{\frac{1}{s}} + c_1\varepsilon^{\alpha-1} + c_2\varepsilon^{\alpha-1} + c_3\varepsilon^{\alpha}|\xi|_\nu\right)\left|W(t, \xi)\right|^2.$$

Since $|\xi|_\nu = 0$ gives an analytic contribution in view of (3.7), it is not restrictive to assume $|\xi|_\nu > 0$. Hence, by setting $\varepsilon := |\xi|_\nu^{-1}$ we get

$$\partial_t \left|W(t, \xi)\right|^2 \leq \left(2\rho'(t)|\xi|_\nu^{\frac{1}{s}} + c_1'|\xi|_\nu^{1-\alpha} + c_3'|\xi|_\nu^{1-\alpha}\right)\left|W(t, \xi)\right|^2.$$

Thus, it follows that for $|\xi|_\nu > 0$ we can write, for some constant $C > 0$,

$$\partial_t \left|W(t, \xi)\right|^2 \leq \left(2\rho'(t)|\xi|_\nu^{\frac{1}{s}} + C|\xi|_\nu^{1-\alpha}\right)\left|W(t, \xi)\right|^2.$$

At this point taking

$$\frac{1}{s} > 1 - \alpha$$

and $\rho(t) = \rho(0) - \kappa t$ with $\kappa > 0$ to be chosen later, for sufficiently large $|\xi|_\nu$ we conclude that

$$\partial_t \left|W(t, \xi)\right|^2 \leq 0,$$

for $t \in [0, T]$ and, for example, without loss of generality, for $|\xi|_\nu \geq 1$. Passing now to V we get

$$\begin{aligned}
\left|V(t, \xi)\right| &= \mathrm{e}^{-\rho(t)|\xi|_\nu^{\frac{1}{s}}} \frac{1}{\det H(t)} \left\|H(t)\right\| \left|W(t, \xi)\right| \\
&\leq \mathrm{e}^{-\rho(t)|\xi|_\nu^{\frac{1}{s}}} \frac{1}{\det} H(t) \left\|H(t)\right\| \left|W(0, \xi)\right| \\
&= \mathrm{e}^{(-\rho(t)+\rho(0))|\xi|_\nu^{\frac{1}{s}}} \frac{\det H(0)}{\det H(t)} \left\|H(t)\right\| \left\|H^{-1}(0)\right\| \left|V(0, \xi)\right|,
\end{aligned} \tag{4.21}$$

where

$$\frac{\det H(0)}{\det H(t)} \| H(t) \| \| H^{-1}(0) \| \le c'.$$

This is due to the fact that $\det H(t)$ is a bounded function with

$$\det H(t) = \lambda_2(t) - \lambda_1(t) \ge 2\sqrt{a_0}$$

for all $t \in [0, T]$ and $\varepsilon \in (0, 1]$, $\| H(t) \| \le c$ and $\| H^{-1}(0) \| \le c$ for all $t \in [0, T]$ and $\varepsilon \in (0, 1]$. Concluding, there exists a constant $c' > 0$ such that

$$\left| V(t, \xi) \right| \le c' \mathrm{e}^{(-\rho(t)+\rho(0))|\xi|_\nu^{\frac{1}{s}}} \left| V(0, \xi) \right|,$$

for all $|\xi|_\nu \ge 1$ and $t \in [0, T]$. It is now clear that choosing $\kappa > 0$ small enough we have that if $|V(0, \xi)| \le c \mathrm{e}^{-\delta |\xi|_\nu^{\frac{1}{s}}}$, $c, \delta > 0$, the same kind of an estimate holds for $V(t, \xi)$. We finally go back to ξ and $\widehat{v}(t, \xi)$. The previous arguments lead to

$$|\xi|_\nu^2 \left| \widehat{v}(t, \xi) \right|^2 + \left| \partial_t \widehat{v}(t, \xi) \right|^2 \le c' \mathrm{e}^{(-\rho(t)+\rho(0))|\xi|_\nu^{\frac{1}{s}}} |\xi|_\nu^2 \left| \widehat{v_0}(\xi) \right|^2 + c' \mathrm{e}^{(-\rho(t)+\rho(0))|\xi|_\nu^{\frac{1}{s}}} \left| \widehat{v_1}(\xi) \right|^2.$$

Since the initial data are both in $\gamma_{\mathcal{L}}^s(G)$ we obtain that

$$|\xi|_\nu^2 \left| \widehat{v}(t, \xi) \right|^2 + \left| \partial_t \widehat{v}(t, \xi) \right|^2 \le c' \mathrm{e}^{\kappa T |\xi|_\nu^{\frac{1}{s}}} \left(C_0 \mathrm{e}^{-A_0 |\xi|_\nu^{\frac{1}{s}}} + C_1 \mathrm{e}^{-A_1 |\xi|_\nu^{\frac{1}{s}}} \right), \tag{4.22}$$

for suitable constants $C_0, C_1, A_0, A_1 > 0$ and κ small enough, for $t \in [0, T]$ and all $|\xi|_\nu \ge 1$. The estimate (4.22) implies that under the hypothesis of Case 2 and for

$$1 \le s < 1 + \frac{\alpha}{1 - \alpha},$$

the solution u belongs to $\gamma_{\mathcal{L}}^s(G)$ in x if the initial data are elements of $\gamma_{\mathcal{L}}^s(G)$.

4.3. Case 3: proof of Theorem 2.5

We now assume that $a(t) \ge 0$ is of class C^ℓ on $[0, T]$ with $\ell \ge 2$. Adopting the notations of the previous cases we want to study the well-posedness of the system (4.6): it follows that Eq. (4.5) can be written as the first order system

$$\partial_t V(t, \xi) = i |\xi|_\nu A(t) V(t, \xi),$$

where V is the column vector with entries V_1 and V_2 and

$$A(t) = \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}.$$

The initial conditions are

$$V(0, \xi) = \begin{pmatrix} i|\xi|_v \widehat{v_0}(\xi) \\ \widehat{v_1}(\xi) \end{pmatrix}.$$

This kind of system and the corresponding second order equation have been studied on $\mathbb{R}^n$ in [21] and [12] obtaining Gevrey well-posedness for $1 \le s < 1 + \ell/2$ and well-posedness in every Gevrey class in case of smooth coefficients. The energy is given by a perturbation of the symmetriser, called *quasi-symmetriser*. The quasi-symmetriser $Q_\varepsilon^{(2)}$ of A (see [21]) is defined as

$$Q_\varepsilon^{(2)}(t) := \begin{pmatrix} 2a(t) & 0 \\ 0 & 2 \end{pmatrix} + 2\varepsilon^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the sequel we collect a few results which are proven in [12] and are essential for the energy estimates below. We refer to Proposition 1, Lemma 1, Lemma 2, and the proof in Section 4.1 in [12].

Proposition 4.1. *There exists a constant $C_2 > 0$ such that*

$$C_2^{-1} \varepsilon^2 |V|^2 \le \big(Q_\varepsilon^{(2)}(t) V, V \big) \le C_2 |V|^2,$$

and

$$\big| \big((Q_\varepsilon^{(2)} A - A^* Q_\varepsilon^{(2)})(t) V, V \big) \big| \le C_2 \varepsilon \big(Q_\varepsilon^{(2)}(t) V, V \big),$$

for all $\varepsilon \in (0, 1)$, $t \in [0, T]$ and $V \in \mathbb{C}^2$. In addition the family of matrices $Q_\varepsilon^{(2)}$ is nearly diagonal and there exists a constant $C > 0$ such that

$$\int_0^T \frac{|(\partial_t Q_\varepsilon^{(2)}(t) V(t, \xi), V(t, \xi))|}{(Q_\varepsilon^{(2)}(t) V(t, \xi), V(t, \xi))} \, dt \le C \varepsilon^{-2/\ell},$$

for all $\varepsilon \in (0, 1]$, $t \in [0, T]$ and all non-zero continuous functions $V : [0, T] \times \mathbb{R}^n \to \mathbb{C}$.

Let us introduce the energy

$$E_\varepsilon(t, \xi) := \big(Q_\varepsilon^{(2)}(t) V(t, \xi), V(t, \xi) \big).$$

By direct computations as in [12] we get

$$\partial_t E_\varepsilon(t, \xi) = \big(\partial_t Q_\varepsilon^{(2)}(t) V(t, \xi), V(t, \xi) \big) + i|\xi|_v \big((Q_\varepsilon^{(2)} A - A^* Q_\varepsilon^{(2)})(t) V, V \big)$$

and therefore by Gronwall lemma and Proposition 4.1, we get

$$E_\varepsilon(t, \xi) \le E_\varepsilon(0, \xi) e^{c(\varepsilon^{-2/\ell} + \varepsilon|\xi|_v)}, \tag{4.23}$$

for some constant $c > 0$, uniformly in t, ξ and ε. By setting $\varepsilon^{-2/\ell} = \varepsilon|\xi|_v$ we arrive at

$$E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi) C_T e^{C_T |\xi|_v^{\frac{1}{\sigma}}},$$

with $\sigma = 1 + \frac{\ell}{2}$. An application of [Proposition 4.1](#) yields the estimate

$$C_2^{-1}\varepsilon^2 |V(t,\xi)|^2 \le E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi) C_T e^{C_T |\xi|_v^{\frac{1}{\sigma}}} \le C_2 |V(0,\xi)|^2 C_T e^{C_T |\xi|_v^{\frac{1}{\sigma}}}$$

which implies

$$|V(t,\xi)| \le C |\xi|_v^{\frac{k}{2\sigma}} e^{C|\xi|_v^{\frac{1}{\sigma}}} |V(0,\xi)|,$$

for some $C > 0$, for all $t \in [0, T]$ and for all ξ. We now go back to $\widehat{v}(t,\xi) = \widehat{u}(t,\xi)_{mk}$ to obtain

$$\left|\widehat{u}(t,\xi)_{mk}\right|^2 \le C^2 |\xi|_v^{\frac{k}{\sigma}} e^{2C|\xi|_v^{\frac{1}{\sigma}}} \left(|\xi|_v^2 |\widehat{u}_0(\xi)_{mk}|^2 + |\widehat{u}_1(\xi)_{mk}|^2\right)$$

and recalling that $|\xi|_v = v_m(\xi)$ and summing over $1 \le m, k \le d_\xi$, we get

$$\left\|\widehat{u}(t,\xi)\right\|_{\mathrm{HS}}^2 \le C^2 \sum_{1 \le m,k \le d_\xi} v_m(\xi)^{\frac{k}{\sigma}+2} e^{2Cv_m(\xi)^{\frac{1}{\sigma}}} \left|\widehat{u}_0(\xi)_{mk}\right|^2$$

$$+ C^2 \sum_{1 \le m,k \le d_\xi} v_m(\xi)^{\frac{k}{\sigma}} e^{2Cv_m(\xi)^{\frac{1}{\sigma}}} \left|\widehat{u}_1(\xi)_{mk}\right|^2. \qquad (4.24)$$

Recall that the initial data u_0 and u_1 are elements of $\gamma_{\mathcal{L}}^s(G)$ and, therefore, there exist constants $A', C' > 0$ such that

$$\left\|e^{A'\sigma - \mathcal{L}(\xi)^{\frac{1}{2s}}} \widehat{u}_0(\xi)\right\|_{\mathrm{HS}} \le C', \qquad \left\|e^{A'\sigma - \mathcal{L}(\xi)^{\frac{1}{2s}}} \widehat{u}_1(\xi)\right\|_{\mathrm{HS}} \le C'. \qquad (4.25)$$

Inserting [(4.25)](#) in [(4.24)](#), taking $s < \sigma$ and $\langle\xi\rangle$ large enough we conclude that there exist constants $C'' > 0$ such that

$$\left\|e^{A'\sigma - \mathcal{L}(\xi)^{\frac{1}{2s}}} \widehat{u}(t,\xi)\right\|_{\mathrm{HS}}^2 \le C'',$$

for all $t \in [0, T]$. By [Proposition 3.1](#) it follows that

$$\sum_{[\xi]\in\widehat{G}} d_\xi \left\|e^{\frac{A'}{2}\sigma - \mathcal{L}(\xi)^{\frac{1}{2s}}} \widehat{u}(t,\xi)\right\|_{\mathrm{HS}}^2 < \infty,$$

i.e. u belongs to $\gamma_{\mathcal{L}}^s(G)$ in x provided that

$$1 \le s < \sigma = 1 + \frac{\ell}{2}.$$

4.4. Case 4: proof of Theorem 2.6

We finally assume $a(t) \geq 0$ and $a \in C^{\alpha}([0, T])$ with $0 < \alpha < 2$. The main difference with respect to Case 2 is that now the roots $\pm\sqrt{a(t)}$ can coincide and are not Hölder of order α but of order $\alpha/2$. For an easy adaptation of the proof of Case 2 in Theorem 2.3 we will equivalently assume that $a \in C^{2\alpha}([0, T])$, $0 < \alpha < 1$ and that the roots are of class C^{α}. We now indicate differences with the proof of Theorem 2.3: again we look for a solution of the system (4.14) of the form

$$V(t, \xi) = e^{-\rho(t)|\xi|^{\frac{1}{s}}} (\det H)^{-1} HW,$$

where $\rho \in C^1([0, T])$ is a real valued function which will be suitably chosen in the sequel,

$$H(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1(t) & \lambda_2(t) \end{pmatrix}$$

and, for $\varphi \in C_c^{\infty}(\mathbb{R})$, $\varphi \geq 0$ with integral 1, we set

$$\begin{aligned} \lambda_1(t) &= (-\sqrt{a} * \varphi_\epsilon)(t) + \varepsilon^{\alpha}, \\ \lambda_2(t) &= (+\sqrt{a} * \varphi_\epsilon)(t) + 2\varepsilon^{\alpha}. \end{aligned} \tag{4.26}$$

Note that λ_1 and λ_2 (where the dependence on ϵ is omitted for the sake of simplicity) are smooth in $t \in [0, T]$ and in addition the following properties hold:

$$\lambda_2(t) - \lambda_1(t) \geq 2\varepsilon^{\alpha},$$

for all $t \in [0, T]$ and $\varepsilon \in (0, 1]$,

$$\left| \lambda_1(t) + \sqrt{a(t)} \right| \leq c_1 \varepsilon^{\alpha}$$

and

$$\left| \lambda_2(t) - \sqrt{a(t)} \right| \leq c_2 \varepsilon^{\alpha},$$

uniformly in t and ε. Arguing as in Case 2 we arrive at the energy estimate

$$\begin{aligned} \partial_t \left| W(t, \xi) \right|^2 &\leq 2 \operatorname{Re}\left(\partial_t W(t, \xi), W(t, \xi) \right) \\ &\leq 2\rho'(t)|\xi|^{\frac{1}{\nu}} \left| W(t, \xi) \right|^2 + 2 \left| \frac{\partial_t \det H}{\det H} \right| \left| W(t, \xi) \right|^2 \\ &\quad + 2 \left\| H^{-1} \partial_t H \right\| \left| W(t, \xi)^2 \right| + |\xi|_{\nu} \left\| H^{-1} A H - \left(H^{-1} A H \right)^* \right\| \left| W(t, \xi) \right|^2. \end{aligned} \tag{4.27}$$

We proceed by estimating

(1) $\frac{\partial_t \det H}{\det H}$,
(2) $\left\| H^{-1} \partial_t H \right\|$,
(3) $\left\| H^{-1} A H - (H^{-1} A H)^* \right\|$.

In analogy with [11] (Subsections 4.1, 4.2, 4.3) we get that for all $T > 0$ there exist constants $c_1, c_2, c_3 > 0$ such that

$$\left| \frac{\partial_t \det H}{\det H} \right| \leq c_1 \varepsilon^1, \qquad (4.28)$$

$$\left\| H^{-1} \partial_t H \right\| \leq c_2 \varepsilon^{-1}, \qquad (4.29)$$

$$\left\| H^{-1} A H - \left(H^{-1} A H \right)^* \right\| \leq c_3 \varepsilon^\alpha, \qquad (4.30)$$

for all $t \in [0, T]$ and $\varepsilon \in (0, 1]$. Hence, combining (4.28), (4.29) and (4.30) with the previous energy we obtain

$$\partial_t \left| W(t, \xi) \right|^2 \leq \left(2\rho'(t) |\xi|_\nu^{\frac{1}{s}} + c_1 \varepsilon^{-1} + c_2 \varepsilon^{-1} + c_3 \varepsilon^\alpha |\xi|_\nu \right) \left| W(t, \xi) \right|^2.$$

Again, it is not restrictive to assume that $|\xi|_\nu > 0$. By setting $\varepsilon := |\xi|_\nu^{-\gamma}$ with

$$\gamma = \frac{1}{1 + \alpha}$$

we get

$$\partial_t \left| W(t, \xi) \right|^2 \leq \left(2\rho'(t) |\xi|_\nu^{\frac{1}{s}} + c_1' |\xi|_\nu^\gamma + c_3' |\xi|_\nu^{1 - \gamma\alpha} \right) \left| W(t, \xi) \right|^2$$

$$\leq \left(2\rho'(t) |\xi|_\nu^{\frac{1}{s}} + C |\xi|_\nu^{1/(1+\alpha)} \right) \left| W(t, \xi) \right|^2.$$

At this point taking

$$\frac{1}{s} > \frac{1}{1 + \alpha}$$

and $\rho(t) = \rho(0) - \kappa t$ with $\kappa > 0$ to be chosen later, we conclude that

$$\partial_t \left| W(t, \xi) \right|^2 \leq 0,$$

for $t \in [0, T]$ and $|\xi|_\nu \geq 1$. Passing now to V and by the same arguments of Case 2 with

$$\frac{\det H(0)}{\det H(t)} \left\| H(t) \right\| \left\| H^{-1}(0) \right\| \leq c \varepsilon^{-\alpha} = c |\xi|_\nu^{\gamma\alpha} = c |\xi|_\nu^{\frac{\alpha}{1+\alpha}}$$

we conclude that there exists a constant $c' > 0$ such that

$$\left| V(t, \xi) \right| \leq c' |\xi|_\nu^{\frac{\alpha}{1+\alpha}} \mathrm{e}^{(-\rho(t) + \rho(0)) |\xi|_\nu^{\frac{1}{s}}} \left| V(0, \xi) \right|,$$

for all $|\xi|_\nu \geq 1$ and $t \in [0, T]$. We finally go back to $\widehat{v}(t, \xi)$ and $\widehat{u}(t, \xi)_{mk}$. We have

$$\nu_m^2(\xi) \left| \widehat{u}(t, \xi)_{mk} \right|^2 \leq c' \mathrm{e}^{(-\rho(t) + \rho(0)) \nu_m(\xi)^{\frac{1}{s}}} \left(\nu_m(\xi)^2 \left| \widehat{u}_0(\xi)_{mk} \right|^2 + \left| \widehat{u}_1(\xi)_{mk} \right|^2 \right),$$

with the constant c' independent of ξ, m and k. Multiplying by $\mathrm{e}^{\delta \nu_m(\xi)^{\frac{1}{s}}}$ and summing over $1 \leq m, k \leq d_\xi$, we get

$$\left\| \mathrm{e}^{\delta(\sigma_{-\mathcal{L}}(\xi))^{\frac{1}{2s}}} \sigma_{-\mathcal{L}}(\xi) \widehat{u}(t, \xi) \right\|_{\mathrm{HS}}^2$$
$$\leq c' \left(\left\| \mathrm{e}^{(-\rho(t)+\rho(0)+\delta)(\sigma_{-\mathcal{L}}(\xi))^{\frac{1}{2s}}} \sigma_{-\mathcal{L}}(\xi) \widehat{u}_0(\xi) \right\|_{\mathrm{HS}}^2 + \left\| \mathrm{e}^{(-\rho(t)+\rho(0)+\delta)(\sigma_{-\mathcal{L}}(\xi))^{\frac{1}{2s}}} \widehat{u}_1(\xi) \right\|_{\mathrm{HS}}^2 \right),$$
(4.31)

for any $\delta > 0$. Since the initial data are both in $\gamma_{\mathcal{L}}^s(G)$, we get that

$$\sum_{[\xi] \in \widehat{G}} d_\xi \left(\left\| \mathrm{e}^{(\kappa T + \delta)(\sigma_{-\mathcal{L}}(\xi))^{\frac{1}{2s}}} \sigma_{-\mathcal{L}}(\xi) \widehat{u}_0(\xi) \right\|_{\mathrm{HS}}^2 + \left\| \mathrm{e}^{(\kappa T + \delta)(\sigma_{-\mathcal{L}}(\xi))^{\frac{1}{2s}}} \widehat{u}_1(\xi) \right\|_{\mathrm{HS}}^2 \right) < \infty$$

for some $\delta > 0$ if κ is small enough. Taking the same sum $\sum_{[\xi] \in \widehat{G}} d_\xi$ of the expressions in (4.31), and using Plancherel's formula, we obtain that

$$\left\| \mathrm{e}^{\delta(-\mathcal{L})^{\frac{1}{2s}}} \mathcal{L} u(t, \cdot) \right\|_{L^2}^2 = \sum_{[\xi] \in \widehat{G}} d_\xi \left\| \mathrm{e}^{\delta(\sigma_{-\mathcal{L}}(\xi))^{\frac{1}{2s}}} \sigma_{-\mathcal{L}}(\xi) \widehat{u}(t, \xi) \right\|_{\mathrm{HS}}^2 < \infty,$$
(4.32)

for κ small enough, for $t \in [0, T]$. This completes the proof of Theorem 2.6.

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