Logarithmic geometry and stacks in resolution of singularities and moduli: Punctured logarithmic invariants

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LMS lecture series

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Degeneration setup and goals

- Consider log smooth family $\mathcal{X} \to B$ of target log schemes,
- and assume the fiber X_0 is simple normal crossings.
  Goals:
- (0) Compare $GW(X_0)$ with $GW(X_b)$,
- (1) Break down $GW(X_0)$
- (2) Write $GW(X_0)$ in terms of invariants of stata of X_0.

(0) Deformation invariance

- $i_b : \{b\} \hookrightarrow B$

Theorem

$$i_b^! \left[\,\overline{\mathcal{M}}\,\right]^{\text{virt}} \quad = \quad \left[\,\overline{\mathcal{M}}_\Gamma(X_b)\,\right]^{\text{virt}}$$

- "Gromov-Witten of $X_b \quad = \quad$ Gromov-Witten of X_0"
-

$$\begin{aligned} \int_{[\overline{\mathcal{M}}_b]^{\text{virt}}} e^*\gamma \quad &= \quad \int_{[\overline{\mathcal{M}}]^{\text{virt}}} e^*\gamma \cup \pi^* b = \int_{[\overline{\mathcal{M}}]^{\text{virt}}} e^*\gamma \cup \pi^* 0 \\ &= \quad \int_{[\overline{\mathcal{M}}_0]^{\text{virt}}} e^*\gamma \end{aligned}$$

(1) The decomposition formula

Theorem (ACGS1)

$$[\overline{\mathcal{M}}_0]^{\mathrm{virt}} \quad = \quad \sum_{f^t : G \to \Sigma_X} m_{f^t}[\overline{\mathcal{M}}_{f^t}]^{\mathrm{virt}}$$

where

- Σ_X, Σ_B *are the* **polyhedral cone complexes with integral structures** *of* X, B*, defined in* [K-K-M-SD]
- f^t *runs over* **vertically rigid** *tropical maps in* Σ_X/Σ_B,
- $\overline{\mathcal{M}}_{f^t} \subset \overline{\mathcal{M}}_0$ *is the subspace of maps with tropicalization* $\succeq f^t$
- *The multiplicity* m_f *is a subtle computation.*

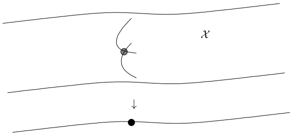
Log GW theory works well because maps to the Artin fan $\mathcal{A}_X$ are logarithmically unobstructed., giving rise to well–behaved virtual fundamental classes.

Towards punctured curves

Orbifolds taught us:

> *The structure of curves should reflect the structure of targets, and vice versa.*

Universal case — degeneration of curves:



- We have $X_0 = Y_1 \cup^D Y_2$, where D is the node,
- for simlicity has coordinates $xy = t$.
- The logarithmic structure M_B on the base is generated by $\log t$
- the logarithmic structure M_x on $\mathcal{X}$ at the node by $\log x, \log y$ with $\log x + \log y = \log t$.

What is the structure of the component $Y_1 = \{y = 0\}$ of X_0?

Punctured curves appear

- What is the structure of the component $Y_1 = \{y = 0\}$ of X_0?
- it is a curve with a logarithmic structure,
- but it is not a log curve since we insisted that log curves are log smooth.
- We call these *punctured curves*.
- Importantly, the element $\log y$ has the property $\alpha(\log y) = 0 \in \mathcal{O}_{Y_1}$.

Idealized log schemes

A punctured curve is an instance of:

Definition (Ogus)

An *idealized log structure* is a log structure $\alpha : M \to \mathcal{O}_X$ along with a monoid ideal $K \subset M$, such that $\alpha(K) = 0 \in \mathcal{O}_X$.
Locally one obtains a morphism $X \to \operatorname{Spec} \mathbb{C}[\bar{M}]/(\bar{K})$. An idealized log scheme is *idealized log smooth* if this morphism is smooth.

In the example, the monoid ideal is generated by $\log y$.

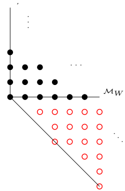
Puncturing a marked curve

Definition

A *puncturing* C° of a marked curve C is a log structure M at a marked point $\{x = 0\}$ with

$$M_S + \mathbb{N}\log x \quad \subseteq \quad M \quad \subsetneq \quad M_S + \mathbb{Z}\log x.$$

In particular the *splitting of a node is a punctrured curve.*



$(S = W)$

Reading the picture

- The embedding of M_S comes from pullback along the projection.
- The vertical projection is *generization* to the generic point of Y_1.
- The horizontal map to $\mathbb{Z}$ is the *contact order*, which here can be negative!

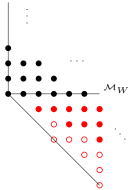
A section would result in another projection, which sends M_S to itself and the vertical generator $\log x$ to some nonzero point of M_S. Do not confuse it with the generization map!

The idealized structure on the curve and base

- Note that any element $m \in M$ *of negative contact order* must satisfy $\alpha(m) = 0$.
- ... any element in the ideal generated satisfies $\alpha(m) = 0$.
- This includes some elements of M_S — this is the so called *puncturing log ideal*.
- This makes S an idealized log scheme!

Maps and prestable punctured maps

- A punctured map is just a log morphism $C^\circ \to X$.
- A punctured log map is said to be prestable if M as above is generated by
  - the marked structure $M_S + \mathbb{N} \log x$ and
  - $f^\flat M_X$.
- example shows a punctured curve where $f^\flat M_X$ is generated by $(2, -1)$.



- This map is not prestable, but had we taken just the solid circles we get a prestable map

Stable punctured maps

- A punctured log map map is stable if it is prestable (a log condition) and the underlying map is stable (a schematic condition).
- *minimal* punctured maps: the logarithmic structure on the base is universal.
- The magical theorem of Gross and Siebert — Remark 1.2.1 — holds as stated, since neither markings nor punctures intervene:

Theorem

The minimal log structure of a stable punctured map is the dual of the lattice of the moduli of tropical punctured maps,

- but what is the tropicalization of a punctured map?

Tropical punctured maps

- Tropicalization maps an element n of a leg to σ_X.
- n takes non-negative values on M, so on $f^\flat M_X$.
- if f prestable, a supporting hyperplane n of a negative face of M is sent to a supporting hyperplane of M_X.

Theorem

Any leg of the tropicalization of a prestable punctured map extends exactly as far as its target cone allows

Type

- The *type* τ of a punctured map is define in exact analogy to the type of a log map.
- It records the graph marked by genus,
- the strata σ of each vertex, edge or leg,
- and the contact order at each edge or leg.
- It can be *decorated* by the curve class associated to each vertex.
- Balancing. a tropical map is the same as balancing a log map.

The space of punctured maps

Theorem ([ACGS])

$\mathcal{M}(X,\tau)$, the stack of minimal stable punctured log maps of decorated type τ, is a Deligne–Mumford stack which is finite and representable over $\mathcal{M}(\underline{X},\underline{\tau})$.

Theorem

The space of prestable punctured maps $\mathfrak{M}(\mathcal{X},\tau)$ in the Artin fan $\mathcal{X}$ of X of undecorated type τ is an idealized log smooth algebraic stack $\mathcal{M}(X,\tau) \to \mathfrak{M}(\mathcal{X},\tau)$ admits a perfect obstruction theory.

Splitting

- denote by $\widetilde{\mathcal{M}}(X,\tau)$ the space of punctured maps with a logarithmic section at each marking and labelled node.
- There are natural evaluation maps $\widetilde{\mathcal{M}}(X,\tau) \to X^n$
- This map is not smooth, nor is it virtually log smooth,
- but it is *ideally virtually log smooth*: it has a perfect logarithmic obstruction theory relative to an appropriate Artin fan version $\widetilde{\mathfrak{M}}(\mathcal{X},\tau)$.
- Fixing a contact order, we basically have

$$Hom(C,X) = Hom(C_1^\circ, X) \times_{Hom(W,X)} Hom(C_2^\circ, X).$$

A bit of glue

Theorem (ACGS2)

Suppose the splitting of type τ along a set of n edges results in type τ'. The following is cartesian:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(X,\tau) & \longrightarrow & \widetilde{\mathcal{M}}(X,\tau') \\ \downarrow & & \downarrow \\ X^n & \longrightarrow & X^n \times X^n \end{array}$$

What this gives

- Idealized log smooth schemes do not in general possess a natural fundamental class — they are not even pure dimensional.
- As a consequence, spaces of punctured stable maps may fail to possess a natural virtual fundamental class.
- Even though Manolache's theorem applies, we are not always handed a class to virtually pull back.
- Examples are given in [ACGS2].

Do not lose hope

- Gross's Remarks on gluing delineates a number of situations where these issues do not arise.
  - First, one only considers *tropically realizable types.*
  - Second, one requires the gluing situation to be *tropically transverse*.
  - Third, one requires that the evaluation maps are *tropically flat*.

  Still this applies in important cases.
- There is further current work addressing this issue.
- See in particular the virtually fundamental classes defined by Battistella–Nabijou–Ranganathan, Johnston.

The end

Thank you for your attention!

Next: lecturers' research talks