Logarithmic geometry and stacks in resolution of singularities and moduli: Resolution of singularities for everyone

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Blowing up

Blowing up

- Say we want to blow up the xyz-space at the origin $x = y = z = 0$.
- For the first chart, x is the exceptional variable, and
- we introduce a transformed variables y', z' satisfying $xy' = y,\ xz' = z$
- For the second chart y is the exceptional, and $yx'' = x, yz'' = z$, etc.
- We have to remember how to patch: $x'' = 1/y'$ but $y = xy'$ and $z'' = z'/y'$.

is there a better way?

Blowing up - the cox construction

- Let $Y = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n], J = (x_1, \ldots, x_k), V(J) = Z$.
- In the one and only chart

$$B = \operatorname{Spec} \mathcal{O}_Y[s, x_1', \ldots, x_k']/(sx_1' - x_1, \ldots, sx_k' - x_k),$$

- s is the exceptional variable, and x_i' are the transformed variables.
- $B_+ = B \smallsetminus V(x_1', \ldots, x_k')$
- B has $\mathbb{C}^*$-action $(s, x_i') \mapsto (t^{-1}s, tx_i')$ preserving B_+.
- The blowup is $Bl_J(Y) = B_+/\mathbb{C}^*$.

From the cox construction to charts

- $B = \operatorname{Spec} \mathcal{O}_Y[s, x_1', \dots, x_k']/(sx_1' - x_1, \dots, sx_k' - x_k)$,
- $B_+ = B \setminus V(x_1', \dots, x_k')$
- The blowup is $Bl_J(Y) = B_+/\mathbb{C}^*$.
- Taking the slice $x_i' = 1$ of the action over the locus $x_i' \neq 0$,
- We have $s = x_i$, so $x_i x_j' = x_j$ giving the old charts.

What is B?

- $B = \operatorname{Spec} \mathcal{O}_Y[s, x_1', \ldots, x_k']/(sx_1' - x_1, \ldots, sx_k' - x_k)$ maps to $\operatorname{Spec} \mathbb{C}[s]$.
- It is the degeneration to the normal bundle:
- where $s = 1$ we have $x_i' = x_i$, so $B_1 = Y$.
- where $s = 0$ we have $x_i = 0$, so B_0 is the normal bundle of Z in Y.

The Cox construction of weighted blowups (Quek-Rydh)

A model weighted blowup has $J = (x_1^{1/w_1}, \ldots, x_d^{1/w_d})$ on $Y = \mathbb{A}^n$.

- Read: "the transform of x_i comes with weight w_i"
- $B = \operatorname{Spec} \mathcal{O}_Y[s, x_1', \ldots, x_k']/(s^{w_1} x_1' - x_1, \ldots, s^{w_k} x_k' - x_k)$.
- It has $\mathbb{C}^*$-action $(s, x_i') \mapsto (t^{-1}s, t^{w_i} x_i')$.
- $B_+ = B \setminus V(x_1', \ldots, x_k')$
- $Bl_J(Y) = [B_+/\mathbb{C}^*]$.
- B is the *degeneration to the weighted normal cone*.

From the Cox construction to charts

- $B = \operatorname{Spec} \mathcal{O}_Y[s, x_1', \ldots, x_k']/(s^{w_1}x_1' - x_1, \ldots, s^{w_k}x_k' - x_k)$.
- It has $\mathbb{C}^*$-action $(s, x_i') \mapsto (t^{-1}s, t^{w_i}x_i')$.
- The weighted blowup is $Bl_J(Y) = [B_+/\mathbb{C}^*]$.
- Taking slice $x_i' = 1$, it is stabilized by $\boldsymbol{\mu}_{w_i} \subset \mathbb{C}^*$.
- It acts via $(s, x_j) \mapsto (\zeta^{-1}s, \zeta^{w_j}x_j)$,
- So the chart over $x_i' \neq 0$ is

$$\left[\frac{\operatorname{Spec} \mathcal{O}_Y[s, x_1', \ldots, x_k']/(s^{w_1}x_1' - x_1, \ldots, s^{w_i} - x_i, \ldots, s^{w_k}x_k' - x_k)}{\boldsymbol{\mu}_{w_i}}\right]$$

Resolution

Resolution of singularities

- A resolution of singularities is a proper birational morphism $X' \to X$ so that X' is smooth.
- I will allow X' to be a DM stack.
- But we may want to destackify at the end - modify X' to get a resolution by a smooth variety.
- There are several concrete ways to do this, belonging to a longer lecture series.
- For instance, Bergh's destackification.

Resolution of curves

- Consider a singular curve C.
- Let $p \in C$ be a singular point, and $C' \to C$ the blowing up.
- $p_a(C') < p_a(C)$, hence at some point you must stop — the curve becomes smooth.

(...weighted blowups do improve efficiency.)

Resolution of surfaces

- Consider the surface $x^2 - y^2 z = 0$
- Its most singular point is $x = y = z = 0$.
- Blow it up and look at the z-chart:
- $x'^2 z^2 - y'^2 z^2 \cdot z = 0$, or $(x'^2 - y'^2 z) z^2 = 0$.
- The z^2 factor is the exceptional. The other factor is exactly as we started!

Exercise

What does this look like with the Cox construction?

- Classical approaches find ways around it.
- I'll avoid that with weighted blowups.

Weighted resolution

- Consider the same surface $x^2 - y^2 z = 0$
- Blow up (x^2, y^3, z^3), proportionally $(x^{1/3}, y^{1/2}, z^{1/2})$.
- $s^6(x'^2 - y'^2 z')$.
- The s^6 factor is the exceptional. The other factor is exactly as we started ...
- But we remove the locus $x' = y' = z' = 0$!!!

Exercise

In characteristic zero, show that this is an improvement! the remaining singularities on the $x' = y' = 0$ locus (where $z' \neq 0$) are normal crossings.

Exercise

Complete the resolution.

Exercise

Repeat for $x^2 - y_1 y_2 y_3 = 0$.

Principalization

Principalization

- A principalization of an ideal sheaf $I \subset \mathcal{O}_Y$ on a smooth variety is a proper birational morphism $Y' \to Y$ such that $I\mathcal{O}_{Y'}$ is exceptional and locally principal.
- We will be interested in a strong principalization, obtained by successively blowing up (using classical or weighted blowings up) of smooth loci containd in $V(I)\mathcal{O}_{Y'}$.

exercise

Show in the two examples that one more blowup results in principalization ("declare X exceptional").

Accidental resolution

- We claim strong princpalization can be used to get resolution.
- Suppose you can embed an irreducible X in a smooth Y.
- Consider $\mathcal{I} = I_X \subset \mathcal{O}_Y$.
- A strong principalization will make $\mathcal{I}$ exceptional.
- It has to blow up the generic point of the proper transform X'.
- But a strong principalization only blows up smooth loci over X, so at that point X' is smooth.

Globalization and functoriality

- In general we only have a local embedding.
- This is addressed by smooth functoriality and the reembedding principle.
- Smooth functoriality demands that if $Y_1 \to Y$ is smooth then principalizaion of $\mathcal{I} \subset \mathcal{O}_Y$ and of $\mathcal{I}\mathcal{O}_{Y_1}$ are the same.
- Reembedding demands that principalization of

$$\mathcal{I} + (z) \quad \subset \quad \mathcal{O}_{Y \times \operatorname{Spec} k[z]}$$

  coincides with that of $\mathcal{I} \subset \mathcal{O}_Y$, namely the same centers are blown up
- Both are automatic in our resolution algorithm.
- Both are anyway desirable!

Invariants and centers

Examples of invariants

- The invariant of $x^2 - y^2 z$ at the origin is $(2, 3, 3)$.
- Rationale: x appears in degree 2, while y and z appear in degree 3.
- The invariant of $x^2 - y_1 y_2 y_3$ at the origin is $(2, 3, 3, 3)$.
- The invariant of $x^5 + x^3 y^3 + y^8$ at the origin is $(5, 15/2)$.
- Rationale: x appears in degree 5,
- and projecting the newton polygon from that point one reaches $15/2$.
- The invariant of $x^5 + x^3 y^3 + y^7$ at the origin is $(5, 7)$.
- This time the point $(0, 7)$ lies below $(0, 15/2)$.

Examples of centers

- The center of $x^2 - y^2 z$ at the origin is (x^2, y^3, z^3).
- Rationale: x appears in degree 2, while y and z appear in degree 3.
- The center of $x^2 - y_1 y_2 y_3$ at the origin is $(x^2, y_1^3, y_2^3, y_3^3)$.
- The center of $x^5 + x^3 y^3 + y^8$ at the origin is $(x^5, y^{15/2})$.
- The center of $x^5 + x^3 y^3 + y^7$ at the origin is (x^5, y^7).

What kind of object is J?

- J is a $\mathbb{Q}$-ideal, an idealistic exponent, a valuative $\mathbb{Q}$-ideal, a Rees algebra.
- following a tradition of encoding rational powers of ideals, always up to integral closure.
- Locally it provides a monomial valuation.

Principalization and resolution

Principalization

This only works in characteristic 0:

Theorem (ℵ-Të-Wł)

$\mathcal{I}\mathcal{O}_{Y'} = E^\ell \mathcal{I}'$ *with* $\mathrm{inv}_{p'}\mathcal{I}' \; < \; \mathrm{inv}_p(\mathcal{I})$.

In other words, after finitely many steps you get principalization, which in turns implies resolution.
We still need to define invariants and centers, and prove the theorem.

Derivatives

Derivatives of an ideal

- We consider for simplicity $\mathcal{I} \subset \mathcal{O}_Y$ where $Y = \operatorname{Spec} k[x_1, \ldots x_n]$.
- Define $\mathcal{D}(\mathcal{I}) = \mathcal{I} + (\partial f/\partial x_i)_{f \in \mathcal{I}, i=1,\ldots,n}$.
- Define* $\mathcal{D}^n(\mathcal{I}) = \mathcal{D}(\mathcal{D}^{n-1}(\mathcal{I}))$.

exercise

Compute $\mathcal{D}(\mathcal{I})$ and $\mathcal{D}^2(\mathcal{I})$ when $\mathcal{I} = (x^2 + y^2 z)$ and when $\mathcal{I} = (x^2 + y_1 y_2 y_3)$.

exercise

Compute $\mathcal{D}(\mathcal{I})$ and $\mathcal{D}^2(\mathcal{I})$ ... when $\mathcal{I} = (x^5 + x^3 y^3 + y^8)$.

***exercise**

What is the right notion in positive characteristics?

exercise

If $Y_1 \to Y$ is smooth and $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, show that $\mathcal{D}(\mathcal{I}_1) = \mathcal{D}(\mathcal{I})\mathcal{O}_{Y_1}$.

Order of an ideal

- Define $\mathrm{ord}_{\mathcal{I}}(p) = \min\{a : \mathcal{D}^a(\mathcal{I})_p = (1)\}$.
- Define $\mathrm{maxord}(\mathcal{I}) = \max_p \mathrm{ord}_{\mathcal{I}}(p)$.

exercise

Convince yourself that ord is upper-semicontinuous.

exercise

Convince yourself that the locus $\{p : \mathrm{maxord}(\mathcal{I}) = \mathrm{ord}_{\mathcal{I}}(p)\}$ is closed.

exercise

Compute $\mathrm{maxord}(\mathcal{I})$ in the three examples.

exercise

If $g : Y_1 \to Y$ is smooth and $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, show that $\mathrm{ord}_{\mathcal{I}_1} = \mathrm{ord}_{\mathcal{I}} \circ g$.

Maximal contact

Maximal contact (Giraud–Hironaka)

- Say $\operatorname{ord}_{\mathcal{I}}(p) = a$. An element $x \in \mathcal{O}_{Y,p}$ is maximal contact at p if $x \in \mathcal{D}^{a-1}\mathcal{I}$ and $\operatorname{ord}_x(p) = 1$.
- In characteristic 0, maximal contact always exists locally at p!
- Indeed $\mathcal{D}^{a-1}\mathcal{I} \neq \mathcal{O}_{Y,p}$ but $\mathcal{D}^{a}\mathcal{I} = \mathcal{O}_{Y,p}$ means that some derivative or some x is a unit.

exercise

Find two maximal contacts when $\mathcal{I} = (x^2 + y^2 z)$, when $\mathcal{I} = (x^2 + y_1 y_2 y_3)$, and when $\mathcal{I} = (x^5 + x^3 y^3 + y^8)$.

exercise

If $g : Y_1 \to Y$ is smooth, $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, and x maximal contact, show that $g^* x$ is maximal contact.

Coefficient ideals

Coefficient ideals

- Say $\mathcal{I}$ has maximal order a.
- The coefficient ideal $C(\mathcal{I}, a)$ is defined as follows.
- Consider $D^{a-i}(\mathcal{I})$ as sitting in degree i, for $i < a$.
- Take the ideal $C(\mathcal{I}, a)$ generated by monomials $f(D^{a-1}(\mathcal{I}), \ldots, D(\mathcal{I}), \mathcal{I})$ of degree $\geq a!$.

exercise

Compute $C(\mathcal{I}, 2)$ when $\mathcal{I} = (x^2 + y^2 z)$ and when $\mathcal{I} = (x^2 + y_1 y_2 y_3)$.

exercise

Compute $C(\mathcal{I}, 5)$ when $\mathcal{I} = (x^5 + x^3 y^3 + y^8)$.

exercise

If $Y_1 \to Y$ is smooth and $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, show that $C(\mathcal{I}_1, a) = C(\mathcal{I}, a)\mathcal{O}_{Y_1}$.

Comments on coefficient ideals

If x_1 is a maximal contact, define $\mathcal{I}[2] = C(\mathcal{I}, a)|_{x_1=0}$.

- Coefficient ideals come to collect the data of all coefficients of elements of the ideal when expanded in maximal contact.
- Villamayor and coauthors Encinas, Bravo, Benito collect this instead in a Rees algebra.
- Key property (Kollár's D-Balancing): Improving $C(\mathcal{I}, a)$ results in improving $\mathcal{I}$.
- Key property (Włodarczyk's Invariance): Maximal contact x are all equivalent for $C(\mathcal{I}, a)$.
- In particular $\mathcal{I}[2]$ is independent of choice of maximal contact x.

Construction of invariants and centers

- Assume $\operatorname{ord}_{\mathcal{I}}(p) = a_1$ with x_1 is a maximal contact.
- Assume $\operatorname{inv}_{\mathcal{I}[2]} = (b_2, \ldots, b_k)$, with center $(x_2^{b_2}, \ldots, x_k^{b_k})$
- Define $a_i = b_i/(a_1 - 1)!$.
- Set $\operatorname{inv}_{\mathcal{I}}(p) = (a_1, \ldots, a_k)$
- Set $J = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k})$.

exercise

Verify the invariants and centers in our examples!

Functoriality of invariants and centers

Theorem (Functoriality, ℵ-Të-Wł)

- *If* $Y_1 \to Y$ *is smooth,* $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$*, and* $p_1 \mapsto p$ *then*

$$\mathrm{inv}_{\mathcal{I}}(p) = \mathrm{inv}_{\mathcal{I}_1}(p_1).$$

- *If* $Y_1 \to Y$ *is smooth,* $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$*, and* $p_1 \mapsto p$ *then*

$$J_1 = J\mathcal{O}_{Y_1}.$$

This implies gluing as well as equivariance.

Some key properties of centers

Characterization of center

Theorem

At each $p \in Y$ the center $J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k})$ is the unique center of maximal invariant such that $J_{\mathcal{I}} \supset \mathcal{I}$ as idealistic exponents.

The invariant of the ideal $\mathrm{inv}_{\mathcal{I}}(p) = (a_1, \ldots, a_k)$ is the invariant of this center.

Warning: remember to take integral closure, so the center (x^2, y^2) contains xy.

Warning: The center is unique, but the variables x_i generating it are not.

- The proof is inductive like most of the results here: one checks that $x_1^{a_1}$ is in the defining equations of the center, and that any lift of the center of $\mathcal{I}[2]$ is as well.
- This can be used as an independent definition and construction (in arbitrary characteristic) of a center.

The support of J

Corollary

Let $M(\mathcal{I})$ be the monomial saturation of $\mathcal{I}$ with respect to $x_1, \ldots, x_k$. Then $J_{\mathcal{I}} = J_{M(\mathcal{I})}$.

This is because $J_{M(\mathcal{I})} \supset J_{\mathcal{I}} \supset M(\mathcal{I}) \supset \mathcal{I}$ and invariants are decreasing.

Theorem (Support)

In characteristic 0:

- *$V(J_{\mathcal{I}})$ is the locus where the invariant is attained, both on Y and on the deformation B to the weighted normal cone.*
- *In particular the invariant persists to the weighted normal cone.*
- *The invariant and center are compatible with smooth morphisms.*

Proofs of neat results ($\aleph$-Të-Wł)

The invariant is upper semicontinuous

- $V(\mathcal{D}^i(\mathcal{I})) = \{p : \mathrm{ord}_{\mathcal{I}}(p) > i\}$.
- so the order is upper semicontinuous.
- Say the maximal order is $a = a_1$.
- x_1 is maximal contact so

$$V(x_1) \supset V(D^{a-1})(\mathcal{I}) = \{p : \mathrm{ord}_{\mathcal{I}}(p) = a\} = \{p : \mathrm{ord}_{C(\mathcal{I},a)}(p) = a!\}.$$

- By induction on dimension $\mathrm{inv}_{\mathcal{I}[2]}$ is upper semicontinuous on $V(x_1)$,
- so $\mathrm{inv}_{\mathcal{I}}$ is upper semicontinuous.

In positive characteristic one must be very careful here!

The maximal locus of $\mathrm{inv}_{\mathcal{I}}$ is the smooth locus $V(J)$

- x_1 is maximal contact so the smooth locus

$$V(x_1) \supset V(D^{a-1})(\mathcal{I}) = \{p : \mathrm{ord}_{\mathcal{I}}(p) = a\} = \{p : \mathrm{ord}_{C(\mathcal{I},a)}(p) = a!\}.$$

- By induction on dimension the maximal locus of $\mathrm{inv}_{\mathcal{I}[2]}$ is $V(x_2, \ldots, x_k)$,
- which is smooth.

In positive characteristic we do not know a general replacement of this.

The J-order and maximal locus

- Given a monomial $z = x_1^{b_1} \cdots x_n^{b_n}$ we define $\mathrm{ord}_J(z) = \sum_{i=1}^k b_i/a_i$.
- It defines a valuation centered at the generic point of $V(J)$.
- By definition, if J is the center associated to $\mathcal{I}$ then $\mathrm{ord}_J(\mathcal{I}) = \mathrm{ord}_J(M(\mathcal{I})) = 1$.
- (Similarly $\mathrm{ord}_{J^{(a-1)!}}(C(\mathcal{I}, a)) = 1$.)
- You can expand $f = \sum_{q \in \mathbb{Q}} f_{J,q}$ with $\mathrm{ord}_J(f_{J,q}) = q$.
- Define $\mathcal{I}_{J,1} = (f_{J,1} | f \in \mathcal{I})$.
- It is the degeneration of $\mathcal{I}$ to the weighted normal cone of J.
- We have $M(\mathcal{I})_{J,1} = M(\mathcal{I}_{J,1})$.

The weighted blowup decreases invariant

Theorem (Sketch, ℵ-Të-Wł)

- *Say $\mathcal{I}$ has invariant $\mathrm{inv}_{\mathcal{I}}(p) = (a_1, \ldots, a_k)$ along smooth X, supporting center J, and we blow up $\bar{J}$.*
- *Let $\mathcal{I}_1$ be the transform of $\mathcal{I}$.*
- *Then the invariant of $\mathcal{I}_1$ is $< (a_1, \ldots, a_k)$ above X.*

- By the Support Theorem this maximum is achieved only inside $V(J(x'))$.
- But we remove $V(x'_1, \ldots, x'_k)$ in the weighted blowup!

Functoriality holds (ℵ-Të-Wł)

- Say $g : Y_1 \to Y$ is smooth, $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, and $p_1 \mapsto p$.
- Derivatives on Y lift to derivatives on Y_1
- $\operatorname{ord}_{\mathcal{I}_1}(p_1) = \operatorname{ord}_{\mathcal{I}}(p)$.
- If $x \in \mathcal{D}^{a-1}(\mathcal{I}) \subset \mathcal{O}_Y$ then $x_1 = g^*x \in \mathcal{D}^{a-1}(\mathcal{I}_1) \subset \mathcal{O}_{Y_1}$.
- If x maximal contact then x_1 maximal contact.
- $g^*C(\mathcal{I}, a) = C(\mathcal{I}_1, a)$.
- $g^*\mathcal{I}[2] = \mathcal{I}_1[2]$
- By induction $\operatorname{inv}_{\mathcal{I}[2]}(p_1) = \operatorname{inv}_{\mathcal{I}[2]}(p)$ with $g^*J_{\mathcal{I}[2]} = J_{\mathcal{I}_1[2]}$
- So $\operatorname{inv}_{\mathcal{I}_1}(p_1) = \operatorname{inv}_{\mathcal{I}}(p)$ and $g^*J_{\mathcal{I}} = J_{\mathcal{I}_1}$.

Still need to show independence of choice of x

Functoriality implies gluing (ℵ-Të-Wł)

- Say $Y = Y_1 \cup Y_2$ is the union of open subschemes, with $Y_{12} = Y_1 \cap Y_2$.
- Say the maximal invariant of $\mathcal{I}$ is $(a_1, \ldots, a_k)$.
- It has centers
  $J_1 = (x_{1,1}^{a_1}, \ldots, x_{1,k}^{a_k})$ on Y_1 and
  $J_2 = (x_{2,1}^{a_1}, \ldots, x_{2,k}^{a_k})$ on Y_2.
- By functoriality $J_1|_{Y_{12}} = J_2|_{Y_{12}}$, so the centers, and their blowings up, glue together.

Functoriality implies equivariance (ℵ-Të-Wł)

- Say G acts on Y preserving $\mathcal{I}$, and let J be the center associated to $\mathcal{I}$.
- We have two maps $G \times Y \to Y$, the projection p_Y and the action a, and $p_Y^*\mathcal{I} = a^*\mathcal{I}$
- These maps are smooth.
- By functoriality $p_Y^* J = a^* J$, so the action lifts to the blowup Y_1.

The re-embedding principle holds (ℵ-Të-Wł)

- Say $X \subset Y \subset Y \times \mathbb{A}^1$.
- The order of $I_{X\subset Y\times\mathbb{A}^1}$ is 1,
- …with maximal contact t - the $\mathbb{A}^1$ parameter.
- $V(t) = Y$ with coefficient ideal $I_{X\subset Y}$,
- …with some invariant $(a_1, \dots, a_k)$ and center $J = (x_1^{a_1}, \dots, x_k^{a_k})$.
- Then $I_{X\subset Y\times\mathbb{A}^1}$
  has invariant $(1, a_1, \dots, a_k)$
  and center $J_1 = (t, x_1^{a_1}, \dots, x_k^{a_k})$.
- (The blowing up of J is the proper transform of the blowing up of J_1.)

Independence of x (Włodarczyk, Kollár)

- if $\mathrm{ord}_{\mathcal{I}}(p) = a$ define $MC = \mathcal{D}^{a-1}(\mathcal{I})$.

Definition (Kollár)

$\mathcal{I}$ is MC-invariant if $\mathcal{D}(\mathcal{I}) \cdot MC \subset \mathcal{I}$.

Exercise

Show that MC invariant ideals satisfy $\mathcal{D}(\mathcal{I})^i \cdot MC^i \subset \mathcal{I}$.

- Coefficient ideals have this property.

Theorem (Włodarczyk)

Let $\mathcal{I}$ be MC-invariant and x, x' be maximal contact. There is a local analytic automorphism $\phi : Y \to Y$ such that $\phi(\mathcal{I}) = \mathcal{I}$ and $\phi^*(x') = x$.

Independence of x (Włodarczyk)

Theorem (Włodarczyk)

Let $\mathcal{I}$ be MC-invariant and x, x' be maximal contact. There is a local analytic automorphism $\phi : Y \to Y$ such that $\phi^*(\mathcal{I}) = \mathcal{I}$ and $\phi^*(x') = x$.

- Extend with compatible parameters $x_2, \ldots, x_n$
- Let $\phi^*(x, x_2, \ldots, x_n) = (x', x_2, \ldots, x_n)$.
- Write $h = x' - x$.
- $f(x') \;=\; f(x+h) \;=\; f(x) + h \cdot \frac{\partial f}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \cdots$
- but $h^i \cdot \frac{\partial^i f}{\partial x^i} \in \mathcal{I}$.

The end

Thank you for your attention!