Logarithmic geometry and stacks in resolution of singularities and moduli: The unreasonable effectiveness of toric varieties

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Log geometry is inevitable

- Some of the best "varieties with boundary" are smooth with NCD.
- Appear necessarily as the best possible degeneration over a curve $X \to B$— semistable: $t = x_1 \cdots x_k$.
- Appear necessarily as the best resolution of singularities
- appear in de Jong's theorem:

Theorem

For any variety X there is a smooth alteration $Y \to X$.

Toric varieties

These are truly the best.

- Strength: equivalence of categories with fans.
- Everything geometric has a polyhedral counterpart.
- Modifications are subdivision, alteration change the lattice, etc.
- Weakness: they are all rational!

Illusie: Logarithmic spaces (according to K. Kato, 1994)

Conference in 1991, a lecture that presented a magical world encompassing toric varieties, with advantages, avoiding drawbacks.
I was influenced by Olsson, Gross, Siebert, Chen, Vistoli, Marcus, Wise, . . .
But at first (de Jong) I stuck with toroidal object.

Sketch of de Jong's theorem

- Choose a projection $X \to B$ which is generically log smooth curve.
- Moduli of maps: there is an alteration $B_1 \to B$ and family of log stable maps

$$\begin{array}{ccc} Y_1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & B. \end{array}$$

  so $Y_1 \to B_1$ is log smooth.
- Inductively resolving $B_2 \to B_1$ get B_2 log smooth,
- so pullback Y_2 is log smooth.
- Combinatorics gives resolution $Y \to Y_2$ giving smooth alteration $Y \to X$.
- In characteristic 0, combinatorics controls the coverings leading to a resolution $X' \to X$ (with de Jong, also Illusie–Temkin, Bergh–Rydh)
- In characteristic 0, combinatorics gives weak semistable reduction for a family of varieties (with Karu)
- . . . and semistable reduction (Adiprasito–Liu–emkin).

End of segment

Next: Combinatorics II