LOGS AND STACKS IN BIRATIONAL GEOMETRY AND MODULI

NOTES FOR THE LMS INVITED LECTURES 2024

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Abstract. Say something here. Something here.

Contents

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sources of funding .

FOUNDATION TOPICS START HERE

1. Introduction

By Dan. Francesca wrote an outline of the whole thing which is useful here.

The story begins with immediately inspiring lectures — two lectures by Vistoli and one by Illusie.

I was a student in Joe Harris's wonderful Moduli of Curves course in 1990, the course that gave rise to [HM98]. Harris was aware that Angelo Vistoli was attending, and provoked him enough to give a series of two lectures introducing algebraic stacks and their place in moduli space. These ended up transforming my career later when Vistoli and I worked on twisted stable maps — but that is a different story told elsewhere.

The summer I graduated I found myself at the Barsotti Memorial Symposium in Padova (1991), where Illusie lectured on *Logarithmic Spaces (according to K. Kato)*. It took longer, but this lecture led to much of the work described in this lecture series.

Both stacks and logarithmic structures introduce *hidden smoothness* into algebraic geometry.

Deligne–Mumford stacks were introduced into algebraic geometry expressly to encode moduli spaces. But from the very beginning it was recognized that, at the same time, they endow familiar singular spaces with a structure that has much of the benefits of smooth spaces. Such is the case with the moduli spaces of stable curves, which, as stacks, are smooth over $\mathbb{Z}$, enabling Deligne and Mumford to deduce their irreducibility in any characteristic from the characteristic 0 case.

Still, it took quite some time to realize that stacks can be used even for the general problem of resolution of singularities. One of our goals here is to retell this story.

Logarithmic structures were introduced into algebraic geometry expressly to bring out hidden smoothness: Kazuya Kato's proverbial "magic powder" that makes any toroidal variety, and any nodal curve, logarithmically smooth.

In this case, what took much longer was to develop a theory of moduli spaces — a task that is still a work-in-progress. One of our goals here is to reach logarithmic and punctured maps, which bring logarithmic geometry into Gromov–Witten theory, just as twisted stable maps did with target stacks.

1.1. **Sequence of lectures.**

(1) Introduction (Dan, 30 minutes)
(2) Stacks 1 (Dan)
(3) Moduli 1 (Pierrick)
(4) Combinatorics 1 (Dhruv)
(5) Combinatorics 2 - Toric geometry (Dhruv)
(6) Stacks 2 (Dan)
(7) GW theory 1 (Francesca)
(8) GW theory 2 (Francesca/Dan)
(9) Log Geometry 1 (Navid)
(10) Log Geometry 2 (Navid/Dan)
(11) Log GW theory I (Hülya)
(12) Resolution 1 (Dan)

(13) Moduli 2 (Pierrick/Dan)
(14) Log GW theory II (Hülya/Dan)
(15) Punctured GW invariants (Dan)
(16) Resolution 2 (Dan)
(17) Log limit linear series (Francesca 30 min)
(18) Punctured maps and mirror symmetry (Pierrick/ Hülya 30 min)
(19) Orbifolds and mirror symmetry (Hülya / Pierrick 30 min)
(20) Log Orbifold correspondence (Navid 30 min)
(21) LMNOP or Log Hilb (Dhruv 30 min)

Goals for April 18: read everything, propose length of lectures, propose division of labor.

2. Stacks, part 1

[1][2]

By Dan. Intil 2.2 a sort of a newly written introduction. From 2.3 taken from the resolution of singularities book. ←1 ←2

2.1. **Origins.** Stack came about from two sources, both relevant to this lecture series.

One source is *hidden smoothness*, where the ideas first came about in geometry. It was long recognized that the quotient of a manifold by a finite group is, in general, singular, but retains many of the good properties of a manifold. In particular, the rational cohomology of a compact orbifold without boundary satisfies Poincaré duality.

This point of view brings about a solid intuition, but is difficult to formalize directly.

The other source, which is far less intuitive, is that of stacks as representing moduli problems. Its advantage is that it is quite natural to formalizing stacks this way.

We will therefore follow a long tradition and describe stacks from the moduli point of view, as certain categories endowed with structure that imbues in them the life of a moduli space.

2.2. **A moduli space is encoded by a category.** We set out to define certain objects — algebraic stacks — that encode moduli problems, and in cases where the modili problem is *representable* one naturally obtains a scheme, or more precisely, a category aturally associated to a scheme.

Think about parametrizing smooth, projective, irreducible curves of genus g. If $\mathcal{M}_g$ were represented by a scheme, then the data of a morphism $S \to \mathcal{M}_g$ would be equivalent to a family of curves $C \to S$ of genus g. The outrageous, incredible, and yet magically successful idea is to give up on representablity and simply *define* an object of $\mathcal{M}_g$ over a scheme S to be a family of curves $C \to S$.

We want to elicit the relationships between these objects. We know what a morphism of schemes $S_1 \to S_2$ is, and gien objects $C_1 \to S_1$ and $C_2 \to S_2$ we similarly know what a morphism $C_1 \to C_2$ *over* $S_1 \to S_2$ should be — the diagram had better be commutative. But keeping in mind that we would like to *parametrize the fibers*, it is natural to insist that

[1] (Hulya) discuss equivalence of categories fibered in groupoids? Specificaloy log curves over log schemes
[2] (Dan) Added in slides

C_1 should be *isomorphic* to the pullback $S_1 \times_{S_2} C_2$, making the following diagram cartesian:

$$\begin{array}{ccc} C_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

Picking out just the pullback arrows — what the founders called *cartesian arrows* — makes our category "moduli-like". Why? consider the *fiber* of this category over the identity $S \to S$. Saying that any morphism is cartesian implies that any morphism lying over the identity is an isomorphism. In the special case where S is a point we are looking at curves with isomorphisms between them; in general we look at families of curves with fiberwise isomorphism. In ancient times one thought of moduli as parametrizing objects *up to isomorphisms.* The categorical framwork is better: we parametrize objects, in a sense, *along with their isomorphisms.*

What remains is the task of making the category sufficiently geometric so that one can act as if it is a scheme.

2.2.1. *Categories fibered in groupoids.* Let us formalize what we have gotten so far. A moduli problem is, in particular, a category $\mathcal{M}$ whose objects should behave like $C \to S$. Any such object has an underlying scheme S over which it lies, and any arrow lies over an arrow of schemes. In other words we should have a *functor* $\mathcal{M} \to \mathcal{S}$ to a suitable base category. This is known as the structure functor. In much of our discussion $\mathcal{S} = \mathrm{Sch}$ is just the category of schemes. Moreover arrows should behave like pullbacks: given an arrow $S_1 \to S_2$ in $\mathcal{S}$ and an object $C_2 \to S_2$ in $\mathcal{M}$ over S_2 (we say *an object in* $\mathcal{M}(S_2)$), there is a suitably unique pullback:

Definition 2.2.2. A functor $\mathcal{F} : \mathcal{M} \to \mathcal{S}$ is a *category fibered in groupoints* if for every $f : S_1 \to S_2$ and every $C_2 \in \mathcal{M}(S_2)$ there is $C_1 \in \mathcal{M}(S_1)$ and an arrow $C_1 \to C_2$ over f, and any other such $C_1' \to C_2$ factor uniquely through an isomorphism $C_1' \to C_1$ over the identity of S_1.

2.2.3. *Schemes as categories fibered in groupoids.* This is all very well for $\mathcal{M}_g$, but stacks are to generalize schemes, so in what way is a scheme a category fibered in groupoids?

The answer, as usual is the taulological answer given by Yoneda: a scheme V corresponds to the category $\mathbf{Sch}_V$ of schemes over V. The functor $\mathcal{F}$ sents a scheeme over V say $T \to V$ to its underlying scource scheme T. And pullbacks are just compositions of arrows.

What's moduli theoretic about this? Again, a scheme is just the parameter space for its own points. You might think of an arrow $f : T \to V$ as parametrizing the family of points in $T \times V$ parametrized by T described by the graph of f.

2.2.4. *Quotients.* How about the quotient of a scheme V by the action of a group-scheme G? Points of a quotient correspond to orbits of the action, and when the action is free these are free orbits,. In other words, an arrow $S \to V/G$ is a principal G-bundle $P \to S$ with an equivariant arrow $P \to V$. [3] [4][5]

3→ 4→ 5→

[3](Dan) upgrade to a definition in slides

[4](Navid) Would it be worth extending the discussion of quotients, perhaps by discussing a few examples? E.g. a cyclic quotient singularity, and seeing how BG sits inside this?

[5](Dan) done a bit in slides!

2.2.5. *Towards stacks: descent of morphisms and descent of objects.* A scheme is obtained by gluing together affine schemes. Similarly, a morphism $S \to V$ in $\mathbf{Sch}_V$ is obtained by gluing together morphisms on an affine cover of V which agree on overlaps. If stacks are to generalize schemes, a topological feature such as this must be required.

We assume given a Grothendieck topology $\mathcal{T}$ on the base category $\mathcal{S}$. For Deligne–Mumford stacks the étale topology suffices. In general one uses either the smooth or *fppf* topology.

A categor fibered in groupoid $\mathcal{M} \to \mathcal{S}$ *satisfies descent for arrows* if given
(AN UPGRADE OF TEH ABOVE FROM EARLIER TEXT)

2.3. **Stacks as moduli.** To really understand what stacks are about we change course. Stacks really come about in order to understand moduli spaces, when the moduli problem is not represented by a scheme.

We will have two key examples:

Example 2.3.1. The moduli stack $\mathcal{M}_g$ of curves of fixed genus $g > 1$, and

Example 2.3.2. the quotient $[V/G]$ of a variety V by a finite group G.

Special cases of Example 2.3.2 are

(1) the quotient $\mathcal{B}G := [\operatorname{Spec} k/G]$ of a *point* by *the trivial action* of G, and
(2) The quotient $V = [V/\{1\}]$ of any variety by the trivial group.

2.3.3. *Moduli of curves as a category.* We consider example 2.3.1.

When we talk about moduli of curves, we want to classify curves of genus g up to isomorphisms.

For our discussion, we work over a field k and fix an integer $g > 1$, and a *curve of genus* g is a smooth projective geometrically integral curve X of genus g over some field extension of k. A *family of curves of genus* g is a projective flat morphism $X \to S$ whose geometric fibers are curves of genus g.

The great observation of Deligne and Mumford is that the moduli space $\mathcal{M}_g$ *is* the category of families of curves of genus g. All that one needs to do is to make this category "geometric".

How do we make families of curves of genus g into a category?

Say $X_1 \to S_1$ and $X_2 \to S_2$ are families of curves of genus g. A *morphism* between them, for the purpose of classification, is a *cartesian* diagram of schemes

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2. \end{array}$$

Note that this implies that the morphism $X_1 \to S_1 \times_{S_2} X_2$ is an isomorphism, whose datum is equivalent to the datum of $X_1 \to X_2$ by the universal property of fibered products.

For the same reason note also that given $X_2 \to S_2$, a family of curves of genus g, and given a morphism $S_1 \to S_2$, the cartesian product $X_1 = S_1 \times_{S_2} X_2$ sits in such a diagram, in a way which is unique up to unique isomorphisms. That's by the universal property of fibered products.

Given a scheme S, the *fiber* of $\mathcal{M}_g$ over the identity morphism of S, denoted $\mathcal{M}_g(S)$, is the subcategory of families $X \to S$ where the morphisms fit over the identity of S:

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S & = & S. \end{array}$$

The definition implies that $X_1 \to X_2$ *is necessarily an isomorphism.* This fits with the idea that we set out to classify curves up to *isomorphisms.*

2.3.4. *A variety is a category.* We consider example 2.3.2(2).

Stacks come to extend the category **Sch** of schemes. On the other hand, stacks come to encode the idea of a moduli space. In what way is a scheme, or a variety V, a moduli space?

The answer is that *any variety is the moduli space of its own points.*

We know what a point on V is. What is a flat family of points? For a scheme S define a family of points on V parametrized by S to simply be a closed subscheme $S' \subset S \times V$ mapping isomorphically to S. Look at the picture - this is the right notion!

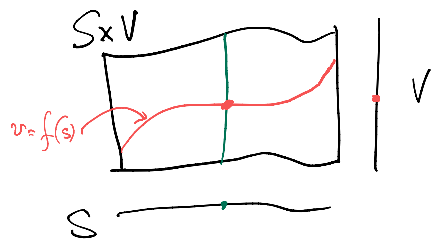

FIGURE 1. The graph of a morphism $f : S \to V$ as a family of points on V parametrized by S

Wait - in which way is this a category? If we have $S_1' \subset S_1 \times X$ and $S_2' \subset S_2 \times X$ then we have a morphism precisely when $S_1' = S_1 \times_{S_2} S_2' \subset S_1 \times X$. Here there is no choice for a map since we are equating $S_1 \times X = S_1 \times_{S_2} S_2 \times X$.

This is equivalent to giving a section $S \to S \times V$, which in turn is equivalent to giving a morphism $S \to V$. In other words,

> **the moduli category of points on V is the category $\mathbf{Sch}_V$ of schemes over V.**

Exercise 2.3.5. *Describe,* as a category, *the fiber* $\mathbf{Sch}_V(S)$ *of the category* $\mathbf{Sch}_V$ *over the identity of a scheme* S.

2.4. **Categories fibered in groupoids.** We can now generalize:

Definition 2.4.1. A functor $F : \mathcal{C} \to \mathbf{Sch}$ from a category $\mathcal{C}$ to the category of schemes makes $\mathcal{C}$ *a category fibered in groupoids* if for every object $X_2 \in \mathcal{C}(S_2)$ and any morphism of schemes $f : S_1 \to S_2$ there is a morphism $\tilde{f} : X_1 \to X_2$ such that $F(\tilde{f}) = f$, which is unique up to unique isomorphism.

Remark 2.4.2. This is not my fault - the notion of groupoid in *a category fibered in groupoids* is not the same as the notion of groupoid in *a groupoid in schemes*. They are close enough to cause likely confusion. Please be careful!

Exercise 2.4.3. *Verify that $\mathcal{M}_g$ is a category fibered in groupoids.*

(You should use the functor that takes a family of curves $C \to S$ to its base scheme S.)

Exercise 2.4.4. *Verify that* $\mathbf{Sch}_V$ *is a category fibered in groupoids.*

(You should use the functor that takes a "family of points" $S \to V$ to its base scheme S.)

Recall that a set Z gives rise to a category whose objects are the elements of Z and arrows are declared to be just the ones needed, namely id_z for all $z \in Z$. Also a category is said to be a set if it is equivalent to a category associated to a set (so it is small and all arrows are unique isomorphisms).

Exercise 2.4.5. *Show that the fibers of* $\mathbf{Sch}_V$ *are sets.*

We say that a category fibered in groupoids is *fibered in sets* if the fibers are sets. The exercise shows that the category of points on a scheme is fibered in sets[6].

2.4.6. *Arrows.* Let $F_1 : \mathcal{C}_1 \to \mathbf{Sch}$ and $F_2 : \mathcal{C}_2 \to \mathbf{Sch}$ be categories fibered in groupoids. A *morphism* or *base-preserving functor* is a functor $G : \mathcal{C}_1 \to \mathcal{C}_2$ with $F_2 \circ G = F_1$.

Exercise 2.4.7. *Show that a morphism* $\mathbf{Sch}_{S_1} \to \mathbf{Sch}_{S_2}$ *is equivalent to a morphism* $S_1 \to S_2$*. Show also it is equivalent to an object of* $\mathbf{Sch}_{S_2}(S_1)$ *and to an element of* $S_2(S_1)$*.*

It thus makes sense to identify S with $\mathbf{Sch}_S$.

For instance, suppose you have a morphism $\xi : \mathbf{Sch}_S \to \mathcal{C}$. Consider the final object $S \stackrel{id_S}{\to} S$ of $\mathbf{Sch}_S$. Its image $\xi(S \stackrel{id_S}{\to} S)$ is an object $\bar{\xi}$ of $\mathcal{C}(S)$, and for any other object $g : T \to S$ of $\mathbf{Sch}_S$ its image is necessarily the pullback of $\bar{\xi}$ by g. Conversely, given an object $\bar{\xi}$ of $\mathcal{C}(S)$ we obtain a morphism $\xi : \mathbf{Sch}_S \to \mathcal{C}$, assigning to $g : T \to S$ the pullback of $\bar{\xi}$ by g.

Exercise 2.4.8. *Draw a big diagram on a big page (or board, or flip chart) explaining this sentence.*

[6] It turns out these are not the only ones.

It now makes sense to identify: morphisms $S \to \mathcal{C}$, in other words morphisms $\mathbf{Sch}_S \to \mathcal{C}$, with objects $\mathcal{C}(S)$.

2.4.9. *Quotients of free actions.* Consider now a scheme X with a *free* action of a group-scheme G having a geometric quotient $X \to Y$. This precisely means $X \to Y$ is surjective and that G acts simply transitively on geometric fibers of $X \to Y$ — we say that $X \to Y$ is a G-torsor or principal G-bundle.

We want to say that Y is the moduli space of orbits on X, which precisely means that its category $\mathbf{Sch}_Y$ is naturally equivalent to the "category of families of orbits", which we must now define.

First note: since the action is free, an orbit in X is a principal G-variety P and an equivariant map $P \to X$.

This allows us to define, for such a free action, a "family of orbits" over a scheme S to be a principal G-bundle $P \to S$ with an equivariant map $P \to X$, a diagram like this

$$\begin{array}{ccc} P & \xrightarrow{G\text{-equivariant}} & X. \\ \downarrow & & \\ S & & \end{array}$$

A morphism from $(P_1 \to S_1, P_1 \to X)$ to $(P_2 \to S_2, P_2 \to X)$ is defined as a cartesian diagram:

$$\begin{array}{ccccc} P_1 & \xrightarrow{G\text{-equivariant}} & P_2 & \xrightarrow{G\text{-equivariant}} & X. \\ \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2 & & \end{array}$$

Note that whenever we have a morphism $S \to Y$ we may form the principal bundle $P := S \times_Y X$, which forms a diagram as above:

$$\begin{array}{ccc} P & \xrightarrow{G\text{-equivariant}} & X \\ \downarrow & & \vdots \\ S & \cdots\cdots\rightarrow & Y \end{array}$$

Conversely, given a principal bundle $P \to S$ and equivariant $P \to X$, the composite map $P \to Y$ is G-invariant. Now S is the categorical quotient of the action of G on P — this follows for instance by flat descent — so the morphism $P \to Y$ factors uniquely through a morphism $S \to Y$. This implies that $P = S \times_Y X$. It follows that

the category of families of orbits on X is equivalent to $\mathbf{Sch}_Y$.

Exercise 2.4.10. *Complete a proof of this statement.*

2.4.11. *Quotients in general.* We come to example 2.3.2 in general.

Now let G act on X, not necessarily freely. We *define* a category fibered in groupoids $[X/G]$ whose objects are diagrams like this

$$\begin{array}{ccc} P & \xrightarrow{G\text{-equivariant}} & X, \\ \downarrow & & \\ S & & \end{array}$$

and whose arrows from $(P_1 \to S_1, P_1 \to X)$ to $(P_2 \to S_2, P_2 \to X)$ are cartesian diagrams

$$\begin{array}{ccccc} P_1 & \xrightarrow{G\text{-equivariant}} & P_2 & \xrightarrow{G\text{-equivariant}} & X. \\ \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2 & & \end{array}$$

Exercise 2.4.12. *(1) Verify that this is a category fibered in groupoids.*
(2) Verify that $[X/G]$ is equivalent to $\mathbf{Sch}_Y$ if and only if G acts freely on X having quotient scheme Y.

To introduce the following exercises, we need a definition that will serve us well later:

Definition 2.4.13. Given categories fibered in groupoids X, S, and Y, and morphisms $\psi_X : X \to Y, \psi_S : S \to Y$, namely morphisms of categories compatible with the functors to $\mathbf{Sch}$, we define the *fibered product* $S \times_Y X$ to be the category whose objects over a scheme T are

(1) morphisms $T \to S$ and $T \to X$, equivalently objects $\xi_S \in S(T)$ and $\xi_X \in X(T)$, and
(2) an isomorphism $\psi_S(\xi_S) \to \psi_X(\xi_X)$.

Exercise 2.4.14. *Write explicitly what this means when S, X are schemes.*

Exercise 2.4.15. Check that this is compatible with pullbacks, making $S \times_Y X$ a category fibered in groupoids.

Exercise 2.4.16. *Going back to where X, S are schemes and $Y = [X/G]$, check that the resulting diagram*

$$\begin{array}{ccc} P & \xrightarrow{G\text{-equivariant}} & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & [X/G] \end{array}$$

is cartesian.

In other words,

> **categories fibered in groupoids allow us to have true quotients in general, extending the notion for free actions.**

Example 2.4.17. Consider now the trivial action of a group-scheme on $\operatorname{Spec} k$. It is customary to write $\mathcal{B}G := [\operatorname{Spec} k/G]$ — it is known as the *classifying stack of* G, at least once it receives the structure of a stack. Objects over a k-scheme S are principal G-bundles $P \to S$, and arrows are cartesian squares.

Exercise 2.4.18. *Given a G-equivariant morphism $X_1 \to X_2$ construct a natural morphism $[X_1/G] \to [X_2/G]$.*

This in particular provides a morphism $[X_1/G] \to \mathcal{B}G$ for any k-scheme X.

Exercise 2.4.19. *Given a homomorphism $H \to G$ and an action of G on X construct a natural morphism $[X_1/H] \to [X_2/G]$.*

2.5. **Stacks.** The category **Sch** of schemes has some useful Grothendieck topologies - in particular there is a sense in which a morphism $\sqcup S_i \to S$ is étale or smooth; and if this morphism is also surjective then it is an étale cover, or a smooth cover (namely a cover in the smooth topology); if S_i are Zariski open subsets you get, of course, a Zariski cover. To avoid saying "Zariski or étale or smooth" all the time we might refer to a $\mathcal{T}$*-covering*, with $\mathcal{T}$ indicating the chosen topology.

2.5.1. *Descent for morphisms of curves.* Let $C_1 \to S$ and $C_2 \to S$ be two families of curves of genus g. What would it take to show that they are isomorphic? Of course this requires writing an S-morphism $C_1 \to C_2$ and an inverse, but to do it concretely — say to give a dumb computer a morphism — we need to use a covering. After all morphisms are locally defined.

Say we have a *Zariski* covering $\sqcup S_i \to S$ and an isomorphism $\phi_i : C_{1i} \to C_{2i}$ of the pullbacks of C_1, C_2 to S_i, then by definition of a morphism this gives an isomorphism $\phi : C_1 \to C_2$ if and only if ϕ_i agree on the intersections: $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$.

It is not trivial, but not hard either, that the same is true for the étale or smooth topologies: if $\sqcup S_i \to S$ is a $\mathcal{T}$-covering then a collection of isomorphisms $\phi_i : C_{1i} \to C_{2i}$ comes from a morphism $\phi : C_1 \to C_2$ if and only if ϕ_i agree on the "intersections": $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$, where $S_{ij} = S_i \times_S S_j$.

It is customary to assign to $C_1 \to S$ and $C_2 \to S$ the functor $\underline{\operatorname{Isom}}_S(C_1, C_2) : \mathbf{Sch}_S \to \text{Sets}$. For an S-scheme T its value is the set $\operatorname{Isom}_T(C_{1T}, C_{2T})$ of isomorphisms of $C_{1T} \to C_{2T}$. The discussion above says that this functor is a $\mathcal{T}$*-sheaf*: for a $\mathcal{T}$-covering $\sqcup T_i \to T$ we have an *equalizer sequence* — the analogue of an exact sequence of sets:

$$\mathrm{Isom}_T(C_{1T}, C_{2T}) \hookrightarrow \prod \mathrm{Isom}_{T_i}(C_{1T_i}, C_{2T_i}) \rightrightarrows \prod \mathrm{Isom}_{T_{ij}}(C_{1T_{ij}}, C_{2T_{ij}}).$$

Exercise 2.5.2. *Consider the case where $C_i \to S$ is a family of curves of genus 0. What does the sequence above say about comparing families of $\mathbb{P}^1$'s over a base S?*

2.5.3. *Descent for curves.* Let S be a scheme. What would it take to construct a family of curves $C \to S$? Again one needs to work locally. But now if we are given families of curves $C_i \to S_i$, their gluing requires the additional data of isomorphisms $\phi_{ji} : C_i|_{S_{ij}} \to C_j|_{S_{ij}}$. And such data ϕ_{ij} must satisfy the proverbial cocycle condition on S_{ijk}.

One can interpret this in terms of a longer, and categorical, "exact sequence", but let's leave that to the ∞-categorists. The traditional, 1-category language is to say that "curves satisfy effective descent", namely the families $C_i \to S_i$ and isomorphisms ϕ_{ji} give rise to $C \to S$, unique up to a unique isomorphism, if and only if the cocycle condition holds.

This holds for the Zariski topology by the definition of a scheme (and set-theoretic gluing). Again it is not trivial, but not hard either, that the same is true for the étale or smooth topologies.

Exercise 2.5.4. *Consider the case where $C_i \to S$ is a family of curves of genus 0. What does this discussion say about comparing families of $\mathbb{P}^1$'s over a base S, in the Zariski or étale topology?*

2.6. **Stacks in general.** Let us now fix $\mathcal{T}$= Zariski, étale or smooth. A category fibered in groupoids $\mathcal{C}$ is a *stack* if $\underline{\mathrm{Isom}}$ functors are $\mathcal{T}$-sheaves and if every descent datum is effective.

The discussion above says that $\mathcal{M}_g$ is a stack in any of these topologies. It is not too hard to show that the same is true for the categories $[V/G]$ we discussed earlier. In particular the category $\mathbf{Sch}_V$ is a stack.

Exercise 2.6.1. *Outline for yourself why indeed $\mathcal{M}_g, \mathbf{Sch}_V$, and $[V/G]$ are stacks in the étale topology.*

2.6.2. *Discussion.* To summarize,

> **stacks allow us to put meaningful topological structures on categories fibered in groupoids, in particular those associated with natural moduli spaces and with quotients.**

This in particular means that objects of our category — or moduli problem — are local in nature. For instance a family $C \to S$ of curves of genus g over S can be recognized as such by restricting it to a covering $\sqcup S_i \to S$. From a practical point of view, this also means that our proverbial dumb computer can work with it, taking charts S_i on S and families $C_i \to S_i$, with appropriate gluing data on the overlaps.

Note however that objects of stacks are one level more complex than sections of sheaves: to give a section $s \in \mathcal{F}(S)$ of a sheaf of sets $\mathcal{F}$ on a scheme S covered by S_i, it is equivalent to give sections $s_i \in \mathcal{F}(S_i)$ which agree on the overlap. The first sheaf axiom says that given s_i such s is unique if it exists — a separatedness condition. The second sheaf axiom says that if s_i agree on overlaps such s does exist — a locality condition. This is all that is needed to glue *sets*, in essence since a bijective map of sets can be broken down to being injective and surjective.

In the case of stacks, we are gluing categories. Keeping the analogy with bijections of sets, here we are concerned with *equivalences* of categories, a slightly more subtle question. Recall that to verify that a functor $\mathcal{G} : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories, one checks three conditions: two conditions on morphisms — that $\mathcal{G}$ is full (surjective on arrows) and faithful (injective on arrows), and a third condition on objects — that $\mathcal{G}$ is essentially surjective.

For stacks, the analogy leads to the two sheaf conditions for arrows, and a third condition on objects, requiring them to glue locally. This is necessarily more subtle — it involves triple intersections and the cocycle condition.

2.7. **Algebraic stacks.**

2.7.1. *Representability.* A stack $\mathcal{C}$ is represented by a scheme S if it is isomorphic to $\mathbf{Sch}_S$. It is representable if it is represented by some scheme. A morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of stacks is *representable* if for any scheme S and any morphism $S \to \mathcal{C}_2$ the fibered product $S \times_{\mathcal{C}_2} \mathcal{C}_1$ is a scheme.[7]

One can show that if T is a scheme, then a morphism $\xi : T \to \mathcal{C}$ is representable if and only if for any scheme S and $\eta : S \to \mathcal{C}$ the sheaf $\mathrm{Isom}_{S\times T}(\xi, \eta)$ is a scheme.[8]

Exercise 2.7.2. *Show that any morphism $S \to \mathcal{C}$ is representable if $\mathcal{C} = \mathbf{Sch}_V$ is the stack represented by a scheme V.*

Exercise 2.7.3. *Show that any morphism $S \to \mathcal{C}$ is representable if $\mathcal{C} = [V/G]$.*

Exercise 2.7.4. *Show, using Hilbert schemes, that any morphism $S \to \mathcal{C}$ is representable if $\mathcal{C} = \mathcal{M}_g$.*

2.7.5. *Smooth and étale morphisms.* A *representable* morphism $\mathcal{C}_1 \to \mathcal{C}_2$ is said to be *smooth* if for every scheme S and $S \to \mathcal{C}_2$ the resulting morphism of schemes $S \times_{\mathcal{C}_2} \mathcal{C}_1 \to S$ is smooth. Similarly for étale, smooth covering, étale covering, proper, etc.

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[7]The standard terminology allows it to be an algebraic space. Our more restrictive notion will suffice for our purposes.

[8]With the algebraic space terminology this is automatic, since reasonable sheaves are algebraic spaces.

2.7.6. *Algebraic stacks.* A stack $\mathcal{C}$ is $\mathcal{T}$*-algebraic* if

(1) for any scheme S, any $S \to \mathcal{C}$ is representable; equivalently the diagonal morphism $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is representable, and
(2) there exists a $\mathcal{T}$-covering $V \to \mathcal{C}$.

Generally, étale-algebraic stacks are known as *Deligne–Mumford stacks*, and smooth-algebraic stacks are known as *Artin stacks*.

Exercise 2.7.7. *Show explicitly that when* $\mathcal{C} = \mathbf{Sch}_V$*, a morphism* $S \to \mathcal{C}$ *is smooth/étale etc. if and only if* $S \to V$ *has the same property.*

Deduce that $\mathbf{Sch}_V$ *is algebraic.*

Exercise 2.7.8. *Show explicitly that when* $\mathcal{C} = [V/G]$*, a morphism* $S \to \mathcal{C}$ *is smooth/étale etc. if and only if* $\mathcal{P} \to V$ *has the same property, where* $\mathcal{P} = S \times_{\mathcal{C}} V$*.*

Show that, if G *is smooth,* $V \to \mathcal{C}$ *is a covering in the smooth topology. Deduce that* $\mathcal{C}$ *is algebraic.*

M. Artin has shown how to relax the requirement that G be smooth.

Exercise 2.7.9. *Every smooth curve of genus 0 has an anticanonical embedding in* $\mathbb{P}^2$*. Considering the Hilbert scheme of conics in* $\mathbb{P}^2$*, show that the locus of smooth conics is open. Show that* $\mathcal{M}_0$ *is the quotient of this scheme by the action of a smooth group scheme, hence is an algebraic stack. Is it a Deligne–Mumford stack?*

Exercise 2.7.10. *Can you repeat the discussion above for* $\mathcal{M}_{1,1}$*, the moduli stack of elliptic curves?*

What happens with $\mathcal{M}_1$*, the moduli stack of curves of genus 1?*

Exercise 2.7.11. *This is the crux of Edidin's paper* [**?**]*, and you might wish to just read Edidin's account:*

(1) Show using Riemann-Roch that the three-canonical linear series separates points and tangent spaces: $\dim H^0(C, O_C(3K_C)) = \dim H^0(C, O_C(3K_C - p - q)) + 2$ *for any two points* p, q*. Deduce that every smooth curve of genus* g *has a complete linear series giving a 3-canonical embedding into a projective space of dimension* $5g - 6$*.*
(2) Show that such curves form a locally closed subscheme H *of the appropriate Hilbert scheme, parametrizing curves of genus* g *of degree* $6g - 6$ *inside* $\mathbb{P}^{5g-6}$*.*
(3) Show that the quotient $[H/G]$ *with* $G = PGL_{5g-5}$ *is isomorphic to* $\mathcal{M}_g$*, as follows:*

- *Given a family of curves $\pi : C \to S$, consider the locally free sheaf $\mathcal{F} = \pi_*\omega_{C/S}^{\otimes 3}$ and its projective frame bundle $\mathcal{P}(\mathcal{F}) \to S$. Show that this is a G-bundle. Show that the pullback of the projectivization $\mathbb{P}(\mathcal{F})$ to $\mathcal{P}$ is a trivial projective bundle, in which the pullback of C embeds as a 3-canonical curve. Show that this gives a G-equivariant morphism $\mathcal{P} \to H$, inducing a morphism $S \to [H/G]$. In turn this defines a morphism $\mathcal{M}_g \to [H/G]$.*
- *There is a universal curve $C_H \to H$ embedded in $H \times \mathbb{P}^{5g-6}$. Given a principal G-bundle $\mathcal{P} \to S$ and equivariant $\mathcal{P} \to H$, consider the pullback $C_{\mathcal{P}}$ of this universal curve. Show that G acts freely on $C_{\mathcal{P}} \to \mathcal{P}$, and the quotient is a family of genus g curves $C \to S$. This in turn defines a morphism $[H/G \to \mathcal{M}_g$.*
- *One needs to show that these morphisms are quasi-inverses of each other (a task rarely seen in writing).*

(4) Deduce that $\mathcal{M}_g$ is algebraic.

2.7.12. *Smoothness of quotients.* Traditionally one defines a morphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ of algebraic stacks to be *smooth* if it is of finite presentation and satisfies the infinitesimal criterion for smoothness. A bit of diagram chasing allows one to circumvent this: the infinitesimal criterion holds if and only if, given a smooth covering $V \to \mathcal{C}_1$, the morphism $\mathcal{C}_1 \to \mathcal{C}_2$ is smooth if and only if the composite $V \to \mathcal{C}_2$ is smooth.

Let us now consider the quotient $[V/G]$ of a k-variety V by a smooth group-scheme G. we conclude that

$[V/G] \to \operatorname{Spec} k$ **is smooth if and only if** $V \to \operatorname{Spec} k$ **is smooth.**

In other words, algebraic stacks are indeed a way to find hidden smoothness in quotients

3. Moduli spaces, part 1

By Pierrick.

The goal of this section is to provide examples of the general notion of moduli space described in Stacks, part 1. These examples will also be used in the following chapters. Since the moduli spaces described in this section parametrize objects with trivial group of automorphisms, we will use the more traditional description of moduli problem as set-valued functors instead of the more general approach based on fibered category in groupoids introduced in Stacks, part 1.

3.1. **The functorial point of view.** We denote by Sch the category of schemes and by Sets the category of sets. For every scheme X, the *functor of points* of X is the functor

$$\begin{aligned} h_X : \mathrm{Sch}^{op} &\longrightarrow \mathrm{Sets} \\ S &\longmapsto \mathrm{Hom}_{\mathrm{Sch}}(S, X) \end{aligned}$$

According to the Yoneda lemma reviewed below, a scheme is uniquely determined by its functor of points.

Lemma 3.1.1 (Yoneda Lemma)**.** *Let* $\mathcal{C}$ *be a category, and let* $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$ *be the category of functors from* $\mathcal{C}^{op}$ *to* Sets*. Then, the functor*

$$\mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$$
$$X \mapsto (h_X : S \mapsto \mathrm{Hom}_C(S, X))$$

is fully faithful.

Exercise 3.1.2. Prove Yoneda lemma.

Many moduli problems take the form of a functor $h : \mathrm{Sch}^{op} \to \mathrm{Sets}$: for every scheme S, $h(S)$ is the set of families of objects parametrized by S, and for every morphism $S \to S'$, the induced morphism $h(S') \to h(S)$ corresponds to a notion of pullback of families.

Definition 3.1.3. A functor $h : \mathrm{Sch}^{op} \to \mathrm{Sets}$ is called representable if there exists a scheme X such that $h = h_X$.

Note that if h is representable, then, by Yoneda lemma, there exists a unique X such that $h = h_X$. If h describes a moduli problem and is representable, that is, $h = h_X$, then we say that X is a *fine moduli space* for the moduli problem. Note that in this case, the identity morphism in $\mathrm{Hom}_{\mathrm{Sch}}(X, X) = h_X(X)$ defines a family parametrized by X referred to as the *universal family* over X. For every scheme S, a family over S is defined by a unique morphism $S \to X$ and is the pullback by this morphism of the universal family over X.

Remark 3.1.4. Representability of a moduli problem is in general obstructed if the parametrized objects have non-trivial automorphisms. In such a case, it is more natural to replace the category of sets by the category of groupoids and to view a moduli problem as a category fibered in groupoids, as in Stacks, part 1.

3.2. **Grassmannian schemes and moduli of marked genus 0 curves.** In this section, we describe two simple examples of fine moduli spaces: the Grassmannian schemes and the moduli spaces of marked genus 0 curves. These moduli spaces can be described in relatively explicit ways.

3.2.1. *Grassmannian schemes.* Fix integers $0 \leq k \leq n$. For every field K, consider the set $\mathfrak{G}rass(k,n)(K)$ of k-dimensional linear subspaces of the n-dimensional vector space K^n. In order to realize $\mathfrak{G}rass(k,n)(K)$ as the set of K-points of an algebraic variety, one first needs to define a "family" version of the notion of k-dimensional linear subspace of K^n. Given any scheme S, we define $\mathfrak{G}rass(k,n)(S)$ as the set of quotients

$$\mathcal{O}_S^{\oplus n} \longrightarrow Q \,,$$

where Q is a locally free sheaf of rank $n-k$. When $S = \mathrm{Spec}\, K$ for a field K, one indeed recovers the previous description of $\mathfrak{G}rass(k,n)(K)$.

Given a morphism $f : S \to S'$ and a quotient map $u : \mathcal{O}_{S'}^{\oplus n} \to Q'$, the pullback $f^\star u : f^\star \mathcal{O}_{S'}^{\oplus n} = \mathcal{O}_S^{\oplus n} \to f^\star Q'$ is also a quotient map. Therefore, one can naturally view

$$S \longmapsto \mathfrak{G}rass(k,n)(S) \,,$$

as a functor

$$\mathfrak{G}rass(k,n) : \mathrm{Sch}^{op} \longrightarrow \mathrm{Sets} \,.$$

Theorem 3.2.2. *For every integers $0 \leq k \leq n$, the functor $\mathfrak{Grass}(k,n)$ is representable by a scheme $G(k,n)$. Moreover, $G(k,n)$ is a smooth projective of dimension $k(n-k)$ over* $\operatorname{Spec}\mathbb{Z}$.

Proof. We sketch a construction of $G(k,n)$ by gluing of affine pieces. By definition, for every base field K, the set of K-points of $G(k,n)$ should be the set of k-dimensional linear subspaces V of K^n. Choosing a basis of V, this set can be described as the set of $k\times n$-matrices M, modulo the action of $GL(k,K)$ by left multiplication. For every $I=(0\leq i_1<\cdots<i_k\leq n)$, the minor p_I of the k-columns of M of indices $i_1,\ldots,i_k$ is a well-defined function on this set, modulo multiplication by a non-zero constant. In particular, $U_I := \{M \,|\, p_I \neq 0\}$ is a well-defined subset. Moreover, if $M\in U_I$, then, up to left multiplication by a unique matrix in $GL(k,K)$, one can assume that M is a $k\times(n-k)$ matrix X^I such that the $k\times k$ sub-matrix consisting of the columns of index $i_1,\ldots,i_k$ is the identity matrix. Hence, U_I is parametrized by the $k(n-k)$ coefficients formed by the $(n-k)$ columns of X^I of index not equal to i_j for any $1\leq j\leq k$. Hence, $U_I\simeq K^{k(n-k)}$. Moreover, if $M\in U^I\cap U^J$, the corresponding matrices X^I and X^J are related by $X^J=(X^I_J)^{-1}X^I$, where X^I_J is the $k\times k$-submatrix of X^I consisting of the columns of index $j_1,\cdots,j_k$.

This construction actually makes sense over $\mathbb{Z}$. For every $I=(0\leq i_1<\cdots<i_k\leq n)$, denote by U_I the affine space $\mathbb{A}^{k(n-k)}$, viewed as the space of $k\times(n-k)$ matrices X^I such that the $k\times k$ columns formed by the k columns of index $i_1,\cdots,i_k$ is the identity matrix. We define the scheme $G(k,n)$ by gluing the $\binom{n}{k}$ affine spaces U_I by $X^J=(X^I_J)^{-1}X^I$, where X^I_J is the $k\times k$-submatrix of X^I consisting of the columns of index $j_1,\cdots,j_k$. Note that this is meaningful over $\operatorname{Spec}\mathbb{Z}$ since matrix multiplication and matrix inverse are respectively polynomial and rational over $\mathbb{Z}$ in the matrix coefficients. One can check that the functor $\mathfrak{Grass}(k,n)$ is indeed represented by the scheme $G(k,n)$.

The scheme $G(k,n)$ is clearly smooth since it is locally isomorphic to affine spaces. Moreover, one can show that $G(k,n)$ is proper by checking the valuative criterion for properness: if R is a valuation ring with fraction field K, the limit of a map $\operatorname{Spec}K\to G(k,n)$ is contained in a chart U_I such that $p_I(\operatorname{Spec}K)$ has the smallest valuation. Actually, one can check that the map

$$(p_I)_I : G(k,n) \longrightarrow \mathbb{P}^{\binom{n}{k}-1} \tag{1}$$

is a closed embedding, and so that $G(k,n)$ is projective over $\operatorname{Spec}\mathbb{Z}$. ♣

Remark 3.2.3. The minors p_I are called the *Plücker coordinates* and the embedding (1) is called the *Plücker embedding.*

Example 3.2.4. Simple examples of $G(k,n)$:

(i) $G(1,n)\simeq\mathbb{P}^{n-1}$.
(ii) The Plücker embedding realizes $G(2,4)$ as a smooth quadric hypersurface in $\mathbb{P}^5$.

Exercise 3.2.5. Find the equation of $G(2,4)$ as a quadric in $\mathbb{P}^5$ in terms of the Plücker coordinates.

The universal family of $G(k,n)$ defines a tautological short exact sequence

$$0\to E\to \mathcal{O}_{G(k,n)}^{\oplus n}\to Q\to 0$$

of vector bundles over $G(k,n)$, where E is a rank k vector bundle over $G(k,n)$, and Q is a rank $(n-k)$ vector bundle over $G(k,n-k)$. The following result describes the tangent bundle of $G(k,n)$ in terms of the tautological bundles E and Q.

Theorem 3.2.6. *The tangent bundle $T_{G(k,n)}$ of $G(k,n)$ over* Spec $\mathbb{Z}$ *is the rank $k(n-k)$ vector bundle $\mathcal{H}om(E,Q)$:*

$$T_{G(k,n)} = \mathcal{H}om(E,Q)\,.$$

Exercise 3.2.7. Prove Theorem 3.2.6.

3.2.8. *Moduli spaces of marked genus 0 curves.* We would like to parametrize smooth genus 0 with n distinct marked point. Given a scheme S, we define $\mathcal{M}_{0,n}(S)$ as the set of flat proper morphisms $\pi : C \to S$ with n sections $s_i : S \to C$, $1 \le i \le n$ having disjoint images, such that every geometric fiber $\pi : C_s \to s$ is a smooth projective curve of genus 0. There is a natural notion of pullback for such families, and so $S \mapsto \mathcal{M}_{0,n}(S)$ naturally defines a functor $\mathcal{M}_{0,n} : \mathrm{Sch}^{op} \to \mathrm{Sets}$.

Theorem 3.2.9. *For every integer $n \ge 3$, the functor $\mathcal{M}_{0,n}$ is representable by a scheme $M_{0,n}$. Moreover, $M_{0,n}$ is irreducible and smooth of dimension $n-3$ over* Spec $\mathbb{Z}$.

Proof. Sketch of proof: describe explictly $M_{0,4}$ using the cross ratio. For $n \ge 5$, realize $M_{0,n}$ as an open subset in the product of $n-3$ copies of $M_{0,4}$. ♣

Example 3.2.10. Simple examples of $M_{0,n}$:

(i) $M_{0,3} = \mathrm{Spec}\,\mathbb{Z}$.

(ii) $M_{0,4} = \mathbb{P}^1 \smallsetminus \{0,1,\infty\}$, where 0, 1, and ∞ are three disjoint sections of $\mathbb{P}^1 \to \mathrm{Spec}\,\mathbb{Z}$.

Remark 3.2.11. For $n < 3$, the representability of $\mathcal{M}_{0,n}$ is obstructed by the existence of non-trivial automorphisms of $\mathbb{P}^1$ fixing n distinct points.

Exercise 3.2.12. Describe the group of automorphisms of $\mathbb{P}^1$ fixing n distinct points for $n=0$, $n=1$, and $n=2$.

3.3. **Hilbert and Quot schemes.**

In this section, we introduce Hilbert schemes, and their generalization given by Quot schemes. These moduli spaces are of fundamental importance in algebraic geometry since they are often used as starting point of constructions of many other moduli spaces.

3.3.1. *Hilbert schemes.* We would like to parametrize all possible closed subschemes $Z \subset X$ in a given scheme X. The Hilbert functor $\mathfrak{Hilb}_X$ assigns to a scheme X the set of closed subschemes $Z \subset X \times S$ which are flat over S.

Assume that X is projective over Spec $\mathbb{Z}$ and fix an ample line bundle L on X.

Definition 3.3.2. For every coherent sheaf E on X, the *Hilbert polynomial* $p_{L,E}$ is the unique polynomial $p_{L,E} \in \mathbb{Q}[x]$ such that, for all $n \in \mathbb{Z}$, we have

$$p_{L,E}(n) = \chi(X, E \otimes L^{\otimes n}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(X, E \otimes L^{\otimes n})\,. \tag{2}$$

Exercise 3.3.3. Prove the existence of the Hilbert polynomial $p_{L,E}$. Hint: reduce to the case where L is very ample, and then consider the restriction of E to an hyperplane section defined by L, and proceed by induction.

Exercise 3.3.4. Show that the Hilbert polynomial $p_{L,E}$ of E is the unique polynomial $p_{L,E} \in \mathbb{Q}[x]$ such that $p_{L,E}(n) = \dim H^0(X, E \otimes L^{\otimes n})$ for all large enough $n \in \mathbb{Z}$.

Exercise 3.3.5. Let $Z \subset X$ be the scheme-theoretic support of E. Show that the Hilbert polynomial $p_{L,E}$ of E has degree the dimension m of Z, and leading coefficient $\frac{d}{m!}$, where d is the degree of Z with respect to L, that is,

$$p_{L,E}(x) = \frac{d}{m!}x^m + o(x^m)$$

Exercise 3.3.6. For $X = \mathbb{P}^m$, $L = \mathcal{O}_{\mathbb{P}^m}(1)$, and $E = \mathcal{O}_Z$, where Z is a degree d hypersurface in $\mathbb{P}^m$, show that

$$p_{L,E}(x) = \binom{x+m}{m} - \binom{x-d+m}{m}$$

Check explicitly in this case that the leading term is given as in Exercise 3.3.5.

A key propery of the Hilbert polynomial is that it stays constant for flat families of coherent sheaves. In particular, the Hilbert polynomial $p_{L,\mathcal{O}_Z}$ of the structure sheaf $\mathcal{O}_Z$ of a subscheme $Z \subset X$ remains constant when Z varies in a flat family. It follows that, for every polynomial $P \in \mathbb{Q}[x]$, there is a well-defined moduli functor $\mathfrak{H}ilb_{X,P}$ for families of subschemes $Z \subset X$ such that $p_{L,\mathcal{O}_Z} = P$, and we have

$$\mathfrak{H}ilb_X = \bigsqcup_{P \in \mathbb{Q}[x]} \mathfrak{H}ilb_{X,P}\,.$$

The following theorem is due to Grothendieck.

Theorem 3.3.7. *Let X be a projective scheme and L an ample line bundle on X. Then, for every polynomial $P \in \mathbb{Q}[x]$ the functor $\mathfrak{H}ilb_{X,P}$ is represented by a projective scheme $Hilb_{X,P}$, called the* Hilbert scheme *parametrizing subschemes of X with Hilbert polynomial P.*

Remark 3.3.8. In Theorem 3.3.7, the Hilbert scheme $Hilb_{X,P}$ is projective, and so in particular of finite type. The latter is already a remarkable fact: the space of all possible subschemes of X is "finite dimensional" once we fix the Hilbert polynomial. Many arguments in modern algebraic geometry rely on this finiteness statement.

Proof. We provide a sketch of the proof of Theorem 3.3.7. It is enough to consider the case $X = \mathbb{P}^m$ and $L = \mathcal{O}(1)$. For every coherent sheaf E and $n \in \mathbb{Z}$, denote $E(n) := E \otimes L^{\otimes n}$. The basic idea is to try to "linarize" the difficult problem to describe the very non-linear problems to describe all subschemes $Z \subset \mathbb{P}^m$, that is, all short exact sequences

$$0 \to I_Z \to \mathcal{O} \to \mathcal{O}_Z \to 0\,.$$

To do this, note that for every $n \in \mathbb{Z}$, we have a short exact sequence

$$0 \to I_Z(n) \to \mathcal{O}(n) \to \mathcal{O}_Z(n) \to 0\,,$$

and so an embedding of vector spaces

$$H^0(\mathbb{P}^m, I_Z(n)) \subset H^0(\mathbb{P}^m, \mathcal{O}(n))\,.$$

Concretely, for every $n \in \mathbb{Z}$, we are looking at the linear subspace of homogeneous degree n polynomials which vanish on Z. Since I_Z is finitely generated, it is clear that this linear subspace determines Z uniquely if n is large enough. The non-trivial result is to show that "large enough" only depends on the Hilbert polynomial of Z, and so can be taken uniformly over Z. The required argument is a cohomological vanishing proved by induction on the dimension and known as Castelnuovo-Mumford regularity. The upshot is that, given $P \in \mathbb{Q}[x]$, there exists n large enough such that, for every Z with Hilbert polynomial P, Z is uniquely determined by the subspace $H^0(\mathbb{P}^m, I_Z(n)) \subset H^0(\mathbb{P}^m, \mathcal{O}_Z(n))$, and moreover, we have a short exact sequence

$$0 \to H^0(\mathbb{P}^m, I_Z(n)) \to H^0(\mathbb{P}^m, \mathcal{O}(n)) \to H^0(\mathbb{P}^m, \mathcal{O}_Z(n)) \to 0\,.$$

In particular, denoting $V := H^0(\mathbb{P}^m, \mathcal{O}(n))$, and $k = \dim V - \dim H^0(\mathbb{P}^m, \mathcal{O}_Z(n)) = \dim V - P(n)$, we obtain an embedding of $\mathrm{H}ilb_{X,P}$ into the Grassmannian of k-dimensional subspaces of V. By Theorem 3.2.2, the Grassmannian is a projective subscheme, and so the representability and projectivity of the Hilbert scheme follow by showing that the image of the embedding in the Grassmannian is closed subscheme of the Grassmannian. ♣

Exercise 3.3.9. Show that for every k and n, the Grassmannian $G(k, n)$ is an example of Hilbert scheme $\mathrm{H}ilb_{\mathbb{P}^{n-1},P}$ for some $P \in \mathbb{Q}[x]$.

Example 3.3.10. Hilbert schemes are not smooth in general. Describe the example of cubic curves in $\mathbb{P}^3$ and briefly mention Murphy's law.

The following result describes the tangent spaces of the Hilbert scheme.

Theorem 3.3.11. *Let Z be a closed subscheme of X, with ideal sheaf I_Z and structure sheaf $\mathcal{O}_Z = \mathcal{O}_X / I_Z$. The tangent space to $Hilb_X$ at the point corresponding to Z is* $\mathrm{Hom}(I_Z, \mathcal{O}_Z)$.

Remark 3.3.12. Given Exercise 3.3.9, Theorem 3.3.11 describing the tangent space of the Hilbert scheme is a generalization of Theorem 3.2.6 describing the tangent space of the Grassmannian.

Exercise 3.3.13. Prove Theorem 3.3.11.

3.3.14. *Quot schemes.* A natural generalization of Hilbert schemes is given by Quot schemes: instead of quotient $\mathcal{O}_X \to \mathcal{O}_Z$, one can consider quotients $E \to Q$ where E is an arbitrary coherent sheaf.

Let X be a projective scheme, L an ample line bundle on X and E a coherent sheaf on X. Fix a polynomial $P \in \mathbb{Q}[x]$. Let $\mathfrak{Quot}_{X,E,P}$ be the functor assigning to a scheme S the set of quotients

$$p_X^\star E \longrightarrow Q\,,$$

with Hilbert polynomial of Q equal to P, and where $p_X : X \times S \to S$ is the projection on S, and Q is a coherent sheaf on $X \times S$, flat over S.

Theorem 3.3.15. *Let X be a projective scheme, L an ample line bundle on X and E a coherent sheaf on X. Then, for every polynomial $P \in \mathbb{Q}[x]$, the functor $\mathfrak{Quot}_{X,E,P}$ is represented by a scheme $Quot_{X,E,P}$.*

Example 3.3.16. For $E = \mathcal{O}_X$, the Quot functor reduces to the Hilbert functor: $Quot_{X,\mathcal{O}_X,P} = Hilb_{X,P}$.

The following result describes the tangent spaces of the Quot scheme.

Theorem 3.3.17. *Let $p : E \to Q$ be a quotient of E. The tangent space to $Quot_{E,X}$ at the point corresponding to $p : E \to Q$ is* $\mathrm{Hom}(\mathrm{Ker}(p), Q)$.

Remark 3.3.18. Given Exercise 3.3.16, Theorem 3.3.17 describing the tangent space of the Quot scheme is a generalization of Theorem 3.3.11 describing the tangent space of the Hilbert scheme.

Exercise 3.3.19. Prove Theorem 3.3.17.

4. Geometry and combinatorics of snc divisors

We will soon learn that statements in logarithmic geometry typically consist of two pieces – one handled by the traditional algebro-geometric canon and another that is combinatorial in nature. We will introduce the combinatorial players in this section.

But before we jump in, let's outline the big picture of what's coming. A smooth variety X with a simple normal crossings divisor D will be our chief instance of a logarithmic scheme. Associated to a pair (X, D) is a *cone complex* Σ_X – an object very similar to a simplicial complex. We will develop a dictionary that associates geometric structures on X to combinatorial ones on Σ_X. In particular, to each *piecewise linear function* φ on Σ_X we will associate a pair $(\mathcal{L}_\varphi, s_\varphi)$ of a line bundle and a section.

Then later on, we'll learn about logarithmic schemes, which rather than funding these correspondences from some given geometry (such as a simple normal crossings divisor) will simply *add to a scheme* the data of a choice of "cone complex" Σ_X to a scheme X. It will do so in such a way that piecewise linear functions again give line bundle-section pairs of this form, subject to natural compatibilities. In fact, this is one among several equivalent definitions of a logarithmic structure.

4.1. **Simple normal crossings pairs.** Let X be a smooth variety.

Definition 4.1.1. A *simple normal crossings* divisor on X is a divisor D with irreducible components $D_1, \dots, D_k$ satisfying the followng two conditions:

(i) each irreducible component D_i is smooth, and
(ii) the divisors meet transversally.

The condition (ii) is stated more precisely, if less succintly, as follows: for every point p in X, the local equation for D is given by $x_1 \cdot \ldots \cdot x_r$ for independent local parameters x_i in the local ring $\mathcal{O}_{X,p}$, with $r \leq n$.

The pair (X, D) is referred to as a *simple normal crossings pair* or *snc pair* for short.

The way to think about condition (ii) is that at every point p in X, some number of the components D_i pass through p. By reordering, we can call these $D_1, \dots, D_r$. In a local neighborhood of p, these are cut out by equations $f_1, \dots, f_r$. The simple normal crossings condition is saying that these f_i, form a partial coordinate system of X near p, and can be augmented to a full system of coordinates by adding in $n - r$ "other" functions. As a consequence, étale locally (or in the analytic topology), there is a neighborhood of p that looks like $\mathbb{A}^r \times \mathbb{G}_m^{n-r}$, and the f_i are identified with the first r coordinate functions.

4.1.2. *A few key examples.* Of course, there are innumerable examples of snc pairs. However, a handful of examples are worth keeping in mind as we work through the topics in these notes.

(i) **Smooth pairs and products.** The simplest example of a smooth pair is (X, D) where X is smooth and D is a smooth divisor. These are called *smooth pairs.* Given two smooth pairs (X, D) and (Y, E) one can construct an *snc pair* by taking their product $X \times Y$ and endowing it with the divisor

$$\pi_X^{-1}(D) \cup \pi_Y^{-1}(E),$$

where π_X and π_Y are the projections onto X and Y respectively. We will shorten the notation to for the pair to $(X \times Y, D + E)$.

(ii) **Hyperplane arrangement pairs.** A very useful class of examples comes from hyperplane arrangements. Let X be $\mathbb{P}^n$ and let $\mathcal{H}$ be a generic union of k hyperplanes $H_1, \dots, H_k$. Generic here just means that all intersections have the expected codimension. The pair $(\mathbb{P}^n, \mathcal{H})$ is an snc pair regardless of the value of k. We will refer to it as a *hyperplane arrangement pair.*

(iii) **Projective bundles.** Another nice class of examples comes from projective bundles. Let B be any smooth variety and let $\mathcal{E}$ be a direct sum of line bundles

$$\mathcal{E} = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_r.$$

Let X be the $\mathbb{P}^r$-bundle associated to $\mathcal{E}$ over B. Because $\mathcal{E}$ is a direct sum of line bundles, there are $r + 1$ canonical "coordinate hyperplane bundles" which, in each $\mathbb{P}^r$-fiber over B, restrict to the coordinate hyperplanes.

If D is the union of these coordinate hyperplanes, then (X, D) is an snc pair. One can play similar games by passing to different fiberwise compactifications of the line bundle, or its associated torus bundle.

These examples are worth keeping in mind as a "testing ground" for intuition.

(iv) **Smooth toric varieties.** We will discuss toric varieties in detail in a series of exercises later on, but for readers that are already familiar with toric varieties, they also provide an important source of examples. Precisely, if X is a *smooth* toric variety and D is a union of invariant boundary divisors, (X, D) is an snc pair.

4.1.3. *A few key non-examples.* It is equally important to have in mind examples of things that are *not* simple normal crossings. The next set of non-examples include some proximate notions to snc pairs.

(i) **Planar triple point.** Let X be $\mathbb{P}^2$ and let D be a union of three lines through a point p. The pair $(\mathbb{P}^2, D)$ is *not* an snc pair. At the point of intersection p, the equations for the lines do not form a regular sequence of parameters, simply because there are 3 equations, but in a 2-dimensional space there should only be 2. In particular, in an snc pair, an intersection of k divisor components is either empty or has codimension k.

(ii) **Merely normal crossings.** Let X be $\mathbb{P}^2$ and let D be a nodal cubic with affine equation given by $y^2 = x^2(x+1)$. Since D is not smooth, the pair $(\mathbb{P}^2, D)$ is not snc.

However, this example is not so far off – it is an example of something that is *normal crossings* but not *simple normal crossings*. A simple way to say this is that formally locally, or locally in the analytic topology, around every point, the pair is isomorphic to a point on the pair $(\mathbb{A}^2, \Delta)$, where Δ is the union of the coordinate axes.

It can be convenient, especially when working with objects like the moduli space of curves, to say this in a slightly more technical way. If the function $x^2(x+1)$ had a square root in $k[x]$, say u, then the equation would read $y^2 - u^2$. This is, in turn, isomorphic to union of the coordinate axes. Although $x^2(x+1)$ is not a square, we can pass to an étale cover on which the square root exists.

Let us actually take a moment to record this definition, because it really is crucial.

Definition 4.1.4. Let X be a smooth variety. A divisor $D \subset X$ is a *normal crossings divisor* if it pulls back to a strict normal crossings divisor on an étale cover of X.

If we are working over the complex numbers are have access to the analytic topology, we can replace the étale cover above with a cover by analytic opens.

(iii) **Regular crossings pairs.** Let X be any pure-dimensional variety (or even more generally, a pure-dimensional scheme of finite type over an algebraically closed field). Let $D = \cup_{i=1}^n D_i$ be a union of effective Cartier divisors.

The pair (X, D) is said to have *regular crossings* if at every point p in X, the local equations for the D_i form a regular sequence. Intuitively, the different D_i intersection "completely" at every point.

Regular crossings pairs play an important role in the interaction between intersection theory and logarithmic geometry, and many of the combinatorial constructions that we outline here for snc pairs will work more generally for regular crossings pairs. Later on, simple normal crossings pairs will be regarded (together with some generalizations) as "smooth objects" in logarithmic geometry, while regular crossings pairs are "flat objects".

(iv) **Toroidal pairs.** We noted that a key set of examples of simple normal crossings pairs come from toric pairs (X, D). Conspicuously absent from our discussion are *singular* toric pairs (X, D), where X is a singular variety and D is the torus invariant boundary divisor. They are not simple normal crossings but they form a larger class of spaces called "toroidal pairs". More generally, these are pairs (X, D) that are étale locally modeled on singular toric pairs such as the ones above.

4.2. **Cone complexes.** The main combinatorial structure associated to a simple normal crossings is its *cone complex*. This combinatorial gadget keeps track of the combinatorics of an snc divisor D and gives combinatorial access to certain aspects of the geometry of (X, D).

The objects themselves are analogous to *simplicial complexes* – one obtains simplicial complexes by taking simplices of various dimension and gluing them along faces. Similarly, cone complexes are obtained by taking *cones* and gluing them along faces. The pictures in

$$FIGUREBELOW$$

should give the reader a picture to keep in mind.

Let us now do this formally. In this section, we will focus on the special case where the cones are *orthants* and come back to the general concept after we have discussed toric varieties.

The *standard k-dimensional orthant* is $\mathbb{R}^k_{\geq 0}$. It comes equipped with a natural collection of *linear functions*:

$$\mathbb{R}^k_{\geq 0} \to \mathbb{R}.$$

The linear functions with integer slopes are the subset of these that carry $\mathbb{N}^k \subset \mathbb{R}_{\geq 0}$ to $\mathbb{Z} \subset \mathbb{N}$. The group of linear functions is isomorphic to $\mathbb{Z}^k$. Inside this group is a monoid $\mathbb{N}^k$, consisting of *non-negative* linear functions.

More generally, an *orthant* is a pair (σ, M) of a topological space together with a finitely generated free abelian group of functions that is isomorphic to the standard orthant such that σ is homeomorphic to $\mathbb{R}^k_{\geq 0}$ and pullback along the homoeomorphic carries M isomorphically to the group of linear functions with integer slopes.

A *face* of an orthant is the vanishing set of a non-negative linear function. For the standard orthant σ, the faces are simply the coordinate axes.

A *morphism* of orthants is a continuous map of topological spaces $\sigma' \to \sigma$ that is compatible with the linear structure. Equivalently, after identifying the orthants with standard ones, the maps are modeled on integer linear maps. A special class of morphisms are isomorphisms onto faces, and we call these *face maps*.

Definition 4.2.1. A *smooth cone complex* Σ is the colimit of a partially ordered set of cones, with transition maps given by face maps. A morphism of cone complexes is a map of underlying topological spaces $\Sigma' \to \Sigma$ such that every cone of Σ' maps to a cone of Σ, and the restriction of the map to each cone of the source is linear.

The geometric relevance of cone complexes come from the fact that there exists a smooth cone complex associated to any simple normal crossings pair (X, D).

We first construct it with a mild simplifying assumption, called strata connectedness, and then in general. The simple case has the advantage of being very explicit, while the second definition we give generalizes well.

Let $D_1, \ldots, D_k$ be the irreducible components of D. We make the assumption that for any subset $I \subset [k]$ the intersection

$$D_I = \bigcap_{i \in I} D_i$$

is either empty or connected[9]. If this condition holds, we say (X, D) *has connected strata*[10].

Construction 4.2.2. Let (X, D) be a pair Now consider the cone complex $\mathbb{R}^k_{\geq 0}$. The r-dimensional cones of this cone complex are in bijection with the size r subsets of $[k]$. Thus, via this bijection, given a pair (X, D) the non-empty intersections D_I pick out a subset of the cones of $\mathbb{R}^k_{\geq 0}$. These form a smooth cone complex. We call this *the cone complex associated to* (X, D) and denote it by $\Sigma(X, D)$.

If (X, D) does not have connected strata then one should take a little more care in defining the cone complex associated to (X, D). We keep the notation from before and let $D_1, \ldots, D_k$ be the irreducible components.

Construction 4.2.3. For each scheme theoretic point $x \in X$, let $I \subset [k]$ be the subset determined by those divisors D_i that contain x. We can associate to each such point x a smooth orthant $\mathbb{R}^{I(x)}_{\geq 0}$ and denote it by σ_x. By convention, if $I(x)$ is empty, we take σ_x to be a point. If there is a specialization $x \rightsquigarrow x'$ then we can identify σ_x with a *face* of $\sigma_{x'}$. The *cone complex associated to* (X, D) is the colimit

$$\Sigma(X|D) := \varinjlim_{x \in X} \sigma_x.$$

Strictly speaking the indexing set is not a partially ordered set. However, one can readily see that the diagram can be replaced with the set of generic points of connected components of strata without changing the colimit.

The above presentation might seem slightly complicated but it is the "correct" way to think about it for the purposes of generalizations. For (X, D) a simple normal crossings pair one can give a more explicit presentation. As in the previous case, there is a map

$$\Sigma(X|D) \to \mathbb{R}^k_{\geq 0},$$

but it is not an isomorphism onto a union of faces. The map is an isomorphism onto its image after restriction to any cone of $\Sigma(X|D)$. However, in general, a point inthe interior of the cone corresponding to $I \subset [k]$ has finitely many preimages, corresponding to the number of connected components in D_I.

[9]Several mathematicians, including the author of this section, have absolutely no idea whether the empty set is connected and what's worse, they are incapable of remembering it even after they are told. So the adjective above may well be redundant, the author may never know.

[10]Note that simple normal crossings pairs need not have connected strata. For example, if $X = \mathbb{P}^2$ and D is a union of distinct curves that meet at least twice, the condition fails. Nevertheless, the condition can always be achieved after blowup, so it is not so restrictive in practice.

4.3. **Variants: toroidal models and self-intersecting divisors.** The cone complex associated to a simple normal crossings pair is a very useful gadget when it comes to manipulating the geometry of a pair, and it functions very similarly to the fan of a toric variety.

In fact, the construction goes much further: one can ask for similar structures that keep track of the combinatorics of more general spaces: toroidal pairs, pairs that are merely normal crossings, regular crossings pairs, and most generally, "logarithmic schemes". Toroidal pairs are the simplest generalization, and their theory is sketched at the end of the guide on toric geometry that follows this chapter.

We will come to these generalizations in due course, but let us note the two key ways in which the combinatorics might be genralized.
(i) If the divisor D can be modeled locally on a toric variety, then at every point $x \in X$ we can take σ_x to be the cone of the toric local model, and the resulting colimit is built up from polyhedral cones that are more complicated than orthants. We will make this precise after we have recalled some of the basic of toric varieties. As noted, you will encounter these in the toric section.
(ii) One can admit the case where D has components that self-intersect, that is if (X, D) is *merely normal crossings*. At every point $x \in X$, one can still associate an orthant to x by using the fact that étale (or analytically, or formally) locally near x, the pair (X, D) is simple normal crossings. However, the resulting colimit need not be equivalent to a colimit over a partially ordered set and one ends up with a *generalized cone complex* or a *cone stack* depending on the chosen formalism.

4.3.1. *Subtleties of self-intersections.* A key example to keep in mind is the following. If D is an irreducible nodal cubic in $\mathbb{P}^2$, then the generic point of the cubic specializes to the singular point in two *different* ways. This leads to two arrows:

$$\mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^2_{\geq 0},$$

which is the inclusion of a ray as the two different axes of $\mathbb{R}^2_{\geq 0}$. The colimit is a "cone complex" consisting of a single vertex, a single ray, and a single two dimension cell.

FIGURE

The example above gives one key way in which self-intersections change the picture: there can be multiple arrows between two cones in the diagram that defines $\Sigma(X, D)$. There is another key way in which self-intersections complicate the picture: there can be *self-arrows* in the diagram. An explicit example may be found in [ACP15, Example 6.1.7] but we explain the qualitative features. It is possible to have a pair (X, D) that is merely normal crossings with D being irreducible but intersecting itself (once). An étale local chart can be given by the simple normal crossings pair $(\widetilde{X} = \mathbb{A}^2 \times \mathbb{G}_m, \widetilde{D})$ where $\widetilde{D}$ is the toric boundary. The étale map $\widetilde{X} \to X$ is unramified of degree 2, and the deck transformation group interchanges the divisors corresponding to the two coordinates in the $\mathbb{A}^2$-direction.

In this situation, the appropriate diagram that is needed to form $\Sigma(X, D)$ is

$$\mathbb{R}^2_{\geq 0} \circlearrowleft,$$

where the arrow is an automorphism that interchanges the two axes of the orthant.[11] One can either take the colimit, which is a "half quadrant" which is the approach followed by [ACP15] or enlarge the category of cone complex to include these new basic objects of "cones with a group action". The latter approach is taken by [CCUW20].

These two examples essentially capture all the phenomena that one needs to account for to go from smooth cone complex associated to simple normal crossings pairs to the appropriate generalization (called *cone stacks* or *generalized cone complexes*) needed to associate analogous objects to toroidal pairs.

Remark 4.3.2. Looking at the constructions above, one might observe that the main structure needed to associate a cone complex-like objects to a scheme is the collection of cones σ_x for $x \in X$ and the maps $\sigma_x \to \sigma_y$ under specialization. This is one of the structures provided by a *logarithmic structure*, which will be explained later in these notes.

4.4. **Foreshadowing the Artin fan.** Let (X, D) be a simple normal crossings pair and let Σ be its cone complex, as above. The cone complex has a well-defined set of *faces*, ordered by inclusion. The set of faces forms a partially ordered set, which we will denote $\mathcal{P}(\Sigma)$. Given a simple normal crossings pair (X, D) has a natural "map"

$$X \to \mathcal{P},$$

which is defined as follows. Given a point x in X, it is contained in a well-defined subset of the divisor components of D, and remembering that the points of $\mathcal{P}$ are indexed by non-empty intersections of the components, the map simply records the deepest (i.e. smallest dimensional) intersection of divisors in which x lies.

There is no obvious map from X to the cone complex Σ, so this makes $\mathcal{P}$ perhaps more natural to think about. But what kind of map is this? Every partially ordered set has a natural topology where the "upper sets", namely the sets that are closed under taking elements that are larger in the poset, are open. Given this topology, one can check the map

$$X \to \mathcal{P},$$

is continuous, where X is given the Zariski topology.

Exercise 4.4.1. When X is a smooth proper toric variety and D is the toric boundary divisor, identify $\mathcal{P}$ with the topological orbit space X/T.

This leads us to a strange situation, where we have a map from an algebro-geometric object to a partially ordered set $\mathcal{P}$. One can ask: *is there a natural algebraic structure on $\mathcal{P}$ making this map into a morphism.*

Remarkably, the answer is *yes*! But in order to find the appropriate algebraic structure, we will need to pass to the world of algebraic stacks. Making this precise is another path that leads inevitavly to the notion of a logarithmic structure.

[11]This is our first hint that there is something stacky about these colimits – the partially ordered set that indexes the colimit is a particularly simple category. If we allow mroe complicated categories, we discover "cone stacks".

4.5. **Geometric correspondences.** Let (X, D) be a simple normal crossings pair and let Σ be its cone complex. The presence of D equips X with a distinguished subvarieties: namely the connected components of the intersections D_I for $I \subset [k]$. We call these *strata*.

Geometric constructions having to with the strata can often be encoded by Σ. We build these up in stages. We start with a basic bijection:

$$\{k\text{-dim'l connected components of } D_I\} \leftrightarrow \{k\text{-dim'l orthants between the rays } \rho_i,\ i \in I\}.$$

This bijection interacts with the Picard group of X. Specifically, to an assignment of integers $(\underline{m}_\rho)$ to the rays we can associate a divisor

$$\sum_\rho m_\rho D_\rho \in \mathrm{Div}(X).$$

This can also be encoded by a *piecewise linear function*

$$\alpha : \Sigma \to \mathbb{R}_{\geq 0}.$$

Indeed given $(\underline{m}_\rho)$ there is a unique continuous function that is (i) linear upon restriction to every orthant in Σ, and (ii) has slope m_ρ upon restriction to ρ.

Remark 4.5.1. Just as piecewise linear functions give divisors, piecewise *polynomial* functions on Σ give rise to cycles (or more honestly, refined cycle classes) of higher codimension.

4.5.2. *Functoriality.* The association of $\Sigma(X, D)$ to (X, D) has some functoriality properties, which, in a sense, generalize the discussion above[12].

Definition 4.5.3. Let (X, D) and (Y, E) be simple normal crossings pairs. A morphism $X \to Y$ is a *morphism of pairs* if the preimage of E is contained in D.

Consider a morphism of pairs $f : (X, D) \to (Y, E)$ between pairs with cone complexes Σ_X and Σ_Y. The *induced morphism*

$$\Sigma(f) : \Sigma_X \to \Sigma_Y$$

is defined as follows. In order to describe the morphism, it suffices to describe the pullback of every piecewise linear function on Σ_Y to Σ_X. Let φ be such a piecewise linear function. It determines a divisor on Y, supported on E, by the discussion above. Since f is a morphism of pairs the pullback of this divisor is supported on D. This again corresponds to a piecewise linear function, and this we have determined the pullback.

Exercise 4.5.4. Let Y be a smooth variety and let D be a union of two divisors meeting transversely. Let X be the blowup at their intersection. Compute Σ_Y, Σ_X, and the map $\Sigma_Y \to \Sigma_X$.

One can be a little more explicit. Let σ be a cone corresponding to a stratum with generic point p_σ. The image of p_σ in Y is contained in a unique minimal stratum, which corresponds to a cone δ in Σ_Y. We determine the map of cones

$$\sigma \to \delta.$$

[12]In essence, the above discussion is about functoriality with target $[\mathbb{A}^1/\mathbb{G}_m]$. The latter is a simple normal crossings pair in the smooth topology. This will come up again later on.

Since we are only interested in the map on these cones, we can pass to the open subschemes obtained by deleting all closed strata that do not meet the one dual to σ, resp. to δ. Shrinking further if necessary, we can assume the divisors D and E become principal on these opens. The map is now pulled back from a monomial map between affine spaces, which is exactly given by a matrix (recording the exponents of the pullbacks coordinate functions).

Another natural set of operations are blowups at strata. Let $W \subset X$ be a stratum of (X, D). By blowing up, we obtain:

$$\pi : X' = \mathrm{Bl}_W X \to X.$$

What's better, the (reduced) preimage D' of $D \subset X$ is also a divisor with simple normal crossings. The divisor D' has one more irreducible component than D does, namely the exceptional divisor E. The intersection pattern also changes: if W is a connected component of D_I the exceptional divisor E meets the strict transforms of D_i for $i \in I$, while $\cap_{i \in I} D_i$ contains one fewer connected component (if (X, D) is strata connected, then the strict transforms no longer intersect).

The cone complex Σ' of the blowup can be described by a *subdivision*. A morphism of cone complexes $\Sigma' \to \Sigma_X$ is a *subdivision* if it is a bijection on the underlying sets.

The basic subdivision is *stellar subdivision*.

Definition 4.5.5. Given a cone σ with primitive generators $u_1, \ldots, u_k$ the **stellar subdivision at** σ is the unique cone complex that refines Σ_X and has precisely one more ray, namely the one spanned by $\sum_{i=1}^k u_i$.

It is a simple exercise from the description above to observe a natural bijection:

$$\{\text{Blowups of } X \text{ along a single stratum}\} \leftrightarrow \{\text{Stellar subdivisions of } \Sigma \text{ along a single cone}\}$$

More generally, using a little bit of toric geometry that we will cover in the next section, this extends to:

$$\{\text{Proper bir. maps of pairs } X' \to X, \text{ with } X' \text{ smooth}\} \leftrightarrow \{\text{Subdivisions } \Sigma' \to \Sigma, \text{ with } \Sigma' \text{ smooth}\}$$

This gives a very powerful way of constructing proper birational models of a given simple normal crossings pair.

For example, one can iterate stellar subdivisions to form new to describe the cone complexes associated to iterated blowups, so iterated blowups along strata (which include both strict transforms and genuinely new strata at each step) are matched with iterated setllar subdivisions.

Example 4.5.6. A particularly useful subdivision of this kind is called *barycentric subdivision*. Given Σ, the barycentric subdivision is obtained by iterated stellar subdivision of all cones in order of descreasing dimension.

Barycentric subdivision can be defined for an arbitrary cone stack, and one particularly useful fact is that the barycentric subdivision of a cone stack is equivalent to a cone complex.

A simpler, though extremely useful, bijection is the bijection between *conical subcomplexes* – namely, unions of cones in Σ – and certain open subschemes of X. A *toroidal open subscheme* of (X, D) is any open obtained by deleting a union of closed strata. There is a natural bijection

$$\{\text{Toroidal open subschemes of } X\} \leftrightarrow \{\text{Conical subcomplexes of } \Sigma.\}$$

4.5.7. *Further geometric operations.* We introduce two additional geometric operations on pairs (X, D) that come from manipulating the cone complex. The first is of a stack theoretic nature: root constructions. We have encountered root stacks in SECTION XYZ. Recall that given a Cartier divisor E on a scheme Y, we can perform the r^{th} root construction of Y along E to obtain a stack $\mathcal{Y} \to Y$ with a Cartier divisor $\mathcal{E}$ that "adds stabilizer μ_r" along E. Precisely, it is the universal object living over Y such that the pullback of E (as a line bundle-section pair) has an r^{th}-root.

On affine space, the operation is particularly simple. Let $Y = \mathbb{A}^n$ and E be the first coordinate hyperplane.

Definition 4.5.8. Let Σ be a cone complex. A *combinatorial root construction* is another cone complex Σ' together with a map

$$\Sigma' \to \Sigma$$

that is (i) a bijection on underlying topological spaces, and (ii) maps each cone σ' surjectively onto a cone σ' by an integer linear map with nonzero, possibly rational, determinant.

A particularly simple instance of this is to take Σ and pick a ray ρ in it. If we choose a lattice point r on ρ, we can construct a combinatorial root construction

$$\Sigma' \to \Sigma$$

as follows. Abstractly $\Sigma = \Sigma$, and the morphism is uniquely specified by sending every ray except ρ identically to itself, and on ρ sending the generator of ρ to the lattice point r.

One way to visualize this is to consider the case $\Sigma = \mathbb{R}^k_{\geq 0}$. If r is the point $r \cdot e_1$, this can be seen as changing the integral points – in the new Σ', which is topologically just $\mathbb{R}^k_{\geq 0}$, the integral points are k-tuples of natural numbers whose first coordinate is divisible by r.

There is a natural bijection

$$\{\text{Generalized root constructions of } X \text{ along } D\} \leftrightarrow \{\text{Combinatorial root constructions of } \Sigma.\}$$

4.6. **The moduli space of curves.** We now come to the key example to which the theory above applies: the pair $(\overline{\mathcal{M}}_{g,n}, \partial_{g,n})$ consisting of the moduli space of curves together with the divisor parameterizing singular curves. A basic result in the deformation theory of curves (and of logarithmic curves) implies that this divisor is merely normal crossings.

Since the divisor is not simple normal crossings, we need the more sophisticated theory of cone stacks alluded to above. Using this, we associate a cone stack denoted $\mathcal{M}_{g,n}^{\text{trop}}$. The theory of cone stacks requires a lot of formal development, but the basic picture is summarized as follows.

A *pre-stable* (g, n)*-graph* is a connected finite graph $\mathbf{G}$, possibly with loops and multiple edges, equipped with a *marking function*:

$$m : \{1, \ldots, n\} \to V(\mathbf{G}),$$

and a genus function

$$g : V(\mathbf{G}) \to \mathbb{Z}_{\geq 0},$$

subject to the constraint that

$$h_1(\mathbf{G}) + \sum_{v \in V(\mathbf{G})} g(v) = g.$$

Here $h_1(\mathbf{G})$ is the first Betti number of the geometric realization of $\mathbf{G}$. A pre-stable (g,n)-graph is called *stable* if at every vertex v with $g(v)=0$, the sum of the valency at v and the size of $m^{-1}(v)$ is at least 3.

The following exercise should be quite enjoyable.

Exercise 4.6.1. The set of (g,n) stable graphs is finite.

It is traditional to visualize the n markings as being "half edges" or "legs" emanating from the vertex determined by the function m.

The object $\mathcal{M}_{g,n}^{\mathrm{trop}}$ is a cone stack associated to the pair $(\overline{\mathcal{M}}_{g,n}, \partial_{g,n})$. The key points to keep in mind in working with this object are the following.

(i) The collection of stable (g,n)-graphs forms a category. The objects are stable (g,n) graphs and morphisms are given by automorphisms and by (sequences of) edge contractions. We note that if a loop edge based at a vertex v is contracted, the genus at the vertex is increased by 1.

(ii) We can associate, to each (g,n)-stable graph $\mathbf{G}$, an orthant $\mathbb{R}_{\geq 0}^{E(\mathbf{G})}$.

(iii) Each graph contraction/automorphism $\mathbf{G}\to\mathbf{G}'$ determines
$$\mathbb{R}_{\geq 0}^{E(\mathbf{G}')} \to \mathbb{R}_{\geq 0}^{E(\mathbf{G})},$$
obtained by setting the length of all contracted edges to 0.

(iv) The cone stack $\mathcal{M}_{g,n}^{\mathrm{trop}}$ is a colimit – in an appropriate category – of the diagram of cones indexed by the category of stable (g,n)-stable graphs. The colimit in the category of topological spaces can be given the structure of a *generalized cone complex* – a gluing together of finite quotients of cones by subgroups of their automorphism groups. The colimit can also be taken in the category of stacks over cone complexes.

(v) One has to remember that there are two types of "weird cells" in $\mathcal{M}_{g,n}^{\mathrm{trop}}$, and they can both be seen in curves of genus 1. The first is the dumbbell graph depicture on the left in the figure below. There are two edge length, both not a preferred labelling of the edges, so the moduli space looks like "$\mathbb{R}^2_{\geq 0}/\mu_2$" where the group action flips the coordinates. The second is the lollipop graph, which has an automorphism that flips the edge, leading to "$\mathbb{R}_{\geq 0}/\mu_2$" where the group acts trivially.

The object $\mathcal{M}_{g,n}^{\mathrm{trop}}$ is a moduli space in its own right. In the construction, for a fixed graph $\mathbf{G}$ we can view the interiors $\mathbb{R}_{>0}^{E(\mathbf{G})}$ of the cells as parameterizing *edge lengths* for the graph $\mathbf{G}$. Indeed, given a point in this cell, we can metrize $\mathbf{G}$ by geometrically realizing an edge e with an interval of length determined by this point. With this in mind, we make the following:

Definition 4.6.2. A pre-stable (g,n) tropical curve Γ is a pre-stable (g,n) graph together with a metrization of the edge lengths
$$\ell : E(\mathbf{G}) \to \mathbb{R}_{>0}.$$
It is *stable* if the (g,n) graph is stable.

Each point of $\mathcal{M}_{g,n}^{\mathrm{trop}}$ (i.e. of the topological colimit above) can be viewed as (g,n)-stable graph, defined up to automorphism, together with a choice of *length* for each edge – that is, a stable (g,n) tropical curve.

The topological colimit $|\mathcal{M}_{g,n}^{\text{trop}}|$ can therefore be viewed as a coarse moduli space of tropical curves. However the colimit carries more structure: the most important fact is that one can map cones into $\mathcal{M}_{g,n}^{\text{trop}}$.

Exercise 4.6.3. Draw the tropical moduli spaces $\mathcal{M}_{1,1}^{\text{trop}}$ and $\mathcal{M}_{1,2}^{\text{trop}}$. Pay special attention to the automorphisms.

In the definition of a tropical curve, the length function ℓ takes value in $\mathbb{R}_{\geq 0}$. We can, more generally, allow the length to take values in an arbitrary monoid.

Definition 4.6.4. Let σ be an orthant (or more generally, a cone) and let S_σ be the monoid of positive linear functions (or that is, the dual cone). A **family of stable** (g,n) **tropical curves over** σ is a stable (g,n)-graph G with marking and genus function as in Definition **??**, and whose length function takes values in S_σ.

To make the connection to actual "families", observe that a point of σ is a monoid homomorphism $\varphi : S_\sigma \to \mathbb{R}_{\geq 0}$. If this homomorphism is applied to the edge length $\ell(e) \in S_\sigma$, we obtain a positive real length for each edge and thus a tropical curve.

By definition of the cone stack $\mathcal{M}_{g,n}^{\text{trop}}$, the collection of morphisms

$$\sigma \to \mathcal{M}_{g,n}^{\text{trop}}$$

is precisely the groupoid of families of stable (g,n) tropical curves over σ.

Exercise 4.6.5. What happens if stability is dropped from the discussion above? Work out the details and explicitly describe the resulting tropical moduli space $\mathfrak{M}_{0,3}^{\text{trop}}$.

4.7. **Maps from tropical curves.** A topic that we will discuss in detail in later parts of these lectures is the theory of *logarithmic stable maps*. In this theory, there is an interaction between the logarithmic structure (i.e. the normal crossings divisor) of the space $\overline{\mathcal{M}}_{g,n}$ of stable curves and the logarithmic structure on a pair (X, D).

The key combinatorial structure is that of a *tropical map*. Let Γ be a (g,n) prestable tropical curve and let $Sigma$. A continuous function $\Gamma \to \mathbb{R}$ is *piecewise linear* if its restriction to every edge/leg of Γ is linear and has integer slope. It will also be useful to recall here that the group of piecewise linear functions on Σ is generated additively by distinguished functions that have slope 1 on some ray and slope 0 along all others. Let us call these "coordinate functions".

Definition 4.7.1. A *prestable tropical map* or simply *tropical map* is a continous map

$$\Gamma \to \Sigma$$

such that the pullback of every piecewise linear function on Σ is piecewise linear on Γ.

One can similarly consider a family of tropical curves over a base σ, and equip the total space with a map to Σ. And just as there is a moduli space of (i.e. a cone stack parameterizing) abstract tropical curves there is also a moduli space parameterizing tropical maps. We will not develop this theory in full detail, but here are the highlights.

For simplicity, let us assume Σ is equal to $\mathbb{R}_{\geq 0}$; the general case works similarly, working with coordinate functions one at a time.

The type of a tropical map. If we fix a map $\Gamma \to \mathbb{R}_{\geq 0}$ there is a natural set of "discrete" data: in addition to the unerlying (g, n) graph of Γ, we can record (i) the cone in $\mathbb{R}_{\geq 0}$ to which each vertex/edge/ray maps to, and (ii) the slope of the map along each edge/leg. We call these the cone decoration and the slope data respectively.

Moduli with fixed type. If we fix all the discrete data, call it Θ, we can consider the "moduli space" σ_Θ of tropical curves Γ together with a map to $\mathbb{R}_{\geq 0}$ with discrete data Θ. To construct it, we will describe via linear equations in a large orthant (and then quotient by a suitable group).

In more detail: fix a prestable graph $\mathbf{G}$ and the cone and slope data (i) and (ii) above, consider the orthant with one coordinate for each edge of $\mathbf{G}$ and with one coordinate for each vertex of $\mathbf{G}$ whose cone decoration is the full $\mathbb{R}_{\geq 0}$. Let us call the cone σ_{big}. Each point in this cone specifies a position for a vertex and a length for an edge. A map $\Gamma \to \mathbb{R}_{\geq 0}$ with these specifications may not exist, but if it exists it is unique. However, fixing the graph, the cone decorations, and the slope data, one can prove the following.

Proposition 4.7.2. *The set of points in σ_{big} that correspond to continuous piecewise linear maps Γ with data Θ is given by a linear subset, i.e. it is a cone.*

Exercise 4.7.3. Prove the proposition.

We have seen that in the case of tropical curves, if we fix a graph $\mathbf{G}$, the automorphism group of $\mathbf{G}$ acts on the cone $\mathbb{R}_{\geq 0}^{E(\mathbf{G})}$. If we fix the combinatorial type Θ of a tropical map, a subgroup of $\mathrm{Aut}(\mathbf{G})$ will respect the additional cone decorations and slope data. The quotient of the cone above by this group $\mathrm{Aut}(\Theta)$, is the local model for the tropical moduli space of maps.

Completing the construction. One can now range over all types Θ; to each type we can associate a cone. It comes with a natural automorphims group and its faces comes from cones of "smaller" types Θ'. Altogether, these can be arranged into a diagram category.

We call the output of this construction $\mathfrak{M}_{g,n}^{\text{trop}}(\mathbb{R}_{\geq 0})$ and refer to it as the *moduli stack of tropical maps to* $\mathbb{R}_{\geq 0}$.

Exercise 4.7.4. Make the above construction precise (or at least as precise as you made it for $\mathcal{M}_{g,n}^{\text{trop}}$).

Exercise 4.7.5. Generalize the construction above to construct the space of maps to an arbitrary cone complex Σ.

4.8. **Some concluding mysteries.** We have phrased things so far via "correspondences" between operations on the cone complex Σ_X and geometric operations on a space X. However, there is a no map as yet of the form:

$$X \text{``}\to\text{''} \Sigma_X.$$

We will, in due course, learn that one can make sense of such a map, somewhat tautologically, as precisely the data of these bijections above. The cleanest way to do this is via Artin stacks. Once we do, we will see that if $\Sigma' \to \Sigma_X$ is a subdivision, then the corresponding blowup can be *defined* as the fiber product

$$X' := X \times_{\Sigma_X} \Sigma'.$$

We will be able to make sense of more general fiber products of this form. One that is exceptionally valuable is the following. Accepting that we don't quite know what maps of this kind are like, one can consider a the diagram:

$$\begin{array}{ccc} & & \mathfrak{M}_{g,n} \\ & & \downarrow \\ \mathfrak{M}^{\mathrm{trop}}_{g,n}(\Sigma_X) & \longrightarrow & \mathfrak{M}^{\mathrm{trop}}_{g,n}. \end{array}$$

The fiber product is a new type of moduli space – birational but distinct from the stack of prestable curves. We will later relearn this object as the *stack of logarithmic maps to the Artin fan*. This construction alone has been at the center of a lot of progress in our understanding of Gromov–Witten theory and the moduli space of curves REFERENCES.

But there are some hints as to what it might mean. Take, for instance, the case of Σ_X. A point of the fiber product F should certainly give an algebraic curve and a tropical curve equipped with a piecewise linear function. The algebraic and tropical curves should be connected in some way. 'But taking this for granted and recalling what we learned about piecewise linear functions and Cartier divisors, one might guess that the fiber product should parameterize a curve together with a Cartier divisor on it. We will see later on that the the fiber product F is an algebraic stack, and it has a map to

$$\mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m]) = \{(C, \underline{p}, L, s) : (C, \underline{p}) \text{ a nodal pointed curve, } L \text{ a line bundle, } s \text{ a section}\}.$$

The space F is arguably the most important object in logarithmic Gromov–Witten theory.

Exercise 4.8.1. Come back to this discussion after you have learned about logarithmic Gromov–Witten theory and re-understand it in the terms described above.

5. Toric geometry: an example-based guide

In the following series of exercises, we will take a look at toric geometry from a perspective that is closley aligned with our goals for learning logarithmic geometry.

5.1. **Monomials, binomials, and toric varieties.** We start by making a few elementary observations about some familiar spaces. The ring of polynomial functions in k variables is isomorphic to the *monoid ring* $\mathbb{C}[\mathbb{N}^k]$, defined as

$$\mathbb{C}[\mathbb{N}^k] = \left\{ \sum_{u \in \mathbb{N}^k} a_u \chi^u, \quad a_u \in \mathbb{C} \right\}.$$

Similarly, the group ring $\mathbb{C}[\mathbb{Z}^k]$ is the *Laurent* polynomial ring in k variables.

These two rings are, respectively, the ring of functions on affine space $\mathbb{A}^k$ and on the algebraic torus $\mathbb{G}_m^k$. Furthermore, the obvious inclusion

$$\mathbb{C}[\mathbb{N}^k] \subset \mathbb{C}[\mathbb{Z}^k]$$

can be understood *geometrically* as the subring of regular functions on $\mathbb{G}_m^k$ that extend to regular functions on $\mathbb{A}^k$. We can examine what happes to the monomial functions on each side:

$$\mathbb{C}^\star \oplus \mathbb{N}^k \subset \mathbb{C}^\star \oplus \mathbb{Z}^k.$$

The group $\mathbb{C}^\star \oplus \mathbb{Z}^k$ is exactly the group of invertible functions on $\mathbb{G}_m^k$, while $\mathbb{C}^\star \oplus \mathbb{N}^k$ is the group of invertible functions on $\mathbb{G}_m^k$ that extend to regular functions on $\mathbb{A}^k$.

Algebraic varieties are constructed by specifying the ring (or sheaf of rings) polynomial functions on them. Toric varieties are constructed by specifying the *monomial* polynomial functions.

Exercise 5.1.1. Consider the quotient algebra

$$\mathbb{C}[x, y, t]/(xy - t^n).$$

For each n find a finitely generated monoid P_n such that $\mathbb{C}[P_n]$ is isomorphic to the algebra above. By choosing generators, draw the points of P_n inside $\mathbb{Z}^2$.

This is the coordinate ring of the A_{n-1} surface singularity. Observe that the groupification of P_n is $\mathbb{Z}^2$ and the inlcusion

$$P_n \subset \mathbb{Z}^2$$

identifies a distinguished immersion

$$\mathbb{G}_m^2 \subset \operatorname{Spec} \mathbb{C}[P_n] =: \mathcal{A}_{n-1}.$$

This is an example of a cyclic quotient surface singularity. One of the objects we will later see really wants to be a stack.

Two key points to take away from the example above. First, the subset of monomial functions

$$\mathbb{C}^\star \oplus P_n \subset \mathbb{C}[P_n]$$

can again be identified with the monoid of invertible functions on the distinguished open torus that extend to a regular functions on the full space $\mathcal{A}_{n-1}$.

Exercise 5.1.2. Consider the square in the height 1 plane in $\mathbb{R}^3$ with coordinates square with vertices $(0,0,1)$, $(0,1,1)$, $(1,0,1)$, $(1,1,1)$. The ray between each of these vertices and $(0,0,0)$ is the cone over this square. Let $C_{1,1} \subset \mathbb{N}^3$ be the monoid of positive integer points in this cone. Draw it.

Identify the ring $\mathbb{C}[C_{1,1}]$ with the quotient algebra

$$\mathbb{C}[x,y,t,u]/(xy-ut).$$

In particular, this is the coordinate ring of the cone over the Segre surface (i.e. $\mathbb{P}^1 \times \mathbb{P}^1$ in its $(1,1)$ embedding). Once again, observe that the inclusion

$$C_{1,1} \subset \mathbb{Z}^3$$

identifies an immersion

$$\mathbb{G}_m^3 \subset \operatorname{Spec} \mathbb{C}[C_{1,1}]$$

of an algebraic torus of dimension 3.

If one takes seriously the principle that schemes with a notion of monomial function are worth studying, the following construction is natural. Let $I \subset \mathbb{C}[\mathbb{N}^k]$ be a *binomial* ideal – that is, an ideal obtained by setting two monomials equal to each other. The schemes obtained from the quotients, namely $\operatorname{Spec} \mathbb{C}[\mathbb{N}^k]/I$ are very interesting, but are a little too wild to work with in practice.

Exercise 5.1.3. Give examples to show that binomial ideals can be reducible, non-reduced, and non-normal.

One of the many equivalent characterizations of affine toric varieties is that they are (normal) integral binomial schemes.

A *toric monoid* P is the submonoid of $\mathbb{Z}^k$ of lattice points in the positive real span of a finite set of vectors $\{v_1, \ldots, v_m\} \subset \mathbb{Z}^k$.

Exercise 5.1.4. Show that the monoid ring $\mathbb{C}[P]$ is irreducible and its Zariski spectrum contains an algebraic torus as a dense open.

5.2. **Global theory.** We have understood the basic theory of affine toric varieties. There are two basic approaches to the global theory: the intrinsic approach via equivariant geometry and the extrinsic approach via gluing constructions. In fact, it is the latter that is more relevant for us, but let us say a few words about the former.

A *toric variety* is a normal equivariant compactification of an algebraic torus – that is, it is a variety X containing a dense torus $T \subset X$ such that the action of T on itself extends to an action of T on X.

The key classification theorem in the theory of toric varieties states that every such variety is obtained by gluing T-stable affine varieties, which are in turn exactly the objects described in the previous section.

Sumihiro's theorem states that any normal algebraic variety X with a torus action can be covered by torus invariant affine opens.

Exercise 5.2.1. By constructing an example, prove that the normality in Sumihiro's theorem cannot be dropped.

In order to globalize, we need to be more careful with keeping our lattices straight. We will fix a free abelian group of rank k and will identify it as the group of pure (i.e. coefficient 1) monomial functions on an algebraic torus T.

The group of algebraic homomorphisms:

$$\mathbb{G}_m \to T$$

is canonically identified with the dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$. After choosing coordinates, they are given by $t \mapsto (t^{a_1}, \dots, t^{a_k})$, for $a_i \in \mathbb{Z}$.

Exercise 5.2.2. Given a monomial function χ on T and a map $\varphi \in N$ defining $\mathbb{G}_m \to T$ the composite is a an element

$$\chi \circ \varphi : \mathbb{G}_m \to \mathbb{G}_m.$$

Viewing this as a rational function on $\mathbb{P}^1$, show that the valuation at 0 is exactly the natural pairing $\langle v, \varphi \rangle$. (Hint: Choose coordinates and express the composite rational map in terms of the exponents appearing in χ and φ.)

Given a toric monoid $P \subset M$ the *dual cone* is the set of points in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ whose value on P is non-negative. It will be denoted σ_P.

The monoid P can be recovered as the monoid of elements in M that are positive on σ_P, so these are essentially equivalent. To make this point honest, we should define a cone in $N_{\mathbb{R}}$ independently of being the dual cone of a monoid. A *cone in* $N_{\mathbb{R}}$[13] is defined to be the intersection of finitely many half-spaces defined by $\langle u, x \rangle \geq 0$ for some fixed $u \in M$.

The dual cone of a monoid has essentially one advantage over the monoid picture: it is naturally *covariant* for morphisms. This should not be surprising: morphisms *out* of a variety (in this case $\mathbb{G}_m$) are naturally covariant.

The next two exercises work towards gluing. Given a cone σ in $N_{\mathbb{R}}$, a hyperplane in $N_{\mathbb{R}}$ is defined by the orthogonal to a fixed $u \in M$. Such a hyperplane H_u is a *supporting hyperplane* if σ lies on the positive side of H_u.

A *face* of σ is the intersection of σ with a supporting hyperplane.

Exercise 5.2.3. Make sense of the faces of the cone σ in $\mathbb{R}^2$ given by the first quadrant. Specifically, make sure it tells you what you expect. To make really sure, repeat the exercise for the dual cones of the monoids P_n defining the $\mathcal{A}_{n-1}$ surfaces described above.

[13]A strictly convex rational polyhedral cone, to be precise.

We come to the key definition. Fix the lattices M and N. A *fan in* $N_{\mathbb{R}}$ Σ is a collection of cones σ in $N_{\mathbb{R}}$, such that the intersection of any two cones is a face of each.

Let P be a toric monoid with dual cone σ. If τ is a face of σ with supporting hyperplane u, then the dual monoid Q of τ is generated by P and $-u$. In turn, there is an open immersion

$$\operatorname{Spec} \mathbb{C}[Q] \to \operatorname{Spec} \mathbb{C}[P]$$

obtained by localizing at the monomial χ^u.

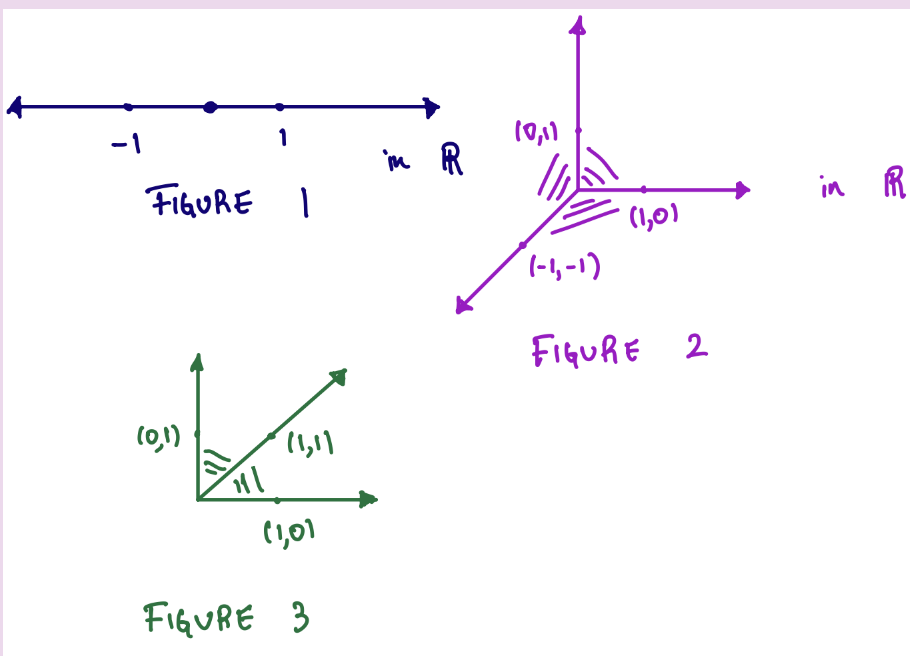
Let Σ be a fan in $N_{\mathbb{R}}$. Its collection of cones are indexed by a partially ordered set – the arrows are given by inclusions of faces. By the discussion above, we obtain a corresponding diagram of affine toric varieties, equipped with open immersions. The *toric variety of a fan* Σ is the colimit

$$X_\Sigma := \varinjlim_{\sigma \in \Sigma} U_\sigma$$

of the affine toric varieties U_σ over σ in Σ.

It follows quickly from this that a toric variety X_Σ is smooth if and only if the primitive elements of the rays of every cone σ are $\mathbb{Z}$-linearly independent.

Exercise 5.2.4. Identify the toric variety of the fan in Figure 1 with $\mathbb{P}^1$. Identify the toric variety of the fan in Figure 2 with $\mathbb{P}^2$. Identify the toric variety of the fan in Figure-3 with the blowup of $\mathbb{A}^2$ at a point.



If X is a toric variety, it turns out to come from a fan Σ_X. The intrinsic description of this fan is as follows. Given a point $v \in N$, there is an associated 1-parameter subgroup

$$\varphi_v(t) : \mathbb{G}_m \to T \subset X.$$

We can ask: (i) does the limit exist as $t \to 0$, and if so (ii) what is it? The answer to (i) picks out a distinguished subset of N, and (ii) produces an equivalence relation on this subset, where two points are equivalent if the corresponding limits are equal. The equivalence relation gives a fan, with cones whose interiors are the set of subgroups that are equivalent.

Exercise 5.2.5. Using the description above, calculate the intrinsic fan of $\mathbb{A}^1$, $\mathbb{A}^2$, $\mathbb{P}^2$, and the blowup of $\mathbb{A}^2$ at the origin. Convince yourself that a product of toric varieties is described by the product of the corresponding fans.

Another consequence of the above characterization of the fan is about properness: a toric variety X_Σ is proper if and only if every lattice point in N lies in some cone of Σ. Note that one direction is obvious! The other direction is true, but we take it on faith.

A fan is called *complete* if every point of N lies in some lattice point.

A well-known trait of toric geometry is that the subject is filled with many interesting geometric bijections. We mention two. First: the T-orbits of a toric variety X_Σ are in natural inclusion-reversing bijection with the cones of Σ. Second: there is a natural inclusion-preserving bijection between the cones of Σ and the affine T-stable open subschemes of X_Σ.

Example 5.2.6. Verify the above bijections for products of projective spaces.

5.3. **Functoriality.** Just as the basic example for a toric variety is an algebraic torus, the basic example of a morphism between toric varieties is an *algebraic homomorphism* of tori:

$$\varphi : T' \to T.$$

These morphisms are precisely given by homomorphisms of lattices

$$M \to M'$$

given by pullback of homomorphisms $T \to \mathbb{G}_m$, or equivalently by

$$N' \to N$$

given by pushforward/post-composition of homomorphisms $\mathbb{G}_m \to T'$.

A *toric morphism* of toric varieties $X' \to X$ is a homomorphism of tori $T' \to T$ that extends to a morphism $X' \to X$ that is compatible with the actions.

Exercise 5.3.1. Let X and X' be toric varieties and let ∂X and $\partial X'$ denote the respective complements of their dense tori. Observe that if $f : X' \to X$ is a toric morphism, then $\varphi^{-1}(\partial X')$ is contained in ∂X. Suppose $f : X' \to X$ is any morphism that satisfies this latter property. Show that it is not necessarily toric. By analyzing your counterexample, relate such morphisms to toric ones.

In combinatorial terms, a morphism $T' \to T$ extends to $X' \to X$ if and only if the induced map

$$N'_\mathbb{R} \to N_\mathbb{R}$$

carries each cone of $\Sigma_{X'}$ into a cone of Σ_X.

Exercise 5.3.2. By using the characterization above, prove that there are no torus equivariant morphisms $\mathbb{P}^n \to \mathbb{P}^m$ unless $n \leq m$. Similarly, show that a complete fan does not admit any torus equivariant morphisms to $\mathbb{A}^1$.

5.4. **Singularity theory and subdivisions.** We have already noted that X_Σ is smooth if and only if for every cone σ in Σ, the rays of σ are a subset of a lattice basis for N.

Exercise 5.4.1. Let $P \subset \mathbb{Z}^n$ be a monoid. Assume that P^{gp} is equal to the full lattice. Prove that there is a unique closed torus orbit p in the associated toric variety, which is a point. Give a basis for the cotangent space $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Let X be a toric variety with fan Σ in $N_\mathbb{R}$. A *modification* is a proper birational morphism $X' \to X$. A *subdivision* of Σ is a new fan Σ' in the same space $N_\mathbb{R}$ equipped with a morphism

$$\Sigma' \to \Sigma$$

that is a *refinement* – the fans share the same support.

Exercise 5.4.2. Prove that a subdivision gives rise to a modification. Harder: In fact, every equivariant modification is a subdivision.

A piece of terminology: a cone σ is called *simplicial* if the generators of its rays are $\mathbb{Q}$-linearly independent.

Exercise 5.4.3. Let σ be a simplicial cone and let X_σ be the associated toric variety. Find a finite and surjective torus equivariant morphism $X' \to X_\sigma$, with smooth domain. We say for ths reason that toric varieties of simplicial cones have "at worst finite quotient singularities".

If you have learned about stacks when you see this: go further and produce a smooth Deligne–Mumford stack $\mathcal{X}_\sigma$ with a proper and birational map $\mathcal{X}_\sigma \to X_\sigma$

An important instance of a projective birational map is *barycentric subdivision*. Let σ be a cone as above, $e_1, \ldots, e_k$ integral generators of its rays. The *barycenter* of σ is the ray $b(\sigma) = \mathbb{R}_{\geq 0} \sum e_i$. The *barycentric subdivision* of a polyhedral complex Δ of dimension m is the minimal subdivision $B(\Delta)$ in which the barycenters of all cones in Δ appear as cones in $B(\Delta)$. It may be obtained by first taking the subdivision centered at the barycenters of m dimensional cones, then the decomposition of the resulting complex centered at the barycenters of the cones of dimension $m-1$ of the *original* complex Δ, etc.

Exercise 5.4.4. Let Σ be any fan. Prove that the fan Σ' obtained by barycentric subdivision of all cones is simplicial. Deduce from this and the previous exercise that every toric variety has an equivariant toric modification by one that has at worst finite quotient singularities

If you have learned about stacks when you see this: go further and deduce that every toric variety admits an equivariant resolution by a smooth toric Deligne–Mumford stack.

Parallel to the singularity analysis of toric varieties, one has a singularity analysis for maps.

Exercise 5.4.5. Give examples of maps of fans $\Sigma' \to \Sigma$ that give rise to dominant morphisms of toric varieties whose fiber dimensions jump. Similarly, give an example of such a morphism where some fibers are non-reduced.

If you have completed the exercise, the following two criteria will come as no surprise. A morphism $\Sigma' \to \Sigma$ is said to be *combinatorially equidimensional* if every cone of Σ' maps surjectively onto a cone of Σ. Such a morphism is said to be *combinatorially reduced* if the inverse image of every lattice point in Σ contains a lattice point of Σ'.

For the following two stated exercises, the real point is to play with enough examples to convince yourself that the statements are plausible.

Exercise 5.4.6. Let $\Sigma' \to \Sigma$ be a morphism of toric varieties. Show that it is dominant if and only if the associated map of vector spaces $N'_{\mathbb{R}} \to N_{\mathbb{R}}$ is surjective. Prove that the associated morphisms $X' \to X$ have equidimensional fibers if and only if $\Sigma' \to \Sigma$ is combinatorially equidimensional. Similarly prove that the fibers of such a morphism are reduced if and only if the map is combinatorially reduced.

The following exercise is half of the "toric weak semistable reduction theorem".

Exercise 5.4.7. Given a morphism of fans $\Sigma' \to \Sigma$, there are subdivisions Δ' and Δ such that the natural induced map

$$\Delta' \to \Delta$$

is combinatorially equidimensional.

One can also make the fibers reduced, but the best way to do this is using toric Deligne–Mumford stacks and root constructions. For a classical approach, you may look up "Kawamata's cyclic covering trick".

5.5. **Cartier divisors.** Recall that on a general scheme X, a Cartier divisor is can be desribed by a collection of principal divisors (so rational functions, well-defined up to scalar) on open sets with the condition that they glue on overlaps (i.e. the choices for the functions can be made to match up to scalar).

In the toric setting, one can make everything torus equivariant: the open sets can be chosen to be T-stable and the rational functions can be chosen to be monomials.

Exercise 5.5.1. Let Σ be a fan in $N_{\mathbb{R}}$. Observing that a character is exactly a linear function on N, argue that a continuous function

$$|\Sigma| \to \mathbb{R}$$

that is linear on each cone of σ gives rise to a Cartier divisor.

It is less obvious, but true, that every Cartier divisor class on a toric variety can be represented by T-invariant Cartier divisor. Part of this includes the statement that the principal divisors are exactly the characters. Together with basic results comparing the Cartier class group with the Picard group, this leads us to the following fact:

$$\mathsf{Pic}(X_\Sigma) = \{\text{Piecewise linear functions on } \Sigma\}/\{\text{Linear functions on } \Sigma\}.$$

Exercise 5.5.2. Verify that this description above correctly computes the Picard group of products of projective spaces. Use this description to compute the integral Picard group of of the $\mathcal{A}_{n-1}$ surface singularity.

5.6. **Toroidal embeddings.** In the lectures, we have come across the notion of an snc pair (X, D). At each point $p \in X$ on such a pair, some number of components of D, say r of them, pass through p. The local parameters cutting out these components give a morphism from an open neighborhood:

$$(f_1, \ldots, f_r) : U_p \to \mathbb{A}^r,$$

carrying p to 0. This is well-defined up to the unit choices for each function. A sufficiently small neighborhood of U_p (in the étale or analytic topology, say) is naturally isomorphic to $\mathbb{A}^{\dim X}$ We can enhance the collection of functions with additional ones until we have a complete set of local parameters. However, these additional parameters do not vanish on D, so more appropriately, we get an isomorphism:

$$U_p \to \mathbb{A}^r \times \mathbb{G}_m^{n-r}.$$

The varieties $\mathbb{A}^r \times \mathbb{G}_m^{n-r}$ are local models for (X, D), and we notice that $\mathbb{A}^r \times \mathbb{G}_m^{n-r}$ are precisely the *smooth affine toric varieties*.

Exercise 5.6.1. Prove the claim above that smooth affine toric varieties are all of the form $\mathbb{A}^r \times \mathbb{G}_m^{n-r}$.

The theory of *toroidal embeddings* generalizes this, by allowing any affine toric variety to play the role of the local model above. Here is the complete definition:

Exercise 5.6.2. A toroidal embedding is a pair (X, D) consisting of a normal variety and a Weil divisor D with open complement U, such that for every point $p \in X$, there exists an open (or étale) neighborhood $\varphi : V_p \to X$ and an étale map

$$\psi_p : V_p \to V_{\sigma(p)}$$

for some toric variety $\sigma(p)$, that is compatible with the respective divisors, i.e.

$$\psi^{-1}T = \varphi^{-1}U$$

where T is the dense torus.

The exercise is to make sense of this definition. In particular convince yourself of the following: (i) given (X, D) and a point $p \in X$, the toric cone $\sigma(p)$ is well-defined, (ii) the interior $X \smallsetminus D$ is automatically smooth, and come up with an example of a toroidal embedding that is neither toric nor snc.

The compatibility of the open set U with the torus T in the local model is crucial – it means that under the natural isomorphism of completed local rings (induced by the étale map), the ideal of functions vanishing on the toric boundary is carried to the ideal of functions vanishing on D.

Exercise 5.6.3. Let X be a (possibly singular) toric variety and let D be a divisor supported on the toric boundary. Prove that not all pairs (X, D) are toroidal embeddings. Classify the divisors D that make (X, D) into a toroidal embedding.

We have seen/asserted above that given a toroidal embedding (X, D) and a point p on X, the local cone $\sigma(p)$ is well-defined.

Exercise 5.6.4. Let p and q be scheme theoretic points of X. Prove that if p specializes to q, then the cone $\sigma(p)$ is a face of the cone $\sigma(q)$.

As a result, given (X, D) we can form a well-defined "cone complex" by taking the limit

$$\Sigma(X, D) := \varinjlim_{p \in X} \sigma(p)$$

where the arrows are given by specialization. The cone complex $\Sigma(X, D)$ bridges the gap between the cone complex associated to an snc pair and the fan associated to a toric variety.

Many of the common properties carry over: piecewise linear functions on $\Sigma(X, D)$ correspond to Cartier divisors, subdivisions correspond to blowups, and there is a certain amount of functoriality. The precise level of functoriality is a subject of ongoing confusion, but the following is true: let $X' \to X$ be a morphism of toroidal embeddings such that the preimage of the divisor on X is contained in the divisor on X'. Assume that the irreducible components of these divisors are normal. Then there is an induced morphism

$$\Sigma(X', D') \to \Sigma(X, D),$$

given by gluing the maps between cones coming from local toric models.

A few warnings are in order: (i) unlike the fan of a toric variety, a pair $\Sigma(X, D)$ is not naturally embedding in a vector space and there is no natural supply of "characters", (ii) the fan of a toroidal embedding $\Sigma(X, D)$ rarely determines X unlike in the toric case, and (iii) the general issue of functoriality of the maps on cone complexes remain something of a mystery.

6. Stacks, part 2

By Dan. This should include whatever notions and constructions are needed for the rest of the series.

Much of this is from the resolution volume.

14→ **6.1. Moduli of curves as an algebraic stack.** [14] The moduli space of curves is a key example where a moduli problem is naturally exhibited as an algebraic stack. One can imagine proving that all the defining criteria of an algebraic stack are satisfied, and this is sometimes done, typically for more difficult situations. But here we have a bag of tricks we can use.

Theorem 6.1.1. *The stack $\overline{\mathcal{M}}_g$ is an algebraic stack.*

Sketch of proof. Every stable curve admits a three-canonical embedding. Consider the Hilbert scheme of 3-canonical nodal curves in projective space. Techniques involving upper-semicontinuity of cohomology allow one to show that this is a locally closed subscheme H_0 of a Hilbert scheme H. The dimension N of the projective space as well as the Hilbert polynomial are determined by Riemann-Roch, hence this is a quasi-projective scheme. There is a natural action of PGL_{N+1} on H_0, and one shows that

$$\overline{\mathcal{M}}_g = [H_0/PGL_{N+1}].$$

♣

Theorem 6.1.2. *The stack $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack.*

Sketch of proof. One can imagine repeating the proof above, and it does work. Alternatively, fix n distinct 1-pointed smooth curves (E_i, q_i) of genus $G \gg g$. There is an equivalence of fibered between $\overline{\mathcal{M}}_{g,n}$ and the closed substack of $\overline{\mathcal{M}}_{g+nG}$ of stable curves consisting of the gluing of a variable stable n-pointed curve $(C, p_1, \ldots, p_n)$ at p_i with each $E_i, q_i)$. ♣

Theorem 6.1.3. *The stack $\overline{\mathfrak{M}}_{g,n}$ of pre-stable curves is an algebraic stack.*

Proof. Again one may struggle with a direct verification... but do not despair! Olsson gave the following simple argument:

Consider the universal family $\overline{\mathfrak{C}}_{g,n} \to \overline{\mathfrak{M}}_{g,n}$ and its fibered product $\overline{\mathfrak{C}}^k := (\overline{\mathfrak{C}}^k_{g,n})_{\overline{\mathfrak{M}}_{g,n}} \to \overline{\mathfrak{M}}_{g,n}$, with k large. It parametrizes k-tuple of points on the fiberes of $\overline{\mathfrak{C}}_{g,n} \to \overline{\mathfrak{M}}_{g,n}$.

There is an open locus where, first, the points are distinct, and do not collide with nodes or the original marked points, and where each component of each prestable curve carries so many of the k points so that it has no automorphism as a marked curve.

If k is large enough this open locus is dense, and, as a category, coincides with an open asubstack $U \subset \overline{\mathcal{M}}_{g,n+k}$ which is actually representable by a quasi-projective scheme, since there are no automorphisms and $\overline{\mathcal{M}}_{g,n+k}$ has p[rojective coarse moduli space!

In other words, we have a smooth morphism $U \to \overline{\mathfrak{M}}_{g,n}$. The fibered product $R = U \times_{\overline{\mathfrak{M}}_{g,n}} U$ is the scheme over $U \times U$ parametrizing isomorphisms between the two universal families of curves, usually constructed as a Hilbert scheme. It follows that $\overline{\mathfrak{M}}_{g,n} = [R \rightrightarrows U]$ is the stack associated to the groupoid given by the action of R on U, so it is algebraic, as needed. ♣

[14](Hulya) I was just looking where the definition of a "stable curve" first appeared in the paper as I want to refer to it, but can not find it – might be good to add somewhere?

6.2. **Stable maps as an algebraic stack.** We are in the business of Gromov–Witten theory, so how about stable maps?

Once again our bag of tricks helps us avoid trouble, in a way which, unfortunately, does not fully extend to orbifold or logarithmic worlds. (The orbifold world is somewhat simoplified using Olsson's bnotion of log twisted curves. The logarithmic world still requires hard and direct arguments. We will ignore things and just assume given that the needed stacks exist.)

Theorem 6.2.1. *Let X be a projective scheme. Then the space $\overline{\mathcal{M}}_{g,n}(X,d)$ of stable maps of degree d in X is an algebraic stack.*

Proof. We consider the forgetful morphism

$$\overline{\mathcal{M}}_{g,n}(X,d) \to \overline{\mathfrak{M}}_{g,n}.$$

Its fiber over a scheme $S \to \overline{\mathfrak{M}}_{g,n}$ is the space of maps $Hom_S(\mathfrak{C}_S, X)$, which is a Hilbert scheme. The following lemma shows that $\overline{\mathcal{M}}_{g,n}(X,d)$ is an algebraic stack. ♣

Lemma 6.2.2 (Give reference)**.** *Assume given a functor $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is an algebraic stack and for every $S \to \mathcal{G}$, with G a scheme, the fibered product $S \times_{\mathcal{G}} \mathcal{F}$ is an algebraic stack. Then $\mathcal{F}$ is an algebraic stack.*

6.3. **Quotients and coarse moduli spaces.** Consider now a scheme X with an action of a group G. A *quotient scheme* $X \to Y = X/G$ is a G-invariant morphism of schemes such that

(1) if $X \to Z$ is another G-invariant morphism of schemes then it factors uniquely as $X \to Y \to Z$, and
(2) For any algebraically closed field K, the map $X(K)/G(K) \to Y(K)$ is bijective.

The first condition shows that a quotient is unique up to unique isomorphism, and the second says that in a sense points of the quotient are orbits of points on X.

Symmetric functions allow you to prove:

Exercise 6.3.1. Let $X = \operatorname{Spec} A$ be an affine scheme and G a finite group. Show that the natural morphism $X \to X/G := \operatorname{Spec} A^G$ is a quotient scheme.

In general quotients do not exist in the category of schemes. If G is finite then quotients exist in the category of algebraic spaces, but this is a cheat: in a sense it is an outcome of the existence of the quotient stack.

Exercise 6.3.2. Let X be a scheme with an action of a group G, and assume a quotient scheme X/G exists. Show that $X \to X/G$ factors uniquely as $X \to [X/G] \to X/G$.

So in a sense X/G is the best schematic approximation (from below) of the stack $[X/G]$.

What can we do in general? Consider a stack $\mathcal{X}$. In analogy to the definition of a quotient, Mumford defined the following:

Definition 6.3.3. A *coarse moduli space* $\mathcal{X} \to X$ is a morphism to a scheme or algebraic space X such that

(1) if $\mathcal{X} \to Z$ is another morphism to a scheme then it factors uniquely as $\mathcal{X} \to X \to Z$, and
(2) For any algebraically closed field K, the map $\mathcal{X}(K) \to X(K)$ is bijective.

This is quite relevant to our volume, especially the following:

Exercise 6.3.4. Let X be a scheme with an action of a group G, and assume a quotient scheme X/G exists. Show that $[X/G] \to X/G$ is a coarse moduli space.

There is a general result. One defines a stack to be *separated* if the same definition as for schemes applies. Keel and Mori [KM97] proved the following fundamental result:

Theorem 6.3.5. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack. Then it admits a separated coarse moduli space $\mathcal{X} \to X$.*

This applies to $\mathcal{M}_g, g \geq 2$ and to $[X/G]$, with G a finite group. It means that any separated Deligne–Mumford stack can be well-approximated (from below) by an algebraic space.

6.4. **Root constructions.** Suppose S is a scheme, and $f \in \gamma(O_S)$ a function. A classical construction is the *n-th order cyclic cover* $S' \to S$ along the divisor $D = V(f)$ given by the equation $y^n = f$. In other words $T = \operatorname{Spec} \mathcal{O}_S[y]/(y^n - f)$.

This is a very useful construction in geometry, but there is something unsatisfying about it: it really does depend on f, not only on $V(f)$: Say you replace f by $f' = uf$, where u is unit. Unless u happens to be an n-th power, the s-schemes $T = \operatorname{Spec} \mathcal{O}_S[y]/(y^n - f)$ and $T' = \operatorname{Spec} \mathcal{O}_S[y]/(y^n - f')$ are simply not isomorphic. This means that one needs to be careful to globalize the construction.

Classically one globalizes the situation by choosing a line bundle L and an isomorphism $L^n \simeq O(D)$, which one rather writes as $\phi : L^{-n} \to \mathcal{I}_D \subset \mathcal{O}_D$. In this case T is the quotient of the algebra $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{L}^{-n}$ by the equations $s^n - f_s$ where s is a local section of L^{-1} and $f_s = \phi(s^n)$ is the corresponding local function vanishing on D.

Exercise 6.4.1. *By choosing a generating section of L, show that* locally *this construction coincides with the scheme T constructed above. What happens if one replaces ϕ by $u\phi$ for a unit u?*

Even putting units aside this still depends on a choice - replacing L by another n-th root of $\mathcal{O}(D)$ changes the cover. Even worse — an n-th root L might not exist globally.

The *Cadman–Vistoli root stack* comes to fix these issues.

Exercise 6.4.2. *(1) Consider the category whose objects over S are line bundles on S and arrows are pullback diagrams. Show that it is equivalent to $\mathcal{B}\mathbb{G}_m$, in particular an Artin stack. Show that the functor sending L to L^n corresponds to the n-th power map $\mathbb{G}_m \to \mathbb{G}_m$.*

(2) *Now consider the category whose objects over S are pairs (L, s) consisting of a line bundle L and a section s. Show that it is equivalent to $\mathcal{A}^1 := [\mathbb{A}^1/\mathbb{G}_m]$ with the natural action. Show that the functor $\mathcal{A}^1 \to \mathcal{A}^1$ sending (L, s) to (L^n, s^n) corresponds to teh n-th power map on $\mathbb{A}^1 \to \mathbb{A}^1$, with the corresponding n-th power map on $\mathbb{G}_m$.*

Exercise 6.4.3. *Consider a scheme V, and a line bundle L on V with section s — for the discussion let us assume giving a Cartier divisor D — and a positive integer n. Let $\sqrt[n]{(V, D)}$ be the category whose objects over S are triples (f, M, ϕ, t) where $f : S \to V$ is a morphism, M an invertible sheaf on S with section t, and $\phi : M^n \to f^*L$ an isomorphism carrying t^n to s. Show that this is an algebraic stack, isomorphic to $V \times_{\mathcal{A}^1} \mathcal{A}^1$, where the map on the right is the n-th power map.*

Exercise 6.4.4. *Suppose D is the vanishing locus of a function g. Let T be the scheme defined at the beginning of the section. Show that (f, M, ϕ, t) is isomorphic to the quotient $[T/\mu_n]$, where μ_n acts on y by multiplication, in particular it does not depend on the choices discussed above..*

6.5. **Coherent sheaves.** This is definitely a topic that deserves thorough attention, but not in a light-touch document such as this. The ideas are natural and mostly extend what you know about varieties smoothly.

As Deligne–Mumford stacks admit étale covers by schemes, they have a well defined élate topology. This in particular means that one can consider sheaves in the étale topology of a Deligne–Mumford stack.

One simple example is the structure sheaf $\mathcal{O}_{\mathcal{X}}$ of a stack $\mathcal{X}$, represented by the structure sheaf on an étale covering $V \to \mathcal{X}$, with trivial gluing data on the overlaps. This manifests again the observation that a stack is just a bunch of rings with rings homomorphisms between them...

One can similarly consider sheaves of $\mathcal{O}_X$-modules, quasicoherent and coherent sheaves of $\mathcal{O}_X$-modules. Ideals, differential forms, and other related notions, such as the spectrum and projective spectrum of a sheaf of algebras, are studied as for schemes.

The situation is not quite as simple for Artin stacks, but this difficulty is better swept under the rug in the present exposition.

6.6. **Stack theoretic $\mathcal{P}roj$.** Let V be a scheme and $\mathcal{A} = \oplus_{m \geq 0} \mathcal{A}_m$ a finitely generated quasi-coherent $\mathcal{O}_V$-algebra, with its natural ideal $\mathcal{A}_+ = \oplus_{m \geq 1} \mathcal{A}_m$. Consider $\operatorname{Spec}_S \mathcal{A}$ with its vertex $V_+ = V(\mathcal{A}_+)$. The grading gives a $\mathbb{G}_m$-action. Define $\mathcal{P}roj_V(\mathcal{A}) := [(\operatorname{Spec}_V(\mathcal{A}) \smallsetminus V_+) \ / \ \mathbb{G}_m]$.

Exercise 6.6.1. Consider the special case where $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra. Show that in this case $\mathcal{P}roj_V(\mathcal{A}) = \mathrm{Proj}_V(\mathcal{A})$, the standard relative projective scheme construction found, for instance, in Hartshorne's book.

6.7. **Weighted projective stacks.** The $\mathcal{P}roj$ construction gives something new already when $V = \operatorname{Spec} k$ is a point. Consider the algebra $\mathcal{A} = k[x1, \ldots, x_n]$, but with the variables x_i placed in degree $w_i > 0$. In other words, the multiplicative group $\mathbb{G}_m$ acts on x_i via $x_i \mapsto t^{w_i} x_i$.

Note that $\operatorname{Spec} \mathcal{A}$ is just affine space, and $V(\mathcal{A}_+)$ is the origin. The quotient *stack* $\mathcal{P}roj_V(\mathcal{A}) := [(\operatorname{Spec}_V(\mathcal{A}) \smallsetminus V_+) \ / \ \mathbb{G}_m]$ is the *weighted projective stack* $\mathcal{P}(w_1, \ldots, w_n)$ of dimension $n-1$. Its coarse moduli space $(\operatorname{Spec}_V(\mathcal{A}) \smallsetminus V_+) \ / \ \mathbb{G}_m$ is the classical weighted projective *space* $\mathbb{P}(w_1, \ldots, w_n)$.

6.8. **Blowups and weighted blowups.** Consider now the situation where $\mathcal{A} = \oplus \mathcal{I}^m$, where $\mathcal{I}$ is a sheaf of ideals. Then $\mathcal{P}roj \mathcal{A} = \operatorname{Proj} \mathcal{A}$ is the blowup of the subscheme $V(\mathcal{I})$ in the classical case. When V and $X = V(\mathcal{I})$ are smooth, namely $\mathcal{I}$ is everywhere generated by a partial local system of parameters $x_1, \ldots, x_k$ on the smooth variety V, we obtain the familiar smooth blowup of X on V.

A somewhat more general *stack theretic* construction is associated with *Rees algebras*. Say the graded pieces of $\mathcal{A} = \oplus \mathcal{I}_m$ are nested ideals sheaves with $\mathcal{I}_m \supset \mathcal{I}_{m+1}$, and multiplication is given by multiplication of ideals, in particular $\mathcal{I}_m \mathcal{I}_n \subset \mathcal{I}_{m+n}$. This is the stack theoretic blowup of the Rees algebra $\mathcal{A}$. Finite generation implies that its coarse moduli space coincides with the blowup of $\mathcal{I}_m$ for some large and divisible m, but the stack has richer structure.

A central object of this book are *smooth weighted blowups*. It is discussed in several sections with different emphases, different levels of generality, and different points of view. We describe here only the local case, where the center is $V(x_1, \ldots, x_k)$, with x_i a partial regular system of parameters, and x_i given weight a_i.

This is obtained by the Rees algebra $\mathcal{A} = \oplus \mathcal{I}_m$ described as follows: one starts with the algebra $\mathcal{B}$ over $\mathcal{O}_V$ where x_i is placed in I_{w_i}. This is already an algebra of ideals, but to make it a Rees algebra one needs to enforce the condition $\mathcal{I}_m \supset \mathcal{I}_{m+1}$, in essence by replacing $\mathcal{B}_m$ by the ideal $\mathcal{I}_m = \sum_{j \geq m} \mathcal{B}_m$.

More natural presentations, involving valuations or extended Rees algebras, are presented in the following chapters.

6.9. **Destackification.** With the exception of the classical method described in [FKP06], there is one point in all the recent work which we have found challenging to explain, and I wish to try to dispell this challenge. The point is that all the new methods start with a *scheme* X and end up with a smooth *stack* $\mathcal{X}'$ resolving it. Evidently algebraic geometers want resolution of singularities to end up with a *scheme*, no matter how useful stacks may be. Have we ended up short of this goal?

Our answer is, and always has been, that the smooth stack $\mathcal{X}'$ always admits a *destackification*, which is an easily understod and computationally feasible task. This task works even in positive characteristics, as long as the stack $\mathcal{X}'$ is *tame*, a condition which is automatic in characteristic 0.

What is destackification?

Definition 6.9.1. Let $\mathcal{X}'$ be a smooth Deligne–Mumford stack. A *destackification* of $\mathcal{X}'$ is a proper birational morphism $\mathcal{X}'' \to \mathcal{X}'$ from another stack, such that the coarse moduli space X'' of $\mathcal{X}''$ is smooth.

The universal property of coarse moduli spaces implies that, if X' is the coarse moduli space of $\mathcal{X}'$, there is a proper birational morphism $X'' \to X'$ induced by $\mathcal{X}'' \to \mathcal{X}'$. In particular $X'' \to X' \to X$ is a resolution of singularities in the classical sense.

One can envision stronger statements. For instance one may wish the stack $\mathcal{X}''$ to also be smooth, and one may wish it to be obtained by a sequence of simple operations like smooth blowups and smooth root constructions. One may further wish $\mathcal{X}'' \to X''$ to be very simple - a sequence of smooth root constructions. All these do hold true, and were developed in different works. For the present discussion all we need is the *existence* of a destackification, and its inherent *simplicity*, in particular it is much simpler than any general algorithm of resolution of singularities.

For the sake of discussion, let us note that all the stacks $\mathcal{X}'$ appearing in our work have at most *finite abelian (and tame) stabilizers*, acting faithfully on tangent spaces. One can say a whole lot in greater generality, but let us stick with this situation for simplicity.

There are several ways to achieve destackification.

6.9.2. *Direct resolution.* Over the years people have devised, again and again, methods for resolving varieties with finite abelian tame quotient singularities, such as the space X'. An early work in this direction is due to Bogomolov [Bog92, Lemma 8.2], which addresses global quotients. Such resolution $X'' \to X'$ is all we need, but in fact it does provide a destackification, with possibly singular $\mathcal{X}''$, if one simply takes $\mathcal{X}'' = X'' \times_{X'} \mathcal{X}'$, whose coarse moduli space is X''.

6.9.3. *Torification and toroidal resolution.* When passing to strict henselizations, the action of the stabilizers of $\mathcal{X}'$ on the tangent spaces can be diagonalized, but this diagonalization is not canonical and cannot be glued to a global structure. However, the works [AdJ97, AKMW02, ATW20b] provide the following result:

Theorem 6.9.4. *The stack $\mathcal{X}'$ admits a canonically defined ideal sheaf $\mathcal{I}_{\mathrm{tor}}$ whose blowing up $\mathcal{X}'_{\mathrm{tor}} \to \mathcal{X}'$, endowed with its exceptional divisor, is toroidal, and the stabilizers on $\mathcal{X}'_{\mathrm{tor}}$ are globally diagonalized with respect to this divisor.*

The ideal $\mathcal{I}_{\mathrm{tor}}$, the so-called *torific ideal*, is obtained locally as the product of ideals of the form $\mathcal{I}_\chi$, where χ runs over the characters of the local stabilizer G_x:

$$\mathcal{I}_\chi \quad = \quad \left(f \in \mathcal{O}^{\mathrm{sh}}_{X,x} \quad : \quad g^*f = \chi(g) \cdot f \quad \forall g \in G_x\right).$$

This implies that the coarse moduli space X'_{tor} is toroidal, and its resolution of singularities $X'' \to X'_{\mathrm{tor}}$, with pullback stack $\mathcal{X}'' = X'' \times_{X'} \mathcal{X}'$, provides a very simple destackification as above.

A similar procedure was devised by Gabber, see [IT14].

6.9.5. *Strong destackification.* The works [Ber17, BR19] provide the strongest destackification result, with all steps being simple operation. It is, however, significantly more costly in computational terms:

Theorem 6.9.6. *There exists a sequence of proper birational maps $\mathcal{X}'' = \mathcal{X}'_n \to \cdots \to \mathcal{X}'_1 \to \mathcal{X}'_0 = \mathcal{X}'$ each of which being either a smooth blowup or a root consstruction along a smooth divisor, and a sequence of root constructions along smooth divisors $\mathcal{X}'' = \mathcal{X}''_n \to \cdots \to \mathcal{X}''_0 = X''$, In particular $\mathcal{X}'' \to \mathcal{X}'$ is a destackification.*

A computer implementation of this algorithm in OSCAR [OSCARdt22] is underway, see [ABB+21].

6.10. **Quotients by groupoids $[R \rightrightarrows V]$.** Kai Behrend gave an elegant description of the stack associated to a groupoid in general. First we note the following: for any scheme X and any étale surjective $V \to X$, one can write $R_V = V \times_X V$ and the two projections give a groupoid $R_V \rightrightarrows V$. If, as suggested above, R_V is to be considered as an equivalence relation on V, then clearly the equivalence classes are just points of X, so we had better define things so that $X = [R_V \rightrightarrows V]$.

Now given a general groupoid $R \rightrightarrows V$, an object over a base scheme B is very much like a principal homogeneous space: it consists of an étale covering $U \to B$, giving rise to $R_U \rightrightarrows U$ as above, together with maps $U \to V$ and $R_U \to R$ making the following diagram (and all its implicit siblings) *cartesian*:

$$\begin{array}{ccc} R_U & \longrightarrow & R \\ \downdownarrows & & \downdownarrows \\ U & \longrightarrow & V \\ \downarrow & & \\ B & & \end{array}$$

There is an important object of $\mathcal{X} = [R \rightrightarrows V]$ with the scheme V as its base: you take $U = R$ above, with the two maps $U \to B$ and $U \to V$ being the source and target maps $R \to V$ respectively. What it does is it gives an *étale covering* $V \to \mathcal{X}$, as required in the definition of an algebraic stack. The existence of such a thing is in fact an axiom required of a fibered category to be a Deligne–Mumford algebraic stack, but since I have not gotten into details you'll need to study this elsewhere. The requirement says in essence that every object should have a universal deformation space.

7. Logarithmic geometry I: divisors and monomials

In our discussion of the moduli space of curves (Section 4.6) we witnessed the key role played by *compactifications*. The compactification

$$\mathcal{M}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}$$

opens up a rich universe of intersection theory, topology, and birational geometry, while also providing a tool to probe the original space $\mathcal{M}_{g,n}$.

However, we are not interested in arbitrary compactifications: following Section 4.1 we focus exclusively on compactifications where the boundary is a normal crossings divisor. This condition is special enough to produce a workable theory, but general enough to ensure such compactifications always exist (this is essentially a consequence of resolution of singularities [Hir64] which we will discuss in Section 15).

Logarithmic geometry is a robust language for keeping track of such compactifications. It achieves this by recording monomials in the functions cutting out the components of the boundary divisor. This is consonant with the function-theoretic perspective which has dominated algebraic geometry since the 1950s. Being a theory of compactifications, logarithmic geometry automatically includes a treatment of *degenerations* as well. We will encounter both perspectives in the sequel.

Disclaimer. This is a user's guide, not a sacred text. We do not develop the theory in its maximal generality, but rather in the generality that we need. Sometimes we are a bit sloppy about assumptions in the pursuit of clarity and its cousin, brevity.

7.1. **Divisorial logarithmic structures: first approximation.** Fix a simple normal crossings pair $(X|D)$ as in Section 4.1 and write the boundary divisor as

$$D = D_1 + \ldots + D_k$$

where each D_i is irreducible. If you find this obscure, think of the pair $(X|\partial X)$ where X is a smooth toric variety, or the pair $(\overline{\mathcal{M}}_{0,n} \,|\, \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n})$. Since X is smooth, each component D_i is a Cartier divisor and is hence cut out by a single function, or more precisely a section of a line bundle:

$$s_i \in H^0(X, \mathcal{O}_X(D_i)).$$

This section s_i is unique up to scaling by units.

Idea. A logarithmic structure on a scheme X is a collection of certain special functions, which we think of as **monomials**.

In our case, we are interested in the functions $s_1^{c_1} \cdots s_k^{c_k}$. We record these as follows. Let $\mathrm{Div}(X)$ denote the set of *generalised Cartier divisors* on X, i.e. pairs consisting of a line bundle and a section:

$$\mathrm{Div}(X) := \{(L, s) : L \text{ is a line bundle on } X, \text{ and } s \in H^0(X, L)\}.$$

We then record our desired monomial functions via the following map:

$$\begin{aligned} \mathbb{N}^k &\to \mathrm{Div}(X) \\ (c_1, \ldots, c_k) &\mapsto (\mathcal{O}_X(c_1 D_1 + \ldots + c_k D_k), s_1^{c_1} \cdots s_k^{c_k}). \end{aligned} \tag{3}$$

This is the prototype of a logarithmic structure. It furnishes X with the chosen monomial functions, with $\mathbb{N}^k$ functioning as the **indexing set**. A morphism of logarithmic structures is required to pull back monomials to monomials.

Monomials are not closed under sums, but are closed under products. Thus the algebraic structure governing monomials is the **monoid**: a set with a single binary operation, satisfying all the axioms of an abelian group except the existence of inverses. Notice that (3) is a map of monoids, and is hence determined by its action on the generators of the free monoid $\mathbb{N}^k$: once we have $s_1, \ldots, s_k$ the monoid operation automatically produces the monomials $s_1^{c_1} \cdots s_k^{c_k}$. The appearance of monoids is the root of the deep interplay between logarithmic and toric geometry, which we explore in Section 8.

7.2. **Divisorial logarithmic structures in full.** The above definition is a valiant first attempt, but it is deficient in several respects. It is inflexible: it cannot be restricted to an open set, or more generally pulled back along a morphism of schemes.

The solution is to replace the global structure with a local one. The indexing monoid $\mathbb{N}^k$ will be replaced by an indexing sheaf $\overline{M}_X$, and the map (3) will be replaced by a map of sheaves.

Step I: Indexing sheaf. We begin by constructing the indexing sheaf $\overline{M}_X$. This is called the **ghost sheaf** or the **characteristic sheaf** of the logarithmic structure. It is a constructible sheaf on X whose stalks are constant on each locally closed stratum of the pair $(X|D)$. To illustrate, choose a point

$$p \in (D_1 \cap D_2) \smallsetminus (D_3 \cup \cdots \cup D_k).$$

We say that p belongs to the locally closed stratum of $(X|D)$ corresponding to the subset $\{1,2\} \subseteq \{1,\ldots,k\}$. We will now define the stalk

$$\overline{M}_X|_p.$$

Remember that this is supposed to index monomials in the s_i. In an open neighbourhood of p, the divisor $D = D_1 + \ldots + D_k$ is indistinguishable from the divisor $D_1 + D_2$. Function-theoretically, the equations $s_3,\ldots,s_k$ are invertible in a neighbourhood of p, and since each s_i is only well-defined up to scaling by units anyway, this means that each of $s_3,\ldots,s_k$ is indistinguishable from the constant function 1. Consequently we should identify

$$s_1^{c_1} s_2^{c_2} s_3^{c_3} \cdots s_k^{c_k} = s_1^{c_1} s_2^{c_2}.$$

In summary: local to p, the only monomials we have are $s_1^{c_1} s_2^{c_2}$. The stalk of the ghost sheaf gives the corresponding **local indexing set**:

$$\overline{M}_X|_p = \mathbb{N}^2.$$

Generalising, let $[k] = \{1,\ldots,k\}$ be the set indexing the divisor components, fix a subset $I \subseteq [k]$, and consider a point in the corresponding locally closed stratum

$$p \in \big(\cap_{i\in I} D_i \big) \smallsetminus \big(\cup_{i \in [k] \smallsetminus I} D_i \big).$$

Then the stalk of the ghost sheaf is given by

$$\overline{M}_X|_p = \mathbb{N}^I$$

and indexes the monomials $\Pi_{i\in I} s_i^{c_i}$. These local stalks glue: if ξ and ξ' are the generic points of locally closed strata, with $\xi' \in \overline{\xi}$, then there is a generisation map

$$\overline{M}_X|_{\xi'} \to \overline{M}_X|_{\xi}.$$

Precisely, ξ and ξ' correspond to subsets $I, I' \subseteq [k]$, and the condition $\xi' \in \overline{\xi}$ is equivalent to $I \subseteq I'$. The generisation map is simply the projection

$$\mathbb{N}^{I'} \to \mathbb{N}^I.$$

This data — stalks on strata, connected by generisation maps — produces what is known as a **constructible sheaf** on X. As you can see, this is a very combinatorial object. For example, its global sections are given by the inverse limit of the diagram of stalks connected by generisation maps:

$$\Gamma(X, \overline{M}_X) = \varprojlim_{p \in X} \overline{M}_X|_p.$$

Being an indexing sheaf for monomial functions, $\overline{M}_X$ is a sheaf of monoids. We have succesfully upgraded our indexing set $\mathbb{N}^k$ to an indexing sheaf $\overline{M}_X$.

Step II: Monomials associated to indices. It remains to upgrade the map (3) associating to each index its actual monomial. This is straightforward: $\mathrm{Div}(X)$ is replaced by $\mathfrak{Div}_X$, the sheaf of generalised Cartier divisors on X. This is defined by

$$\mathfrak{Div}_X(U) = \mathrm{Div}(U)$$

for every open $U \subseteq X$. The scary font is to warn you that this is no ordinary sheaf: it is actually a symmetric monoidal stack over X. This extra layer of theory has to do with the fact that each s_i is only well-defined up to scaling by units, and we need some way of keeping track of different choices. Ultimately this turns out to be important, but we won't worry about it too much in these lectures.

Putting everything together, we have a sheaf $\overline{M}_X$ indexing the monomials, and a morphism recording the monomial associated to each index:

$$\overline{M}_X \to \mathfrak{Div}_X.$$

This generalises, and we finally arrive at our general definition:

Definition 7.2.1. A **prelogarithmic structure** on a scheme X consists of a constructible sheaf of monoids $\overline{M}_X$ on X, and a morphism of symmetric monoidal stacks:

$$\delta \colon \overline{M}_X \to \mathfrak{Div}_X. \tag{4}$$

To align with existing notation, we write M_X for a prelogarithmic structure. At the moment this is simply a wrapper for the data $(\overline{M}_X, \delta)$.

This "definition" is really a theorem, due to Borne–Vistoli [BV12] and anticipated by Olsson [Ols03].[15] The "usual" definition of a prelogarithmic structure looks quite different. We will discuss the equivalence of the two definitions in Section 7.4, where we will also explain what distinguishes a logarithmic structure from a prelogarithmic structure (don't worry about the "pre" for now: the above definition captures the key ideas).

7.3. **Non-divisorial examples.** The key benefit of Definition 7.2.1 is that it pulls back: given a morphism $f \colon Y \to X$ of schemes and a prelogarithmic structure M_X on X, we obtain a prelogarithmic structure $M_Y := f^\star M_X$ on Y as follows. The ghost sheaf is taken to be the pullback:

$$\overline{M}_Y := f^{-1}\overline{M}_X$$

and the map $\overline{M}_Y \to \mathfrak{Div}_Y$ is obtained by composing the map $f^{-1}\overline{M}_X \to f^{-1}\mathfrak{Div}_X$ with the map

$$\begin{aligned} f^{-1}\mathfrak{Div}_X &\to \mathfrak{Div}_Y \\ (L, s) &\mapsto (f^\star L, f^\star s). \end{aligned}$$

This is a bit abstract, and is best understood via examples. We now present two. Both are non-divisorial, and hence fall outside the scope of the previous section. They share a common theme: both are obtained from a divisorial prelogarithmic structure by pullback (and as a result encode information concerning a deformation of the underlying scheme). In a sense which will be made precise in Section 8.2.4 , all sensible prelogarithmic structures can be obtained from a divisorial prelogarithmic structure by pullback.

[15] This was in turn inspired by earlier work of Deligne and Faltings [Fal90], see [Kat89, Complement 1].

7.3.1. *Logarithmic points.* Consider $X = \mathbb{A}^1_t$ equipped with the prelogarithmic structure associated to the divisor $D = (t = 0)$. This means that ghost sheaf $\overline{M}_{\mathbb{A}^1}$ has stalk $\mathbb{N}$ at the origin and 0 elsewhere, and the map $\overline{M}_{\mathbb{A}^1} \to \mathfrak{Div}_{\mathbb{A}^1}$ is given by

$$\begin{aligned} \mathbb{N} &\to \operatorname{Div}(U) \\ 1 &\mapsto (\mathcal{O}_U, t) \end{aligned}$$

for every open set $U \subseteq \mathbb{A}^1_t$ containing the origin. Consider now the inclusion of the origin

$$\iota\colon \operatorname{Spec}\Bbbk \hookrightarrow \mathbb{A}^1_t.$$

We pull back the prelogarithmic structure from $\mathbb{A}^1_t$ to $\operatorname{Spec}\Bbbk$. The ghost sheaf $\overline{M}_{\operatorname{Spec}\Bbbk}$ has a single stalk, namely $\mathbb{N}$. Since the function t on $\mathbb{A}^1_t$ restricts to 0 on $\operatorname{Spec}\Bbbk$, the map $\delta\colon \overline{M}_{\operatorname{Spec}\Bbbk} \to \mathfrak{Div}_{\operatorname{Spec}\Bbbk}$ is defined on the unique nonempty open subset of $\operatorname{Spec}\Bbbk$ by:

$$\begin{aligned} \delta\colon \mathbb{N} &\to \operatorname{Div}(\operatorname{Spec}\Bbbk) \\ n &\mapsto \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \end{aligned}$$

The scheme $\operatorname{Spec}\Bbbk$ equipped with this prelogarithmic structure is known as the **standard logarithmic point**. We denote it $(\operatorname{Spec}\Bbbk, \mathbb{N})$. Conceptually, it is the scheme $\operatorname{Spec}\Bbbk$ equipped with the monomial function t. This function happens to vanish identically on $\operatorname{Spec}\Bbbk$, but this is unimportant: the logarithmic structure still treats it as a monomial. We see that for a logarithmic scheme there can actually be *more monomials than functions*. Notice in particular that this logarithmic structure does not arise from any divisor on $\operatorname{Spec}\Bbbk$. Rather, the logarithmic structure is "remembering" the fact that $\operatorname{Spec}\Bbbk$ once sat inside the larger scheme $\mathbb{A}^1_t$.

This generalises: we can embed $\operatorname{Spec}\Bbbk$ as the origin in $\mathbb{A}^k$ and pull back. The resulting prelogarithmic structure can be defined intrinsically on $\operatorname{Spec}\Bbbk$ by taking $\overline{M}_{\operatorname{Spec}\Bbbk} = \mathbb{N}^k$ and taking δ to be the map

$$\begin{aligned} \delta\colon \mathbb{N}^k &\to \operatorname{Div}(\operatorname{Spec}\Bbbk) \\ m &\mapsto \begin{cases} 1 & \text{if } m = (0, \ldots, 0), \\ 0 & \text{if } m \neq (0, \ldots, 0). \end{cases} \end{aligned}$$

Again, the prelogarithmic structure is remembering the fact that $\operatorname{Spec}\Bbbk$ once sat inside a larger scheme. We denote this prelogarithmic scheme $(\operatorname{Spec}\Bbbk, \mathbb{N}^k)$.

7.3.2. *Logarithmic curves.* Now consider a curve consisting of two irreducible components meeting at a node:

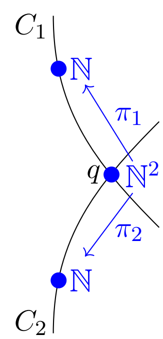
$$C = C_1 \cup_q C_2.$$

We can smooth out the node in a one-parameter family, producing a smooth surface S and a flat morphism

$$\begin{array}{c} S \\ \big\downarrow{\scriptstyle \pi} \\ \operatorname{Spec}\Bbbk[\![t]\!] \end{array}$$

whose general fibre is a smooth curve and whose central fibre is identified with C. The central fibre $\pi^\star(0) = C_1 + C_2$ is a simple normal crossings divisor in S and we equip S with the divisorial prelogarithmic structure. We then pull this back along the inclusion $C \hookrightarrow S$ of the central fibre. This produces a prelogarithmic structure on the curve C whose ghost sheaf is illustrated below:

(5)



The stalks are interpreted as follows:

- At the node q the stalk of the ghost sheaf is $\mathbb{N}^2$. This indexes monomials in the functions z_1, z_2 cutting out the irreducible components C_1, C_2. Pure powers $z_1^{c_1}$ and $z_2^{c_2}$ define interesting functions, but any multiple of the product $z_1 z_2$ will vanish identically. Nevertheless, the logarithmic structure encodes *all* these monomials, vanishing and nonvanishing.
- Away from the node, the stalk of the ghost sheaf is $\mathbb{N}$. This indexes powers of the function z_i cutting out the irreducible component C_i on which we find ourself. This function vanishes identically on the given irreducible component, similar to what we saw for logarithmic points above.

As with the logarithmic points above, we think of the prelogarithmic structure here as remembering the fact that C once sat inside a large scheme, namely S. In this case, it is simply remembering the fact that C came with a smoothing.

In this specific example, the prelogarithmic structure does not actually carry any extra information: there is a unique way to smooth out a curve with a single node in such a way that the total space S is smooth. There are two generalisations we can consider, both leading to prelogarithmic structures carrying interesting information:

(1) We could allow the total space S to be singular (see Section 7.6). This amounts to allowing different speeds of node smoothing. This speed will be recorded in the logarithmic structure.

(2) We could start with a more complicated curve. If C contains multiple nodes, then in a one-parameter family we get to decide the relative speeds of the node smoothings. Again these speeds will be recorded in the logarithmic structure.

These two extensions concern two orthogonal notions of speed: we refer to them as the **absolute exponential speed** and the **relative multiplicative speed**, respectively. We will explore both of these: the former in Section 7.6.4 and the latter in Exercise 8.8.6.

7.4. **Units and logarithmic structures.** Let us return to Definition 7.2.1 of a prelogarithmic structure. We already noted that this differs from the "usual" definition, see e.g. [Kat89, Ogu18]. We now explain the connection.

Definition 7.4.1 (The "usual" definition)**.** A **prelogarithmic structure** on a scheme X consists of a sheaf of monoids M_X on X and a morphism of monoid sheaves

$$\alpha\colon M_X \to \mathcal{O}_X$$

where $\mathcal{O}_X$ is viewed as a sheaf of monoids under multiplication. A prelogarithmic structure is a **logarithmic structure** if and only the restriction of α induces an isomorphism:

$$\alpha^{-1}(\mathcal{O}_X^\star) \xrightarrow{\cong} \mathcal{O}_X^\star. \tag{6}$$

Unlike $\overline{M}_X$, the sheaf M_X is not constructible, and is rarely even finitely-generated. The relationship between M_X and $\overline{M}_X$ is encoded in a short exact sequence. Given a logarithmic structure, the isomorphism (6) gives an inclusion $\mathcal{O}_X^\star \hookrightarrow M_X$ and we obtain a short exact sequence of monoid sheaves:

$$1 \to \mathcal{O}_X^\star \to M_X \to \overline{M}_X \to 0.$$

In short:

- M_X indexes all monomials, including unit factors.
- $\overline{M}_X$ indexes monomials modulo units.

The condition for a prelogarithmic structure to be a logarithmic structure implies that the units in M_X are precisely a copy of $\mathcal{O}_X^\star$:

$$M_X^\star = \alpha^{-1}\mathcal{O}_X^\star \cong \mathcal{O}_X^\star.$$

This means that nontrivial elements of $\overline{M}_X$ should not index units. This leads to the following:

Definition 7.4.2. Let $M_X = (\overline{M}_X, \delta)$ be a prelogarithmic structure in the sense of Definition 7.2.1. This prelogarithmic structure is a **logarithmic structure** if and only if

$$\delta(m) \cong (\mathcal{O}_U, 1) \Rightarrow m = 0$$

for every open set $U \subseteq X$ and section $m \in \overline{M}_X(U)$.

All prelogarithmic structures we have considered so far have in fact been logarithmic structures. Every prelogarithmic structure induces a logarithmic structure which satisfies a universal property. Using Definitions 7.2.1 and 7.4.2 this is very easy: simply quotient $\overline{M}_X$ by the kernel of δ. For the construction using Definition 7.4.1, see [Kat89, Section 1.3].

7.5. **Morphisms.** Since logarithmic structures encode monomials, it stands to reason that a morphism of logarithmic schemes should pull back monomials to monomials. The definition consists of two steps: we first encode how the indices pull back, and then encode an identification between the associated monomial functions.

Definition 7.5.1. A **morphism** of logarithmic schemes $f\colon (X, M_X) \to (Y, M_Y)$ consists of a morphism of schemes $f\colon X \to Y$, a morphism of sheaves

$$f^\flat\colon f^{-1}\overline{M}_Y \to \overline{M}_X \tag{7}$$

and a system of isomorphisms in $\mathfrak{Div}_X$

$$f^\star\delta(m) \cong \delta(f^\flat m) \tag{8}$$

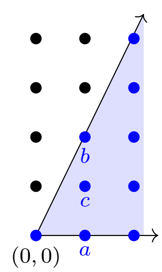
for all sections $m \in \overline{M}_X$, compatible with the monoid structure. In fancy language, the data of (7) and (8) consist of the data of a 2-commuting diagram of monoidal stacks:

$$\begin{array}{ccc} f^{-1}\overline{M}_Y & \xrightarrow{f^\flat} & \overline{M}_X \\ \downarrow{\scriptstyle \delta} & \Rightarrow & \downarrow{\scriptstyle \delta} \\ f^{-1}\mathfrak{Div}_Y & \xrightarrow{f^\star} & \mathfrak{Div}_X. \end{array}$$

Note that the isomorphisms (8) are *part of the data*: sometimes there is a unique choice, but sometimes there is continuous moduli of choices. We will encounter this phenomenon when we explore morphisms of logarithmic schemes, in Exercises 8.8.1 and 8.8.2.

7.6. **Singular monoids.** We have introduced monoids and explained how they are well-adapted to studying monomials. However, we haven't used monoids to their full potential: so far we have only considered free monoids $\mathbb{N}^k$. This'd be a bit like working in algebraic geometry and only ever considering smooth varieties (in turn, this'd be a bit like visiting the Philippines and spending all your time in the resort; you'd enjoy a well-deserved break, but you'd be missing out on so much).

7.6.1. *A singular monoid.* Consider the subset $Q \subseteq \mathbb{Z}^2$ consisting of the lattice points contained in the following convex cone:



Convexity implies that this is closed under addition, and hence forms a monoid. As a monoid, it is generated by $a = (1,0), b = (1,2), c = (1,1)$. These satisfy the single relation $a + b = 2c$. This establishes an isomorphism:

$$Q \cong \mathbb{N}^3_{abc}/(a + b = 2c).$$

One way to see that Q is not isomorphic to a free monoid is to consider the monoid ring $\Bbbk[Q]$. This exponentiates[16] the linear relation amongst generators, giving

$$\Bbbk[Q] \cong \Bbbk[x, y, z]/(xy = z^2).$$

The corresponding variety has a singularity at $(0,0,0)$, and hence $\Bbbk[Q]$ is not isomorphic to the monoid ring $\Bbbk[\mathbb{N}^k] \cong \Bbbk[x_1, \ldots, x_k]$ arising from a free monoid. We think of Q as a **singular monoid**.

[16]We see here a possible origin of the term "logarithmic geometry." We pass from monoids to rings by exponentiating the generators and relations, and conversely given a ring defined using binomial relations, we pass to the associated monoid by taking the logarithm of the generators and relations.

7.6.2. *A singular logarithmic scheme.* We have constructed a singular monoid Q. What is this good for? Remember that we wish to use our monoids to index monomials. Consider the variety:

$$W = \operatorname{Spec} \Bbbk[Q] = (xy = z^2) \subseteq \mathbb{A}^3_{xyz}.$$

The variety W contains a bounary divisor $(z = 0)$ which decomposes into two irreducible components:

$$(z = 0) = (z = x = 0) \cup (z = y = 0) \subseteq W.$$

We wish to equip W with the divisoral logarithmic structure corresponding to the divisor $(z = 0)$. Since W is singular, this falls outside the scope of the simple normal crossings divisorial logarithmic structures constructed in Section 7.2.

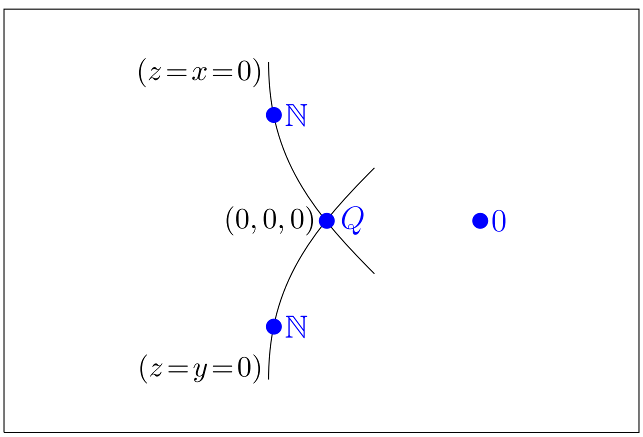
In this more general setting, the monomials we wish to index are precisely the functions which vanish away from the divisor $(z = 0)$. Of course we have the function z itself, but we also have the functions x and y, since

$$z \neq 0 \Rightarrow x \neq 0 \text{ and } y \neq 0.$$

Therefore the logarithmic structure encodes precisely the monomials in the functions x, y, z. Crucially, however, these functions have a dependency, namely $xy = z^2$ (this did not occur for simple normal crossings divisors). Instead of the indexing monoid $\mathbb{N}^3$, we thus obtain the indexing monoid

$$\mathbb{N}^3_{abc}/(a + b = 2c) \cong Q.$$

This gives the stalk of the ghost sheaf at the singular point $(0, 0, 0)$. The other stalks are illustrated in the following diagram:



$W = (xy = z^2) \subseteq \mathbb{A}^3_{xyz}$

Exercise 7.6.3. Describe the two generisation maps $Q \to \mathbb{N}$ for the ghost sheaf above.

More directly, and closer in spirit to Definition 7.4.1, the logarithmic structure on W is given by specifying a submonoid of the coordinate ring, namely the image of the map

$$Q \to \Bbbk[Q].$$

Although W is singular, when it is equipped with this logarithmic structure it becomes *logarithmically smooth*. This is an instance of the phenomenon of hidden smoothness in logarithmic geometry (see Section 14.1).

7.6.4. *Logarithmic curves revisited.* In the previous section we constructed a singular monoid Q and studied the associated variety $W = \operatorname{Spec} \Bbbk[Q]$. Here is an important situation in which Q arises naturally.

Consider again the nodal curve $C = C_1 \cup_q C_2$ from Section 7.3.2. As before we will take a one-parameter smoothing of C:

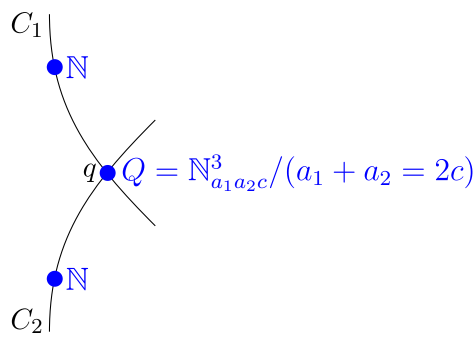
$$\begin{array}{c} S \\ \downarrow \\ \operatorname{Spec} \Bbbk[\![t]\!]. \end{array}$$

However, unlike before, we will not assume that the total space S is smooth. Rather we will assume that, local to the node $q \in C$, S is defined by the equation

$$z_1 z_2 = t^2.$$

Therefore, locally, S is isomorphic to the variety $W = \operatorname{Spec} \Bbbk[Q]$ considered in the previous section. Here t is the equation of the central fibre, while z_1, z_2 are the equations of the components C_1, C_2 inside the central fibre C. Compared to the previous model ($z_1 z_2 = t$), we think of the new model ($z_1 z_2 = t^2$) as smoothing the node *slower*: when $t^2 = 0$ the old model is already smooth, whereas the new model doesn't become smooth until $t^3 = 0$, which we think of as requiring that t is further from 0.

As in Section 7.3.2, we equip S with the divisorial logarithmic structure corresponding to the central fibre and then pull it back to the central fibre. The logarithmic structure on S is essentially described in the previous section. When we pull back to the central fibre C, we obtain a logarithmic structure whose ghost sheaf is:



Thus we have two different logarithmic structures on the same underlying curve C. They can be distinguished already at the level of ghost sheaves. They both encode smoothings of C, but differ because these smoothings have different (exponential) speeds.

In Section 7.3.2 we also had monomials in z_1, z_2, t. However the relation $t = z_1 z_2$ allowed us to eliminate a variable and express all our monomials as monomials in z_1, z_2. This is how we ended up with free monoids. In the current setting we cannot eliminate any variables.

A more complicated example is explored in Exercise 8.8.6.

7.7. **History and references.** Logarithmic structures were introduced in unpublished work of Fontaine and Illusie, motivated by arithmetic geometry. The theory was first written down and developed by K. Kato [Kat89]. F. Kato studied logarithmic deformation and the important case of logarithmic curves [Kat00]. The subject was revolutionised by the work of Olsson connecting logarithmic structures to algebraic stacks [Ols03]. The unusual definition of logarithmic structures given above, and its equivalence with the usual definition, appears in work of Borne and Vistoli [BV12]. For modern references, see [ACG+10] or [Ogu18].

8. Logarithmic geometry II: toric models and Artin fans

8.1. **Toric varieties.** The treatment of the monoid Q in Section 7.6.1 may have felt familiar: we have seen very similar constructions in our treatment of toric varieties. In fact, there is a bijective correspondence:

$$\{\text{toric monoids}\} \longleftrightarrow \{\text{normal affine toric varieties}\}$$
$$Q \longleftrightarrow \operatorname{Spec} \Bbbk[Q].$$

A monoid Q is **toric** if it appears as

$$Q = \sigma^\vee \cap M$$

where M is a lattice and $\sigma \subseteq N_\mathbb{R}$ is a convex rational polyhedral cone. A monoid is toric if and only if it is finitely-generated, torsion-free, integral, and saturated; for a list of terminology see [Che14b, Appendix A]. Given a toric monoid $Q = \sigma^\vee \cap M$ we write

$$U_\sigma = \operatorname{Spec} \Bbbk[Q]$$

for the corresponding affine toric variety. This variety carries a natural collection of monomial functions, given by the image of the map:

$$Q \to \Bbbk[Q].$$

Thus every affine toric variety U_σ carries a natural logarithmic structure. This is precisely the divisorial logarithmic structure corresponding to the boundary $\partial U_\sigma \subseteq U_\sigma$ given by the complement of the dense torus. In fact, $\operatorname{Spec} \Bbbk[Q]$ is the *universal* scheme which carries monomial functions indexed by Q. Because of this, affine toric varieties form the fundamental building blocks for all logarithmic schemes. This is expressed through the notion of toric models, which we now turn to.

8.2. **Toric models.** Given a logarithmic scheme (X, M_X) and a point $p \in X$ we can extract a monoid, namely the stalk of the ghost sheaf:

$$Q = \overline{M}_X|_p.$$

If we insist that all stalks are toric monoids, then Q is dual to a cone σ and produces an affine toric variety:

$$U_\sigma = \operatorname{Spec} \Bbbk[Q] = \operatorname{Spec} \Bbbk[\overline{M}_X|_p].$$

Thus, we have a cone and a corresponding affine toric variety attached to every point $p \in X$. We now explore these in more detail. Some of the following will be familiar from Sections 4 and 5.

8.2.1. *Simple normal crossings pairs.* We begin in a familiar setting. Consider, as in Section 7.2, a simple normal crossings pair

$$(X|D = D_1 + \ldots + D_k).$$

Fix a subset $I \subseteq [k]$ and a point p in the corresponding locally-closed stratum:

$$p \in (\cap_{i\in I} D_i) \setminus \left(\cup_{i\in[k]\setminus I} D_i\right).$$

Let $n = \dim X$. Since the pair $(X|D)$ is simple normal crossings, there is an open neighbourhood $V \subseteq X$ of p and an isomorphism

$$V \cong \mathbb{A}^n$$

which sends p to the origin and identifies the divisor components $\{D_i : i \in I\}$ with the first $|I|$ coordinate hyperplanes. The divisorial logarithmic structure on V is pulled back from the divisorial logarithmic structure on $\mathbb{A}^n$ corresponding to the first $|I|$ coordinate hyperplanes. Unless $|I| = n$ this is not quite the toric boundary, but this is easily remedied. We replace V by a slightly smaller open set and obtain an isomorphism

$$V \cong \mathbb{A}^{|I|} \times \mathbb{G}_{\mathrm{m}}^{n-|I|}$$

which sends p to the point $(0,\ldots,0,1,\ldots,1)$. Again this identifies the divisor D with the first $|I|$ coordinate hyperplanes, but now this is the entire toric boundary! To summarise, X is covered by open sets V equipped with isomorphisms

$$V \cong U_\sigma \tag{9}$$

which identify D with the toric boundary ∂U_σ. Here σ is a rational polyhedral cone and U_σ is the associated affine toric variety. The isomorphism (9) is called a **local toric model**.

8.2.2. *Toroidal embeddings.* In the above the cones σ are all smooth, but there is no reason to impose this. This leads us to the following definition, which dates back to the earliest days of toric geometry but has been somewhat forgotten over the years.

Definition 8.2.3 ([KKMSD73])**.** A **toroidal embedding** is a pair $(X|D)$ such that, locally on X there exists an isomorphism to an affine toric variety

$$V \cong U_\sigma$$

which identifies D with the toric boundary ∂U_σ. In other words, a toroidal embedding is a pair $(X|D)$ produced by patching together local toric models.

8.2.4. *Logarithmic schemes.* In the previous definition, requiring that the isomorphism identifies the boundary divisors is equivalent to requiring that it identifies the associated divisorial logarithmic structures. Viewing a general logarithmic structure as the pullback of a divisorial one, we are led to the following definition:

Definition 8.2.5. An **fs logarithmic scheme**[17] (X, M_X) is a logarithmic scheme such that, locally on X, there exists a morphism to an affine toric variety

$$V \to U_\sigma \tag{10}$$

[17] "Fs" stands for "fine and saturated" which itself stands for "coherent, integral and saturated". "Coherent" is the condition that finitely-generated local models exist, whilst "integral and saturated" is a condition on the monoids giving those models, which guarantees that they are toric.

such that the logarithmic structure $M_X|_V$ is the pullback of the logarithmic structure on U_σ corresponding to ∂U_σ. The morphism (10) is called a **local toric model**. Note that it is not necessarily an isomorphism.

How do we find the local toric model U_σ around a point $p \in X$? For simple normal crossings pairs, the corresponding monoid is given by $\mathbb{N}^I$ which is the stalk of the ghost sheaf. More generally, given any logarithmic scheme we consider:

$$Q = \overline{M}_X|_p.$$

The local toric model near $p \in X$ is then given by the affine toric variety $\operatorname{Spec} \Bbbk[Q]$. The local toric model

$$V \to \operatorname{Spec} \Bbbk[Q] \tag{11}$$

is obtained from the generalised Cartier divisors associated to the elements of Q, by restricting to an open set on which they trivialise. The morphism (11) is referred to as an **atomic chart**, see [MW22, Proposition 2.2.2.5].

8.3. **Artin cones.** Fix an fs logarithmic scheme (X, M_X). The local toric models for (X, M_X) are precisely that: local. There is no sensible way to glue the toric varieties U_σ into a toric variety Z and obtain a global morphism $X \to Z$. This is because each local toric model is not unique: given a local toric model

$$V \to U_\sigma$$

and any element of the dense torus T_σ we can use the action $T_\sigma \curvearrowright U_\sigma$ to obtain another, distinct model. Artin cones provide a formalism for producing unique local models, and mediate the close relationship between logarithmic geometry and the theory of stacks.

Definition 8.3.1. Given a cone σ the associated **Artin cone** is the stack quotient

$$\mathcal{A}_\sigma := [U_\sigma / T_\sigma]$$

where U_σ is the toric variety associated to σ and $T_\sigma \curvearrowright U_\sigma$ is the dense torus.

The stack quotient was defined in Section 2.4.11, but don't worry too much about the formal definition right now; we will demystify it shortly. Consider instead what happens when we postcompose the local toric model with the quotient morphism:

$$V \to U_\sigma \to \mathcal{A}_\sigma.$$

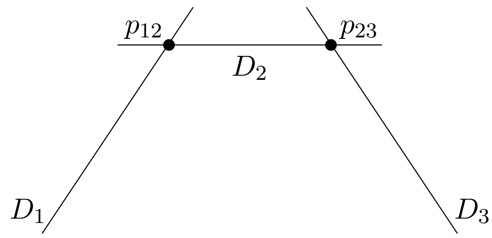
However the quotient $\mathcal{A}_\sigma$ is defined, any sensible definition must identify all the previously distinct models, giving rise to a *unique* morphism

$$V \to \mathcal{A}_\sigma. \tag{12}$$

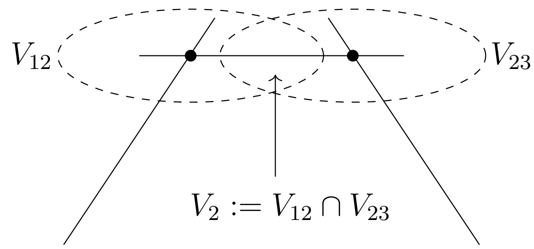
This is not technically a toric model since $\mathcal{A}_\sigma$ is not a toric variety. However it clearly exhibits toric aspects: $\mathcal{A}_\sigma$ is constructed directly from the cone σ. Moreover $\mathcal{A}_\sigma$ carries a logarithmic structure induced by the boundary $\partial \mathcal{A}_\sigma = [\partial U_\sigma / T_\sigma]$, and pulling back this logarithmic structure along (12) produces the logarithmic structure $M_X|_V$.

Example 8.3.2. Consider a simple normal crossings divisor of the following form:

(13)



We choose open sets V_{12}, V_{23} around the closed points p_{12}, p_{23}, such that V_{ij} only intersects the locally closed strata containing p_{ij}. Then $V_2 := V_{12} \cap V_{23}$ is an open set around an interior point of D_2:



Examining the local models, we obtain *unique* local maps to Artin cones:

$$V_{12} \to [\mathbb{A}^2/\mathbb{G}_{\mathrm{m}}^2],$$
$$V_{23} \to [\mathbb{A}^2/\mathbb{G}_{\mathrm{m}}^2],$$
$$V_2 \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}].$$

We have actually already encountered these maps, in disguise. Recall that the morphism $\overline{M}_X \to \mathfrak{Div}_X$ in Section 7.2 encoded a collection of generalised Cartier divisors on X. On the other hand, a map

$$X \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$$

is precisely the data of a generalised Cartier divisor (L, s) on X. This follows formally from the definition of the stack quotient, but it is also easy to justify intuitively. A section of a line bundle is equivalent to a collection of local functions which differ by units on overlaps (these units define the transition functions for the line bundle). Since a function is the same thing as a morphism to $\mathbb{A}^1$ we see see that a collection of local functions differing by units on overlaps (and well-defined up to global multiplication by a unit) is precisely the same thing as a global map $X \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$.

Thus, the above maps $V_{ij} \to [\mathbb{A}^2/\mathbb{G}_{\mathrm{m}}^2] = [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]^2$ precisely encode the monomials in the equations of the local divisor branches, as discussed in Section 7.2. We thus observe an equivalence of data

$$\{\text{toric models}\} \longleftrightarrow \{\text{monomial functions}\}$$

which unites Sections 7 and 8. Philosophically this equivalence makes sense: because toric varieties are the universal schemes equipped with monomial functions, giving a toric model is equivalent to giving a collection of monomial functions.

In summary, we have modified the notion of "toric model" to mean a map to an Artin cone, instead of a map to an affine toric variety. Doing so produces a *unique* toric model around each point. Uniqueness is the key ingredient which we now use to glue the local models $\mathcal{A}_\sigma$ into a global model. To achieve this, we first need to understand how the local cones σ vary as we move around X. This is captured in the notion of tropicalisation.

8.4. **Tropicalisation.** We saw the divisorial case of this construction in Sections 4 and 5.

8.4.1. *Construction.* Recall from Section 8.2.4 that the cones σ defining the local models $\mathcal{A}_\sigma$ are dual to the stalks of the ghost sheaf. Given generic points ξ, ξ' of locally closed strata in X, with $\xi' \in \overline{\xi}$, there is a generisation map:

$$\overline{M}_X|_{\xi'} \to \overline{M}_X|_{\xi}.$$

By assumption the stalks are toric monoids, hence the above map dualises to a map of cones:

$$\sigma' \leftarrow \sigma.$$

A key fact, which follows from the definition of fs logarithmic scheme (Definition 8.2.5), is that $\sigma \to \sigma'$ is always a *face inclusion*.

Exercise 8.4.2. Prove that $\sigma \to \sigma'$ is a face inclusion.

Ranging over generic points of locally closed strata in X, we obtain a diagram of cones connected by face inclusions. We refer to this as the **tropicalisation** of the logarithmic scheme (X, M_X) and denote it:

$$\mathrm{Trop}(X, M_X) \qquad \text{or} \qquad \Sigma(X, M_X).$$

8.4.3. *Examples and properties.* The tropicalisation is an abstract cone complex: it resembles the fan of a toric variety, however it does not come with a preferred embedding into a vector space. Two simple cases are already familiar:

(1) Consider a logarithmic scheme (X, M_X) arising from a simple normal crossings divisor $D \subseteq X$. Then $\mathrm{Trop}(X, M_X)$ is the cone over the dual intersection complex of D.

(2) Consider a logarithmic scheme (X, M_X) arising from a toric pair $(X, \partial X)$. This is included in the previous case if X is smooth, but otherwise it is new. Then $\mathrm{Trop}(X, M_X)$ is the fan of X, viewed as an abstract cone complex (i.e. without its preferred embedding into $N \otimes \mathbb{R}$).

By construction, the cones of $\mathrm{Trop}(X, M_X)$ are in inclusion-reversing bijection with the strata of (X, M_X), generalising the orbit-cone correspondence in toric geometry. Many other features of fans generalise, most notably: subdivisions of the tropicalisation induce birational modifications of the scheme, and piecewise-linear functions on the tropicalisation induce generalised Cartier divisors on the scheme (see Section 4.5 for a discussion in the case of simple normal crossings paris). These are crucial techniques, underpinning the myriad applications of logarithmic geometry to moduli theory.

8.4.4. *Tropicalising a logarithmic curve.* Recall the logarithmic curve (C, M_C) constructed in Section 7.3.2. This was obtained by choosing a smoothing $\pi \colon S \to \operatorname{Spec} \Bbbk[\![t]\!]$ of C, taking the divisorial logarithmic structure on S with respect to the central fibre C, and then pulling back along the inclusion $C \hookrightarrow S$ of that central fibre.

Equipping the base $\operatorname{Spec} \Bbbk[\![t]\!]$ with the logarithmic structure corresponding to $(t=0)$ we see that π pulls back monomial functions to monomial functions, hence gives a logarithmic morphism. Restricting to the central fibre, we obtain a logarithmic morphism

$$(C, M_C) \to (\operatorname{Spec} \Bbbk, \mathbb{N}) \tag{14}$$

where the target is the standard logarithmic point (see Section 7.3.1). We will now tropicalise this map: tropicalisation is functorial, so we will obtain a map between the tropicalisations.

The target is easy: the ghost sheaf constitutes a single stalk, namely $\mathbb{N}$. The corresponding cone is $\sigma = \mathbb{R}_{\geqslant 0}$ and therefore

$$\operatorname{Trop}(\operatorname{Spec} \Bbbk, \mathbb{N}) = \mathbb{R}_{\geqslant 0}.$$

The source is almost as easy. The stalks of the ghost sheaf are illustrated in (5). There is a single maximal cone $\sigma = \mathbb{R}^2_{\geqslant 0}$ attached to the node, and we obtain:

$$\operatorname{Trop}(C, M_C) = \mathbb{R}^2_{\geqslant 0}.$$

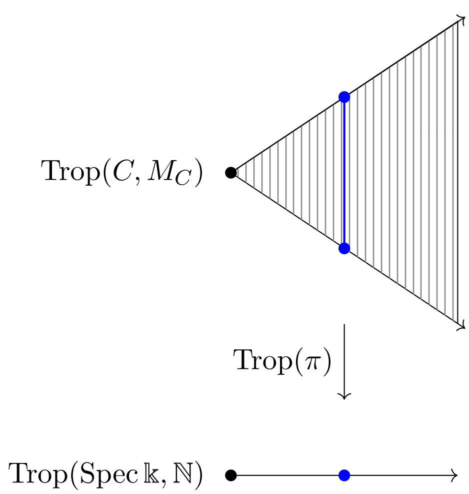
What about the map between tropicalisations? The logarithmic morphism (14) induces a pullback of stalks of ghost sheaves. Since $(t=0) \subseteq \operatorname{Spec} \Bbbk[\![t]\!]$ pulls back to the reduced union of the two central fibre components in S, the pullback map on ghost sheaves is:

$$\begin{aligned} \mathbb{N} &\to \mathbb{N}^2 \\ 1 &\mapsto (1,1). \end{aligned}$$

Dualising, we obtain the following map of cones:

$$\begin{aligned} \operatorname{Trop}(C, M_C) = \mathbb{R}^2_{\geqslant 0} &\to \mathbb{R}_{\geqslant 0} = \operatorname{Trop}(\operatorname{Spec} \Bbbk, \mathbb{N}) \\ (a_1, a_2) &\mapsto a_1 + a_2. \end{aligned}$$

We illustrate this as follows:



A fibre is indicated in blue. This fibre is a tropical curve: a finite metrised graph. Notice that the graph is precisely the dual graph of C: the vertices correspond to the irreducible components, the edge corresponds to the node. The fact that the edge contracts as we move towards zero in the base records the fact that the node smooths out in the family $S \to \operatorname{Spec} \Bbbk[\![t]\!]$.

We conclude that tropicalising a *logarithmic* curve produces a *tropical* curve. This is an instance of **faithful tropicalisation**. Note that tropicalising a *single* logarithmic curve gave rise to a *family* of tropical curves. This is because, as already discussed in Section 7.3.2, the logarithmic structure captures remnants of a smoothing.

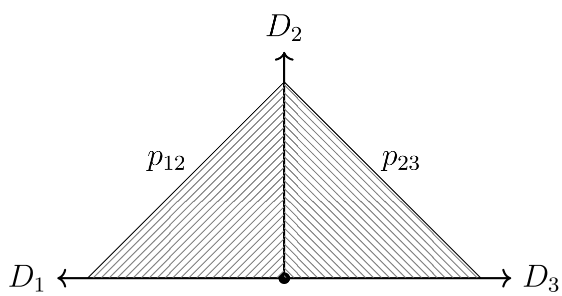
8.5. **Artin fans.** We now use the data of the tropicalisation to glue the local Artin cones discussed in Section 8.3. The result will be a global object, called the **Artin fan**. This is constructed, in a canonical way, from the tropicalisation. To see how it works, consider a simple normal crossings divisor as in Example 8.3.2. We first construct the tropicalisation. There are generisation maps of monoids

$$\begin{array}{ccccccccc} \overline{M}_{\xi_1} & \longleftarrow & \overline{M}_{p_{12}} & \longrightarrow & \overline{M}_{\xi_2} & \longleftarrow & \overline{M}_{p_{23}} & \longrightarrow & \overline{M}_{\xi_3} \\ \| & & \| & & \| & & \| & & \| \\ \mathbb{N} & \xleftarrow{\pi_2} & \mathbb{N}^2 & \xrightarrow{\pi_1} & \mathbb{N} & \xleftarrow{\pi_1} & \mathbb{N}^2 & \xrightarrow{\pi_2} & \mathbb{N} \end{array}$$

giving rise to face inclusions of cones:

$$\mathbb{R}_{\geqslant 0} \xhookrightarrow{i_2} \mathbb{R}^2_{\geqslant 0} \xhookleftarrow{i_1} \mathbb{R}_{\geqslant 0} \xhookrightarrow{i_1} \mathbb{R}^2_{\geqslant 0} \xhookleftarrow{i_2} \mathbb{R}_{\geqslant 0}$$

The tropicalisation is give by the following cone complex. Each cone is labelled by the corresponding stratum in (X, M_X):



This cone complex is constructed by gluing two copies of $\mathbb{R}^2_{\geqslant 0}$ along a face $\mathbb{R}_{\geqslant 0}$. Likewise, the Artin fan will be constructed by gluing two copies of $[\mathbb{A}^2/\mathbb{G}^2_{\mathrm{m}}]$ along an $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. The trick here is to recognise that

$$[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] = [(\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}})/\mathbb{G}^2_{\mathrm{m}}] = [(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1)/\mathbb{G}^2_{\mathrm{m}}].$$

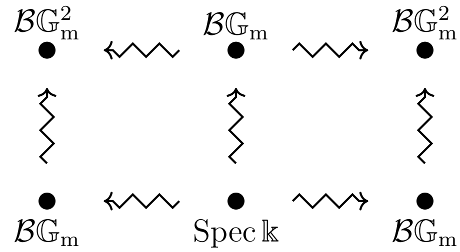
From this, we obtain *open* embeddings, which we use to glue:

$$[\mathbb{A}^2/\mathbb{G}^2_{\mathrm{m}}] \hookleftarrow [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \hookrightarrow [\mathbb{A}^2/\mathbb{G}^2_{\mathrm{m}}]. \tag{15}$$

Gluing via (15), we obtain an Artin stack contaning both copies of $[\mathbb{A}^2/\mathbb{G}^2_{\mathrm{m}}]$ as dense opens. We refer to the result as the **Artin fan** of (X, M_X) and denote it

$$\mathcal{A}_{(X,M_X)}.$$

It is profitable to think of this as a finite topological space, whose points correspond to the cones in $\mathrm{Trop}(X, M_X)$ and carry isotropy depending on the dimension of the cone. In our example we have



where the arrows indicate specialisation. Now, the unique local models $V \to \mathcal{A}_\sigma$ constructed in Section 8.3 glue to a unique global morphism

$$X \to \mathcal{A}_{(X,M_X)}$$

which encodes the logarithmic structure M_X, in the sense that M_X is the pullback of the toric logarithmic structure on $\mathcal{A}_{(X,M_X)}$.

Remark 8.5.1. The construction of the Artin fan from the tropicalisation can be reversed. Modulo certain subtleties and caveats, [CCUW20, Theorem 3] establishes an equivalence of categories:

$$\{\text{cone complexes}\} \longleftrightarrow \{\text{Artin fans}\}.$$

8.6. **History and references.** As remarked, toroidal embeddings date to the origins of toric geometry [KKMSD73, Oda81]. Artin fans were introduced by Abramovich–Wise in [AW18] based on ideas of Olsson [Ols03]. They play a key role in Gromov–Witten theory where they function as *universal targets* (more on this later). For an overview of tropicalisations, Artin fans, and related structures, see [ACM+16].

8.7. **Where to from here?** The best way to gain an appreciation for logarithmic geometry is to see it in action. Applications are manifold, and later lectures will focus on two important instances of very different flavours: Gromov–Witten theory and resolution of singularities.

8.8. **Exercises.** These exercises pertain to both Section 7 and Section 8.

Exercise 8.8.1 (Morphisms of divisorial logarithmic schemes)**.** Fix two simple normal crossings pairs equipped with their associated divisorial logarithmic structures:

$$(X|D) = (X|D_1 + \ldots + D_k), \qquad (Y|E) = (Y|E_1 + \ldots + E_l).$$

Given a morphism of schemes $f\colon X \to Y$, prove the following:

(1) If $f^{-1}E \subseteq D$[18], there is a unique enhancement of f to a logarithmic morphism.
(2) If $f^{-1}E \not\subseteq D$, there is no enhancement of f to a logarithmic morphism.

Exercise 8.8.2 (Morphisms of logarithmic points)**.** Recall that $(\operatorname{Spec}\Bbbk, Q)$ denotes the logarithmic point with ghost sheaf given by the monoid Q.

[18]Set-theoretically, i.e. ignoring the non-reduced structure of $f^{-1}E$.

(1) Show that there exists a unique logarithmic morphism
$$(\operatorname{Spec}\Bbbk, \mathbb{N}) \to (\operatorname{Spec}\Bbbk, 0).$$
(2) Show that there is no logarithmic morphism
$$(\operatorname{Spec}\Bbbk, 0) \to (\operatorname{Spec}\Bbbk, \mathbb{N}).$$
(3) Show that there exists a $\mathbb{G}_{\mathrm{m}}$ of distinct logarithmic morphisms
$$(\operatorname{Spec}\Bbbk, \mathbb{N}) \to (\operatorname{Spec}\Bbbk, \mathbb{N})$$
covering the identity $\mathbb{N} \to \mathbb{N}$ on ghost sheaves.
(4) Find the moduli of distinct logarithmic morphisms
$$(\operatorname{Spec}\Bbbk, \mathbb{N}) \to (\operatorname{Spec}\Bbbk, \mathbb{N}^k)$$
covering the sum map $\mathbb{N}^k \to \mathbb{N}$. Interpret this in terms of tangent vectors in $\mathbb{A}^k$.

Exercise 8.8.3 (Logarithmic regularity)**.** Fix a logarithmic scheme (X, M_X) and a point $x \in X$. Working locally, each element $m \in \overline{M}_{X,x}$ defines a generalised Cartier divisor, whose line bundle we may trivialise to obtain a function
$$f_m \in \mathcal{O}_{X,x}$$
well-defined up to units. We define the **logarithmic ideal at** x as follows:
$$I_{X,x} := (f_m \mid m \in \overline{M}_{X,x} \setminus \{0\}).$$
The **logarithmic nucleus at** x is the vanishing of this ideal:[19]
$$\mathbb{V}(I_{X,x}) \hookrightarrow \operatorname{Spec}\mathcal{O}_{X,x}.$$
(1) Let $X = \mathbb{A}^2_{xy}$ and let $D \subseteq X$ be the union of the two coordinate axes. Equip X with the associated divisorial logarithmic structure. Compute the logarithmic nucleus $\mathbb{V}(I_{X,x})$ for all $x \in X$ and thereby show that $\mathbb{V}(I_{X,x})$ is always non-singular.

We now introduce a mildly pathological logarithmic structure on $X = \mathbb{A}^2_{xy}$ which differs from the divisorial logarithmic structure considered above. The union D of the two coordinate axes is defined by a single equation:
$$D = \mathbb{V}(xy) \subseteq \mathbb{A}^2_{xy} = X.$$
We define a logarithmic structure on X encoding this single equation. To achieve this we take $\overline{M}_X = \mathbb{N}_D$, with associated monomial function:
$$\mathbb{N} \to \operatorname{Div}(X)$$
$$1 \mapsto (\mathcal{O}_X, xy).$$
(2) Contrast this with the divisorial logarithmic structure considered in (1). The difference is visible already at the level of ghost sheaves.
(3) Compute the logarithmic nucleus $\mathbb{V}(I_{X,x})$ for all $x \in X$ and thereby find an $x \in X$ such that $\mathbb{V}(I_{X,x})$ is singular.

A logarithmic scheme is **logarithmically regular** at $x \in X$ if $\mathbb{V}(I_{X,x})$ is non-singular and:

$$\dim_x X = \dim_x \mathbb{V}(I_{X,x}) + \mathrm{rk}\, \overline{M}_{X,x}.$$

To quote Ogus, a logarithmic scheme is logarithmically regular if "its singularity is completely accounted for by its logarithmic structure" [Ogu18, III.1.11].

(4) Let $X = (\mathrm{Spec}\, \Bbbk, \mathbb{N})$ be the standard logarithmic point. Compute $\mathbb{V}(I_{X,x})$ and thereby show that X is not logarithmically regular.

(5) Let X be a normal toric variety (not necessarily smooth) equipped with the divisorial logarithmic structure associated to the toric boundary. Prove that X is logarithmically regular.

Regularity is the logarithmic notion of smoothness. This is smoothness in the absolute sense, which in logarithmic geometry means smoothness of the structure morphism $X \to (\mathrm{Spec}\, \Bbbk, 0)$ to the trivial logarithmic point.

Relative notions of smoothness are also important: for example, logarithmic curves are typically singular over the trivial logarithmic point $(\mathrm{Spec}\, \Bbbk, 0)$, but smooth over the standard logarithmic point $(\mathrm{Spec}\, \Bbbk, \mathbb{N})$. We will encounter these relative notions in Section 14.

Exercise 8.8.4 (Logarithmic differentials)**.** There is an alternative characterisation of logarithmic regularity/smoothness, namely that the sheaf of **logarithmic differentials** is locally free. This exercise explores logarithmic differentials on singular affine toric varieties, thereby showing that these spaces are logarithmically smooth.

(i) Given a ring R, remind yourself how the R-module Ω_R of Kähler differentials is defined.

Now take $R = \Bbbk[x, y, z]/(xy - z^2)$ and let $X = \mathrm{Spec}\, R$ (this is a singular toric variety).

(ii) Find a presentation for Ω_R.

(iii) Prove that Ω_R is not locally free, and hence that X is not smooth. (Hint: compare the stalks $\Omega_R \otimes_R R/\mathfrak{m}$ for different maximal ideals $\mathfrak{m} \lhd R$.)

As in Section 7.6, we equip X with the logarithmic structure corresponding to the toric boundary ∂X. This gives rise to an R-module $\Omega_R^{\log}$ of logarithmic differentials. Instead of formally defining this module, we present an ad hoc method for computing it:

- **Generators:** $dx/x, dy/y, dz/z$ (differentials with "logarithmic poles" along D).
- **Relations:** Obtained from the relation(s) in Ω_R by dividing through by arbitrary monomials in x, y, z (provided that the result is still an R-linear combination of the above generators).

Using this method:

[19] If you are unhappy with Spec of the local ring, simply choose a small open set U around x and replace $\overline{M}_{X,x}$ and $\mathcal{O}_{X,x}$ with $\overline{M}_X(U)$ and $\mathcal{O}_X(U)$ respectively.

(iv) Write down a presentation for the R-module $\Omega_R^{\log}$ and thereby show that it is free of rank 2.

It follows that $X = \operatorname{Spec} R$ equipped with the toric logarithmic structure is logarithmically smooth (although X itself is singular). Now repeat the above steps for the following affine toric varieties:

(v) $R = \Bbbk[x, y, z]/(xy - z^n)$ for $n \geqslant 2$.

(vi) $R = \Bbbk[x, y, z, w]/(xy - zw)$.

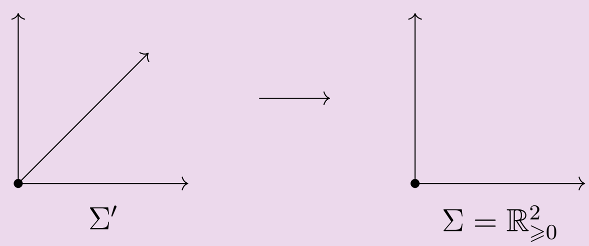
The way we have introduced logarithmic differentials is a bit ad hoc, though tremendously useful in practice. There is of course a more conceptual approach.

(vii) Look up the formal definition of logarithmic differentials (see e.g. [ACG+13, Proposition 3.4]) and relate it to the presentations you found above.

Exercise 8.8.5 (Logarithmic modifications)**.** Consider the logarithmic curve (C, M_C) from Section 7.3.2. Recall from Section 8.4.4 that its tropicalisation is a quadrant:

$$\Sigma = \Sigma(C, M_C) = \mathbb{R}^2_{\geqslant 0}.$$

Subdivide this tropicalisation by introducing a diagonal ray:

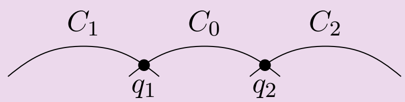


This induces a **logarithmic modification** $(C', M_{C'}) \to (C, M_C)$, generalising the correspondence between birational modifications and subdivisions discussed in Section 4.5. The logarithmic modification is defined by fibring over the induced map of Artin fans:

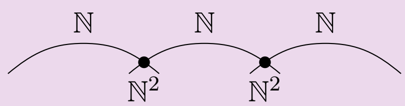
$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & \square & \downarrow \\ \mathcal{A}_{\Sigma'} & \longrightarrow & \mathcal{A}_{\Sigma}. \end{array}$$

Describe the underlying scheme C'.

Exercise 8.8.6. Let C be a nodal curve consisting of a chain of three $\mathbb{P}^1$s:



We will enhance C to a logarithmically smooth curve over the standard logarithmic point $(\operatorname{Spec} \Bbbk, \mathbb{N})$. We take the ghost sheaf to be



with the obvious generisation maps. This sheaf arises by choosing a regular smoothing of C (a smooth surface $S \to \mathbb{A}^1$ whose general fibre is smooth and whose central fibre is C), taking the divisorial logarithmic structure associated to the central fibre, and pulling back to the central fibre.

(1) Describe the global sections of $\overline{M}_C$.

(2) Using the above interpretation via regular smoothings, describe the generalised Cartier divisors associated to each global section of $\overline{M}_C$. (For an alternative approach, describe them by imposing the condition given in Definition 7.4.2.)

(3) Conclude that there is a $\mathbb{G}_{\mathrm{m}}$ worth of pairwise non-isomorphic logarithmic structures with this ghost sheaf.

(4) Generalise the above to arbitrary chains of rational curves.

Exercise 8.8.7. Fix a normal toric variety X with dense torus action $T \curvearrowright X$ and toric boundary $D = \partial X$. We write $(X|D)$ to denote X equipped with the logarithmic structure corresponding to D.

(i) Prove that the Artin fan of the logarithmic scheme $(X|D)$ is identified with the global quotient:

$$\mathcal{A}_{X|D} = [X/T].$$

(ii) Deduce that, for any two Hirzebruch surfaces $\mathbb{F}_a$ and $\mathbb{F}_b$ we have

$$[\mathbb{F}_a/T] \cong [\mathbb{F}_b/T].$$

(iii) Give an alternative proof of the above fact using toric GIT (i.e. without patching an open cover).

Exercise 8.8.8 (Bad monoids)**.** We explore pathological monoids, gradually introducing conditions which lead to an intrinsic characterisation of **toric monoids**. All monoids will be finitely-generated, commutative and unital.

You should employ two different ways of constructing monoids: (i) via a presentation (quotients of free monoids by relations make sense), and (ii) as a submonoid of a lattice (not necessarily the integral points of a rational polyhedral cone). If you need help, a list of interesting monoids is provided at the end of the exercise.

(i) Given a monoid Q, construct the **groupification** Q^{gp} (this is the universal abelian group admitting a map $Q \to Q^{\mathrm{gp}}$).

A monoid Q is **integral** if the map $Q \to Q^{\mathrm{gp}}$ is injective.

(ii) Find an example of a monoid which is not integral.

(iii) Prove that if $\Bbbk[Q]$ is an integral domain than Q is integral. Show that the converse does not hold and interpret the discrepancy.

An integral monoid $Q \hookrightarrow Q^{\mathrm{gp}}$ is **saturated** if and only if the following condition holds: for all $q \in Q^{\mathrm{gp}}$, if $nq \in Q$ for some $n \in \mathbb{N}$, then $q \in Q$.

(iv) Find an example of a non-saturated monoid.

(v) (Bonus) Related saturatedness of Q to integral closedness of $\Bbbk[Q]$.

A monoid Q is **sharp** if the only invertible element is the identity: $Q^{\star} = \{0\}$.

(vi) Prove that if Q is integral, saturated and sharp then Q^{gp} is torsion-free. Show that saturatedness is essential here.

A monoid is **toric** if it is integral, saturated, and sharp.

(vii) Prove that if Q is toric then $Q \cong \sigma^{\vee} \cap M$ for a strictly convex rational polyhedral cone σ.

For a general monoid, the map $Q \to Q^{\mathrm{gp}}$ induces a morphism $\operatorname{Spec} \Bbbk[Q^{\mathrm{gp}}] \to \operatorname{Spec} \Bbbk[Q]$.

(viii) Describe this morphism for a toric monoid, then describe it for the non-toric examples you have found above. Always keep in mind the important role played by monomial functions.

Here is a short list of relevant monoids:

- $\mathbb{N}^2/(e_1 + e_2 = e_1)$.
- $\mathbb{N}^2/(2e_1 = 2e_2)$.
- $\mathbb{N} \smallsetminus \{1\}$ (exercise: find a presentation for this monoid).

9. Moduli spaces, part 2

By Pierrick
Moduli spaces 2: Moduli spaces of varieties.
Prerequisite: Toric geometry, resolutions of singularities.

9.1. **Introduction.** The main goal of this section is to describe some aspects of the moduli theory of varieties. The main issue is the construction of separated and proper moduli spaces. Concretely, given a family $X^{\circ} \to C^{\circ}$ of "nice" varieties that one tries to parametrize, over a punctured smooth curve $C^{\circ} = C \smallsetminus \{0\}$, one would like to be able to extend it uniquely into a family $X \to C$ of "nice" varieties. By the valuative criteria for separation and properness, the existence of such an extension is related to the properness of the moduli space of "nice" varieties, whereas the uniqueness of the extension is related to the separation of the moduli space.

Given a smooth family $X^\circ \to C^\circ$ over a punctured smooth curve, it might not be possible to find an extension $X \to C$ with smooth central fiber. For example, over $\mathbb{C}$, there is a purely topological obstruction given by the monodromy, viewed as an element in the mapping class group of the general fiber [TO DO: ADD EXPLANATIONS]. Correspondingly, moduli spaces of smooth varieties are typically non-proper. For example, the moduli spaces $\mathcal{M}_{g,n}$ of n-marked genus g curves are non-proper for $(g,n) \neq (0,3)$.

Since one cannot in general extend a smooth family $X^\circ \to C^\circ$ into a smooth family $X \to C$, one can ask how to guarantee the existence of such an extension by allowing the mildest possible types of singularities in the central fiber and in the total space of the extended family. A general answer to this question is provided, in characteristic zero, by the semistable reduction theorem, reviewed in §9.2. The semistable reduction theorem is an existence result and there is no uniqueness result for such an extension. In particular, this result is not enough to construct separated and proper moduli spaces.

To construct separated and proper moduli spaces, one would like to produce a "canonical" extension starting from any of the extensions given by the semistable reduction theorem. This question is can be reformulated as asking for a "canonical" birational model of the total space of the one-parameter family. For varieties of (log-) general type, such model can be taken as the "canonical model" of the total space in the sense of birational geometry. We describe in §9.3 how the notion of stable curve can be recovered from the birational geometry of the 2-dimensional total space of one-paramter families of curves. In §9.4, we introduce KSBA stable varieties, which are higher dimensional versions of the stable curves, and we state the existence of separated and proper moduli spaces of KSBA stable varieties, generalizing the Deligne–Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$. Finally, we present in §9.5 an explicit combinatorial description of moduli spaces of KSBA stable toric varieties. In particular, we will illustrate the particularly nice interplay between the general notion of KSBA stable variety coming from birational geometry, and combinatorial objects naturally attached to toric varieties, such as the secondary fan and the secondary polytope.

9.2. **Semistable reduction.** We first state the general semistable reduction theorem of Kempf–Knudsen–Mumford–Saint-Donat [KKMSD73]. We then illustrate the general theorem in the context of toric families of toric varieties, where the main issues can be reformulated combinatorially.

9.2.1. *The semistable reduction theorem.* Let C be a smooth curve, $0 \in C$ be a marked point, and $C^\circ := C \setminus \{0\}$ the corresponding punctured curve. We consider families of varieties with base C. As reviewed in §9.1, a smooth family $X^\circ \to C^\circ$ might not extend to a smooth family $X \to C$. In the following definition, we define a class of "semistable families", which contains the class of smooth families, but allow some very restricted class of singularities for the central fiber and the total space.

Definition 9.2.2. A *semistable family* over the marked curve $(C,0)$ is an onto morphism $f : X \to C$, with smooth total space X, and with central fiber $X_0 := f^{-1}(0)$ which is reduced, and with smooth irreducible components crossing normally.

We can now state the semistable reduction theorem of Kempf–Knudsen–Mumford–Saint-Donat [KKMSD73].

Theorem 9.2.3. *Let k be a field of characteristic 0. Let $(C, 0)$ be a smooth marked curve over k, and $f : X \to C$ be an onto morphism of varieties over k such that the restriction $X^\circ \to C^\circ$ is smooth. Then, there exists a finite morphism $\pi : C' \to C$, with $\pi^{-1}(0) = \{0'\}$, and a projective morphism $p : X' \to X \times_C C'$, such that:*

(i) *p is an isomorphism over $(C')^\circ$,*

(ii) *the morphism $f' : X' \to C'$, obtained as the composition $X' \xrightarrow{p} X \times_C C' \to C'$, is a semistable family.*

By Hironaka's resolution of singularities, given $f : X \to C$ as in Theorem 9.2.3, there exists a projective morphism $p : X' \to X$, which is an isomorphism over C°, such that X' is smooth, and the reduced central fiber $(f')^{-1}(0)_{red}$ of $f' := f \circ p : X' \to C$ is a simple normal crossing divisor on X'. What is missing to obtain the semistable reduction theorem is to have a reduced central fiber $(f')^{-1}(0)$. It is for this last step that a finite ramified cover $C' \to C$ is needed in general.

Example 9.2.4. The following example is the simplest illustration of the role of the finite base change $C' \to C$ in the semistable reduction theorem. Consider $X = \mathbb{A}^1$, $C = \mathbb{A}^1$, and

$$\begin{aligned} f : \mathbb{A}^1 &\longrightarrow \mathbb{A}^1 \\ x &\longmapsto x^n \,, \end{aligned}$$

for some $n > 1$. Then, $f^{-1}(0)$ is the subscheme of $X = \mathbb{A}^1$ defined by $x^n = 0$, that is, a non-reduced 0-dimensional subscheme of length n. Under the base change

$$\begin{aligned} \pi : C' = \mathbb{A}^1 &\longrightarrow C = \mathbb{A}^1 \\ t &\longmapsto t^n \,, \end{aligned}$$

we have $X' := X \times_C C' = \mathbb{A}^1$, and the base change $f' : X' \to C'$ is given by

$$\begin{aligned} \pi : X' = \mathbb{A}^1 &\longrightarrow C' = \mathbb{A}^1 \\ x &\longmapsto x \,, \end{aligned}$$

with obviously reduced central fiber $(f')^{-1}(0)$.

9.2.5. *Example: toric varieties.* We illustrate Theorem 9.2.3 in the context of toric varieties. In fact, to prove Theorem 9.2.3 for toric morphisms is the heart of the proof of the general case, and many foundational notions in toric geometry were introduced by Kempf–Knudsen–Mumford–Saint-Donat in [KKMSD73].

Let P be a compact lattice polytope in $\mathbb{R}^n$. Denote by $C(P)$ the cone in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ over $P \times \{1\}$. Denote by $\overline{\mathcal{X}}_P$ the $(n+1)$-dimensional affine toric variety with fan the strictly convex cone $C(P)$. The projection $C(P) \to \mathbb{R}_{\geq 0}$ on the last factor of $\mathbb{R}^n \times \mathbb{R}$ induces a toric morphism

$$f_P : \overline{\mathcal{X}}_P \longrightarrow \mathbb{A}^1 \,.$$

Denoting $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$, we have $f_P^{-1}(\mathbb{G}_m) = \mathbb{G}_m \times \mathbb{G}_m^n = \mathbb{G}_m^{n+1} \subset \overline{\mathcal{X}}_P$. In particular, f_P is smooth over $\mathbb{G}_m$

Example 9.2.6. The polytope P is a standard simplex of dimension n, that is, P is of volume $\frac{1}{n!}$, if and only if the affine toric variety $\overline{\mathcal{X}}_P$ is non-singular.

Let $\mathcal{P}$ be an integral lattice polyhedral decomposition of P. Then, the cone $C(\mathcal{P})$ over $\mathcal{P}$ is a fan with support $C(P)$. We denote by $\mathcal{X}_{\mathcal{P}}$ the corresponding toric variety and $\mathcal{X}_{\mathcal{P}} \to \overline{\mathcal{X}}_P$ the corresponding toric morphism.

Definition 9.2.7. An integral lattice polyhedral decomposition $\mathcal{P}$ of P is *unimodular* if all the n-dimensional faces of $\mathcal{P}$ are standard simplices.

The decomposition $\mathcal{P}$ is unimodular if and only if the toric variety $\mathcal{X}_{\mathcal{P}}$ is non-singular.

Exercise 9.2.8. For $n = 2$, show that every compact lattice polytope admits a unimodular decomposition. Hint: Pick's formula.

Exercise 9.2.9. For $n \geq 3$, show that there exists compact lattice polytope which do not admit unimodular decompositions.

The following key result of [KKMSD73] guarantees that unimodular decompositions always exist after rescaling P.

Theorem 9.2.10. *Let P be a compact lattice polytope in $\mathbb{R}^n$. Then, there exists an integer $N \in \mathbb{Z}_{>0}$ such that the scaled polytope NP admits an unimodular decomposition.*

Theorem 9.2.10 implies the semistable reduction theorem for the toric morphism $f_P : \overline{\mathcal{X}}_P \to \mathbb{A}^1$. Indeed, $f_{NP} : \overline{\mathcal{X}}_{NP} \to \mathbb{A}^1$ is the order N base change of f_P, and, if $\mathcal{P}$ is an unimodular decomposition of NP, then $\mathcal{X}_{\mathcal{P}} \to \overline{\mathcal{X}}_{NP}$ is a simple normal crossing resolution of $\overline{\mathcal{X}}_{NP}$ with reduced central fiber.

Remark 9.2.11. For every P, the affine toric variety $\overline{\mathcal{X}}_P$ has canonical singularities. If $\mathcal{P}$ is an unimodular resolution of P, then $\mathcal{X}_{\mathcal{P}} \to \overline{\mathcal{X}}_P$ is a crepant resolution of $\overline{\mathcal{X}}_P$. The existence of polytopes without unimodular decompositions in dimension $n \geq 3$ implies the existence of canonical singularities without crepant resolutions.

9.3. **Stable reduction for curves and birational geometry of surfaces.** The key to the existence of the moduli stack $\overline{\mathcal{M}}_{g,n}$ of n-marked genus g stable curves as separated and proper Deligne–Mumford stack is the stable reduction theorem: possibly after a finite base change, every one-parameter family of stable curves over a punctured curve admit a unique extension over the puncture with a stable central fiber. In this section, we describe how to understand this result in terms of birational geometry of the surface obtained as total space of the one-parameter family of curves.

Let $(C, 0)$ be an affine smooth marked curve and let $f : X \to C$ be an onto morphism such that the restriction $f : X^\circ \to C^\circ$ away from 0 is a smooth projective family of curves of genus $g \geq 2$. By the semistable reduction theorem, modulo a finite base change of $(C, 0)$ and without changing $X^\circ \to C^\circ$, one can assumme that X is smooth and that the central fiber $X_0 := f^{-1}(0)$ is reduced normal crossing in X. However such an X is very far from unique. For example, blowing-up a point contained in X_0 produces a new X with the same properties. For moduli purposes, one would like to find a distinguished X so that the corresponding central fiber X_0 could be viewed as "the" limit of the family $X^\circ \to C^\circ$. In other words, one would like to find a distinguished surface among all the surfaces birational to X. The birational geometry of surfaces provide an answer: every smooth surface of general type admits a unique "canonical model" characterized by the property that its canonical divisor is ample. Note that such result is usually stated for projective surfaces, but there is a direct generalization applying to surfaces projective over an affine curve, as $X \to C$. Since the general fibers are curves of genus $g \geq 2$, X is a surface of general type over C, and so admits a canonical model $X_{can} \to C$ given explicitly by

$$X_{can} = \mathrm{Proj}_C \bigoplus_{n \geq 0} f_\star(K_X^{\otimes n}) \,. \tag{16}$$

Birational geometry of surfaces guarantees that $\bigoplus_{n\geq 0} f_\star(K_X^{\otimes n})$ is a sheaf of *finitely generrated algebras*, and so the above Proj construction indeed defines X_{can} as an algebraic surface. The canonical model X_{can} can be constructed explicitly starting from X, then successively contracting all rational (-1)-curves, and then successively contracting all rational (-2)-curves. Unlike X, the canonical model X_{can} is typically singular: the contraction of a chain of n (-2)-curves leads to an A_n surface singularity locally given by

$$xy = t^{n+1} \,.$$

In general, canonical models of projective surfaces of general types of "canonical singularities", given by DuVal/ADE quotient singularities and obtained by contracting collections of (-2)-curves with ADE Dynkin dual graphs. In our relative situation, only type A singularities appear.

Now is the key point: one can check that the central fiber $X_{can,0}$ of the canonical model X_{can} is always a stable curve. Indeed, contracting the (-1)-curves amounts to contracting rational components of X_0 with only one special point, and contracting the (-2)-curves amounts to contracting rational components of X_0 with only two special points. In other words, the correct notion of "stable curve" is whatever appeared as central fiber of canonical models of one-parameter families of smooth curves.

9.4. **Canonical models and KSBA stable varieties.** In 1988, motivated by progress in higher-dimensional birational geometry, Kollár and Shepherd-Barron [KSB88] suggested to generalize the logic summarized in the previous section 9.3 from curves to higher dimensional varieties – see also Alexeev [Ale96] for the case of pairs. In the last 30 years, a general theory of KSBA (Kollár–Shepherd-Barron–Alexeev) moduli spaces has been established, as described in the book [Kol23].

Looking at central fibers of canonical models of one-parameter families of smooth varieties of general type leads to the notion of *stable pair*, generalizing stable curves to higher dimension.

Definition 9.4.1. Let X be a projective variety and D a $\mathbb{Q}$-divisor on X. The pair (X, D) is *stable* if:

i) (X, D) has semi-log-canonical (slc) singularities.
ii) $K_X + D$ is $\mathbb{Q}$-Cartier and ample.

We now define moduli spaces of stable poairs. Let $\mathbf{w} = (w_1, \ldots, w_n)$ be a sequence of weights $0 < w_i \leq 1$, and $v \in \mathbb{Q}_{>0}$. Let $\mathcal{M}_{d,\mathbf{w}}(v)$ be the moduli space of d-dimensional stable pairs (X, D), endowed with a decomposition $D = \sum_{i=1}^n w_i D_i$ where D_i are effective reduced divisors, and such that $(K_X + D)^d = v$. The correct definition of a "family" of stable pairs is actually quite sublte – see [Kol23].

Theorem 9.4.2. *The moduli space* $\mathcal{M}_{d,\mathbf{w}}(v)$ *is a proper Deligne–Mumford stack.*

Example 9.4.3. For $d = 1$, $\mathbf{w} = (1, \dots, 1)$ and $v = 2g-2+n > 0$, we have $\mathcal{M}_{d,\mathbf{w}}(v) = \overline{\mathcal{M}}_{g,n}$.

9.5. **Secondary fan and moduli space of KSBA stable toric varieties.** In this section, we present an explicit example of the KSBA moduli spaces $\mathcal{M}_{d,\mathbf{w}}(v)$ in the context of toric varieties.

Let P be a lattice polytope in $\mathbb{R}^n$, that is the convex hull of finitely many points in $\mathbb{Z}^n$. Denote by $P_\mathbb{Z}$ the set of integral points in P, and $|P_\mathbb{Z}|$ its cardinality. We first define a fan $\widetilde{Sec}(P)$ in the real vector space $\mathbb{R}^{|P_\mathbb{Z}|}$. For every point $h = (h_p)_{p\in P_\mathbb{Z}} \in \mathbb{R}^{|P_\mathbb{Z}|}$, the lower convex hull of the points (p, h_p) in $P \times \mathbb{R}$ is a piecewise linear function φ_h on P. The domains of linearity of φ_h define a polyhedral decomposition $P = \cup_i P_i$ into lattice polytopes P_i. Moreover, each polytope P_i is naturally marked by the subset Q_i of integral points $p \in (P_i)_\mathbb{Z}$ such that (p, h_p) belongs to the graph of φ_h over P_i. Hence, for every $h \in \mathbb{R}^{|P_\mathbb{Z}|}$, we obtain a decomposition of P into marked polytopes (P_i, Q_i).

Definition 9.5.1. A decomposition of P into marked polytopes (P_i, Q_i) is called *regular* if it is induced by some $h \in \mathbb{R}^{|P_\mathbb{Z}|}$.

For every regular decomposition D of P, the closure in $\mathbb{R}^{|P_\mathbb{Z}|}$ of the set of points h inducing the regular decomposition D of P is a cone C_D in $\mathbb{R}^{|P_\mathbb{Z}|}$. These cones fit together into a complete fan $\widetilde{Sec}(P)$ on $\mathbb{R}^{|P_\mathbb{Z}|}$.

If f is an affine linear function on $\mathbb{R}^n$, then the points $h = (h(p))_{p\in P_\mathbb{Z}}$ and $(h(p)+f(p))_{p\in P_\mathbb{Z}}$ define the same polyhedral decomposition of P, and so are contained in the same cone of $\widetilde{Sec}(P)$. In other words, the fan $\widetilde{Sec}(P)$ is naturally invariant under the action on $\mathbb{R}^{|P_\mathbb{Z}|}$ of the space $\mathbb{R}^{n+1}$ of affine linear functions on $\mathbb{R}^n$, and so defines a complete fan $\mathrm{Sec}(P)$ in $\mathbb{R}^{|P_\mathbb{Z}|}/\mathbb{R}^{n+1} \simeq \mathbb{R}^{|P_\mathbb{Z}|-n-1}$, called the *secondary fan*. By construction, the cones of the secondary fan are indexed by the regular decompositions of P. The maximal dimensional cones are indexed by the regular triangulations of P, that is, the regular decompositions of P into simplices.

Let Y be a projective toric variety, with boundary toric divisor D, and L an ample line bundle. Denote by P the corresponding momemtum polytope, and $\mathcal{M}_P$ the toric variety with fan the secondary fan $\mathrm{Sec}(P)$.

Lemma 9.5.2. *Let H be a divisor in the linear system of L. Then the pair $(Y, D + \epsilon H)$ is semi-log-canonical for $0 < \epsilon << 1$ if and only if H is does not contain a 0-dimensional strata of D.*

Proof. TO DO. Recall first that the pair (Y, D) is always canonical since $K_Y + D = 0$. ♣

Let $\mathcal{M}_{(Y,D,L)}$ be the KSBA compactification of the moduli space of semi-log-canonical pairs $(Y, D + \epsilon H)$. It is a proper Deligne-Mumford stack, with projective coarse moduli space.

Theorem 9.5.3. *The toric variety $\mathcal{M}_P$ is the normalization of the coarse moduli space of $\mathcal{M}_{(Y,D,L)}$.*

Proof. TO DO: Include a sketch of proof. Explain the link between polyhedral decompositions of P and irreducible components of degenerations of $(Y, D + \epsilon H)$. ♣

For recent non-toric generalizations of Theorem 9.5.3 motivated by mirror symmetry, see [HKY20] and [AAB24].

10. Preamble: Towards stable maps

By Dan

10.1. **Kontsevich's formula.** THIS TEXT IS TAKEN FROM: Lectures on GW invariants of orbifolds

Let us step back to the story of Gromov–Witten invariants. Of course these first came to be famous due to their role in mirror symmetry. But this failed to excite me, a narrow-minded algebraic geometer such as I am, until Kontsevich [Kon95] gave his formula for the number of rational plane curves.

This is a piece of magic which I will not resist describing.

10.1.1. *Setup.* Fix an integer $d > 0$. Fix points $p_1, \ldots p_{3d-1}$ in general position in the plane. Look at the following number:

Definition 10.1.2.

$$N_d = \# \left\{ \begin{array}{l} C \subset \mathbb{P}^2 \text{ a rational curve,} \\ \deg C = d, \text{ and} \\ p_1, \ldots p_{3d-1} \in C \end{array} \right\}.$$

Remark 10.1.3. One sees that $3d - 1$ is the right number of points using an elementary dimension count: a degree d map of $\mathbb{P}^1$ to the plane is parametrized by three forms of degree d (with $3(d+1)$ parameters). Rescaling the forms (1 parameter) and automorphisms of $\mathbb{P}^1$ (3 parameters) should be crossed out, giving $3(d+1) - 1 - 3 \ = \ 3d - 1$.

10.1.4. *Statement.*

Theorem 10.1.5 (Kontsevich)**.** *For $d > 1$ we have*

$$N_d = \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 > 0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1 - 2} - d_1^3 d_2 \binom{3d-4}{3d_1 - 1} \right).$$

Remark 10.1.6. The first few numbers are

$$N_1 = 1, \ N_2 = 1, \ N_3 = 12, \ N_4 = 620, \ N_5 = 87304.$$

The first two are elementary, the third is classical, but N_4 and N_5 are nontrivial.

Remark 10.1.7. The first nontrivial analogous number in $\mathbb{P}^3$ is the number of lined meeting four other lines in general position (the answer is 2, which is the beginning of Schubert calculus).

10.2. **Set-up for a streamlined proof.**

10.2.1. $\overline{\mathcal{M}}_{0,4}$. We need one elementary moduli space: the compactified space of ordered four-tuples of points on a line, which we describe in the following unorthodox manner :

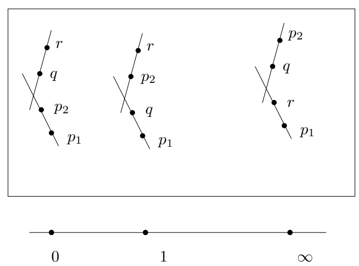
$$\overline{\mathcal{M}}_{0,4} = \overline{\left\{ p_1, p_2, q, r \in L \;\middle|\; \begin{array}{l} L \simeq \mathbb{P}^1 \\ p_1, p_2, q, r \text{ distinct} \end{array} \right\}}$$

The open set $\mathcal{M}_{0,4}$ indicated in the braces is isomorphic to $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, the coordinate corresponding to the cross ratio

$$CR(p_1, p_2, q, r) = \frac{p_1 - p_2}{p_1 - r}\frac{q - r}{q - p_2}.$$

The three points in the compactification, denoted

$$\begin{aligned} 0 &= (p_1, p_2 \mid q, r), \\ 1 &= (p_1, q \mid p_2, r), \text{ and} \\ \infty &= (p_1, r \mid p_2, q), \end{aligned}$$



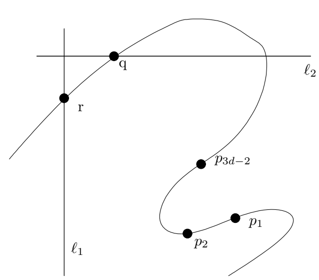
describing the three different ways to split the four points in two pairs and position them on a nodal curve with two rational components.

10.2.2. *A one-parameter family.* We now look at our points $p_1, \ldots p_{3d-1}$ in the plane.

We pass two lines ℓ_1, ℓ_2 with general slope through the last point p_{3d-1} and consider the following family of rational plane curves in $C \to B$ parametrized by a curve B:

- Each curve C_b contains $p_1, \ldots p_{3d-2}$ (but not necessarily p_{3d-1}).
- One point $q \in C_b \cap \ell_1$ is marked.
- One point $r \in C_b \cap \ell_2$ is marked.

In fact, we have a family of rational curves $C \to B$ parametrized by B, most of them smooth, but finitely many have a single node, and a morphism $f : C \to \mathbb{P}^2$ immersing the fibers in the plane.



10.2.3. *The geometric equation.* We have a cross-ratio map

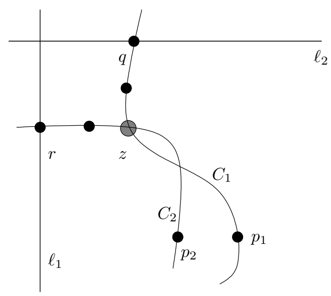
$$\begin{aligned} B &\xrightarrow{\lambda} \overline{\mathcal{M}}_{0,4} \\ C &\mapsto CR(p_1, p_2, q, r) \end{aligned}$$

Since points on $\mathbb{P}^1$ are homologically equivalent we get

$$\deg_B \lambda^{-1}(p_1, p_2|q, r) = \deg_B \lambda^{-1}(p_1, q|p_2, r).$$

10.2.4. *The right hand side.* Now, each curve counted in $\deg_B \lambda^{-1}(p_1, q|p_2, r)$ is of the following form:

- It has two components C_1, C_2 of respective degrees d_1, d_2 satisfying $d_1 + d_2 = d$.
- We have $p_1 \in C_1$ as well as $3d_1 - 2$ other points among the $3d - 4$ points $p_3, \ldots p_{3d-2}$.
- We have $p_2 \in C_2$ as well as the remaining $3d_2 - 2$ points from $p_3, \ldots p_{3d-2}$.
- We select one point $z \in C_1 \cap C_2$. where the two abstract curves are attached.
- We mark one point $q \in C_1 \cap \ell_1$ and one point $r \in C_2 \cap \ell_2$.



For every choice of splitting $d_1 + d_2 = d$ we have $\binom{3d-4}{3d_1-2}$ ways to choose the set of $3d_1 - 2$ points on C_1 from the $3d - 4$ points $p_3, \ldots p_{3d-2}$. We have N_{d_1} choices for the curve C_1 and N_{d_2} choices for C_2. We have $d_1 \cdot d_2$ choices for z, d_1 choices for q and d_2 for r. This gives the term

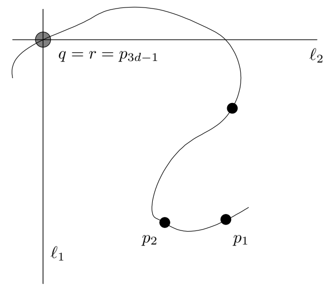
$$\deg_B \lambda^{-1}(p_1, q|p_2, r) = \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 > 0}} \binom{3d-4}{3d_1-2} \cdot N_{d_1} N_{d_2} \cdot d_1 d_2 \cdot d_1 \cdot d_2.$$

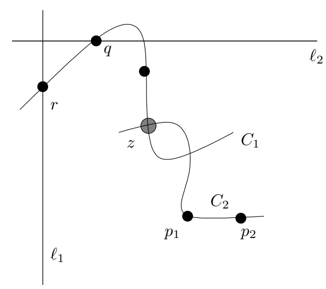
A simple computation in deformation theory shows that each of these curves actually occurs in a fiber of the family $C \to B$, and it occurs exactly once with multiplicity 1.

10.2.5. *The left hand side.* Curves counted in $\deg_B \lambda^{-1}(p_1, p_2|q, r)$ come in two flavors: there are *irreducible* curves passing through $q = r = \ell_1 \cap \ell_2$ This is precisely N_d.

Now, each *reducible* curve counted in $\deg_B \lambda^{-1}(p_1, p_2|q, r)$ is of the following form:

- It has two components C_1, C_2 of respective degrees d_1, d_2 satisfying $d_1 + d_2 = d$.
- We have $3d_1 - 1$ points among the $3d - 4$ points $p_3, \ldots p_{3d-2}$ are on C_1.
- We have $p_1, p_2 \in C_2$ as well as the remaining $3d_2 - 2$ points from $p_3, \ldots p_{3d-2}$.
- We select one point $z \in C_1 \cap C_2$. where the two abstract curves are attached.
- We mark one point $q \in C_1 \cap \ell_1$ and one point $r \in C_1 \cap \ell_2$.





For every choice of splitting $d_1+d_2=d$ we have $\binom{3d-4}{3d_1-1}$ ways to choose the set of $3d_1-1$ points on C_1 from the $3d-4$ points $p_3,\dots p_{3d-2}$. We have N_{d_1} choices for the curve C_1 and N_{d_2} choices for C_2. We have $d_1\cdot d_2$ choices for z, d_1 choices for q and d_1 for r. This gives

$$\begin{aligned}\deg_B \lambda^{-1}&(p_1,p_2|q,r)\\ &= N_d + \sum_{\substack{d=d_1+d_2\\ d_1,d_2>0}} \binom{3d-4}{3d_1-1}\cdot N_{d_1}\ N_{d_2}\cdot d_1d_2\cdot d_1^2\end{aligned}$$

Equating the two sides and rearranging we get the formula. ♣

Gromov–Witten theory allows one to systematically carry out the argument in general, without sweeping things under the rug as I have done above.

The next lecture series will show how to do it!

11. Gromov–Witten theory I: geometry of the moduli & genus zero invariants

11.1. **Moduli of stable maps.** We recall the moduli functors introduced in Section 6.2 $\mathcal{M}_{g,n}(X,\beta)$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)$ parametrizing family of maps from smooth pointed curves and stable maps respectively.

More precisely:

$$\mathcal{M}_{g,n}(X,\beta)(S) := \{f\colon C/S\to X\times S \ :\ \substack{C\xrightarrow{\pi}S \text{ is smooth with geometric fibers of genus } g\\ (p_1,\dots,p_n)\colon S\to C \text{ markings}\\ f_{s,*}[C_s]=\beta}\ \}.$$

Where f_1 and f_2 are isomorphic if $f_2=f_1\circ\alpha$ for $\alpha\colon C\to C$ an authomorphism of C over S We say that f has an authorphism if there exist $\alpha\colon C\to C$ such that $f=f\circ\alpha$

Example 11.1.1. Consider $f \in \mathcal{M}_0(\mathbb{P}^2, 2[L])$ given by $f([t_0 : t_1]) = [0 : t_0^2 : t_1^2]$. Then f has an authorphism of order 2 $\alpha([t_0 : t_1]) = [t_0 : -t_1]$

In order to compute enumerative invariants we will need to perform intersection theoretic computations on the parameter spaces. To do so, we want to work with *proper* moduli spaces.

The solution is to allow both the curve and the map to degenerate:

$$\overline{\mathcal{M}}_{g,n}(X,\beta)(S) := \{f : C/S \to X \times S \ : \ \substack{C \xrightarrow{\pi} S \ \text{ is flat with geometric fibers at most nodal (connected) curves of genus } g \\ (p_1,\dots,p_n)\colon S \to C \ \text{ smooth markings} \\ f_{s,*}[C_s]=\beta \\ f_s \text{is stable}} \}$$

where $f_s : C_s \to X$ is said *stable* if for any irreduible component $D \subseteq C$ *contracted by* f, i.e. $f(D) = pt$ the curve D marked with $\{p_1, \dots, p_n\} \cap D$ and with the nodes $D \cap \overline{C \smallsetminus D}$ is *stable* in the Deligne-Mumford sense, namely it has at least 3 special point if D is rational and at least 1 if it has genus one.

Remark 11.1.2. The *stability* ensures that $f : C \to X$ has finite autorphism group, i.e. $|\operatorname{Aut}(f)| < \infty$.

One can rephrase stability of a family $f : C/S \to X$ of stable maps as follows: the line bundle $\omega_\pi(\sum \sigma_i) \otimes f^*\mathcal{O}_X(1)$ is ample, where $\pi : C \to S$, σ_i are the n distinct sections and $O_X(1)$ is an ample line bundle on X.

We learnt that moduli functors are rarely representable to schemes due to the presence of automorphisms, but for the moduli space of stable maps the next best thing holds:

Theorem 11.1.3 (Kontsevich)**.** *$\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a proper, Deligne-Mumford stack.*

11.2. **Tautological maps.** Moduli spaces of stable maps, much like moduli spaces of curves comes equipped with tautological morphisms:

forgetful morphism: The morphism

$$\mathrm{ft}_i : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$$

forgets the i-th marking and stabilize the map (See Figure 11.2).

For $i = n+1$, this is the natural projection from the universal curve over the moduli space. We will often use the alternative notation $\overline{\mathcal{C}}_{g,n+1}$ for the universal curve and denote by

$$F : \overline{\mathcal{C}}_{g,n+1} \to X \times \overline{\mathcal{M}}_{g,n+1}(X,\beta)$$

the universal map.

evaluation morphisms: For each marking p_i we get an evaluation map defined by

$$ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X, \qquad ev_i(f : (C, p_1, \dots, p_n) \to X) := f(p_i);$$

gluing morphisms: Let $\beta_1, \beta_2, g_1, g_2$ and n_1, n_2 be such that $\beta_1 + \beta_2 = \beta$, $g_1 + g_2 = g$ and $n_1 + n_2 = n$ then we get

$$\overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times_X \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$$

where the maps defining the fiber product over X are the evaluations ev_{n_1+1} and ev_{n_2+1} (See Figure 11.2).

Figure 2. Forgetful morphism

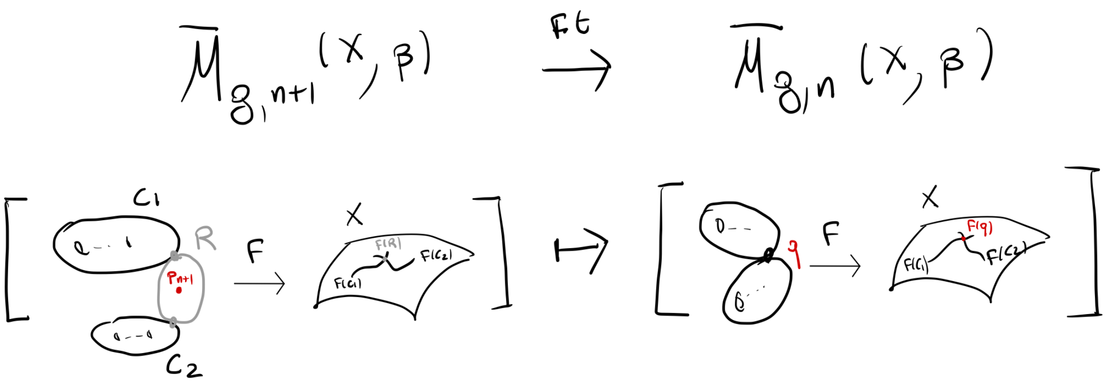
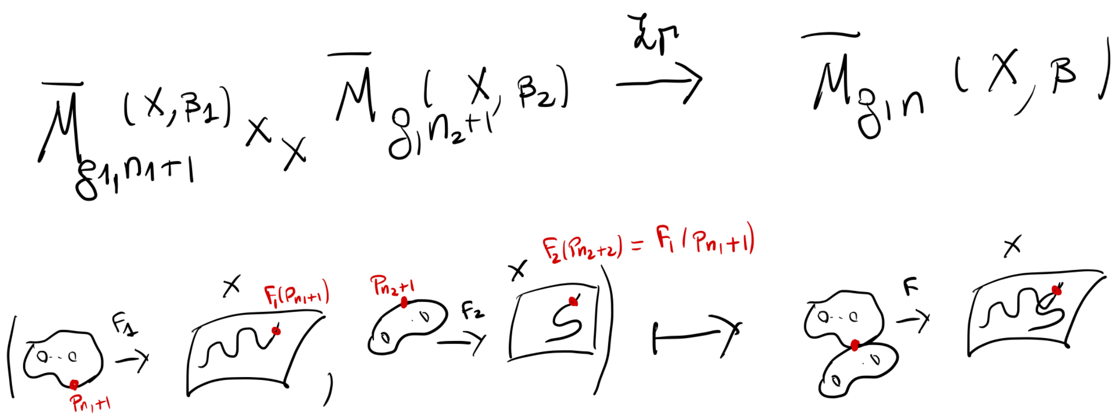


Figure 3. Gluing morphism



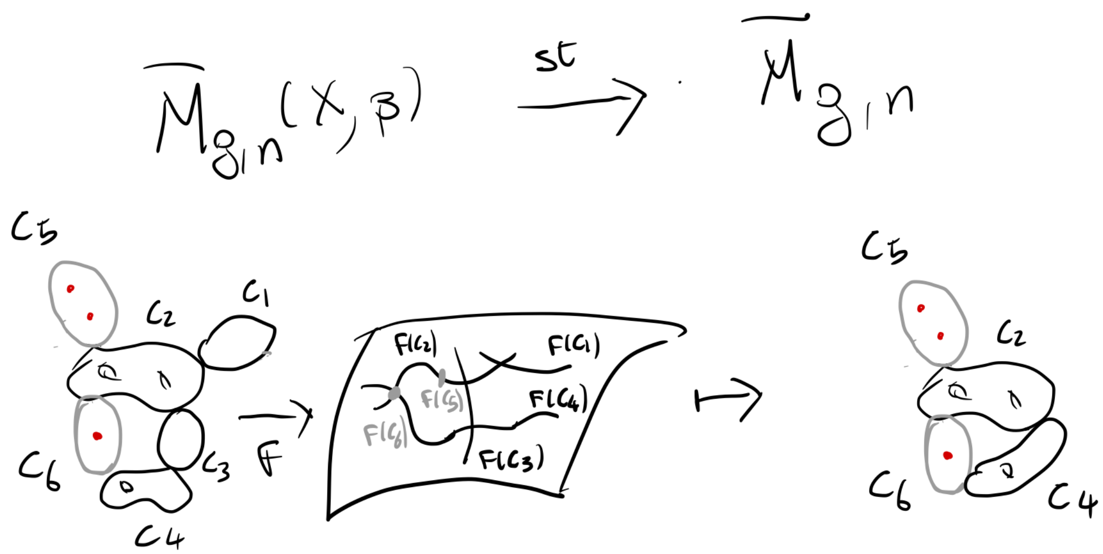
stabilization morphisms: The morphism

$$\mathrm{st}\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$$

is defined forgetting the map and stabilizing the curve (See Figure 11.2).

Exercise 11.2.1. Define the forgetful morphism and the stabilization morphism in families.

FIGURE 4. Stabilization morphism



Hint On a family $C \xrightarrow{\pi} S$ the line bundle $\omega_\pi(\sum x_i)$ fails to be π-ample precisely on those components that need to be contracted. Furthermore, the locus Furthermore, the locus of rational tails $R \subset C$ is a divisor.

If we are happy with the Artin stack $\mathfrak{M}_{g,n}$ of pre-stable curve, we can define a forgetful morphism

$$p \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to \mathfrak{M}_{g,n}$$

which just forgets the map; as we saw in Section6.2 $\mathfrak{M}_{g,n}$ is an algebraic stack so we have tools to understand it.

11.2.2. *Naive definition of the invariants.* Using the *markings* we can now impose our meeting conditions. Indeed we have *evaluation maps*

$$ev_i \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X, \qquad ev_i(f \colon (C, p_1, \dots, p_n) \to X) := f(p_i);$$

which allow us for example to describe the loci of maps where $f(p_i) \in V_i$ for some fixed subvariety V_i, as the intersection $\bigcap ev_i^{-1}(V_i)$.

Assuming that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is *smooth* and proper, the solution to our enumerative problem can then be expressed as the degree of the zero homology class

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)] \cap \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$$

for a_i the cohomology class corresponding to the constraint V_i. Notice that this class has degree 0 when $\sum \deg(a_i) = \dim(a_i)$.

11.3. **Geometry of moduli of maps.**

Warning. *Although proper, the moduli spaces $\overline{\mathcal{M}}_{g,n}(X, \beta)$ are in general far from smooth! They have several irreducible components, these components have different dimensions, and thus talking of the* fundamnetal class *of the moduli space does not quite make sense; it is not even clear in which homological degree, that shoukd be the dimension, such a class should live.*

Example 11.3.1. The moduli space $\overline{\mathcal{M}}_1(\mathbb{P}^2, 3[L])$ has 3 irreducible components M, D_1, D_2 of dimensions, respectively, $9, 109$. M is the closure of the locus of smooth elliptic curves embeed as cubis in $\mathbb{P}^2$; D_1 is the component generically parametrizing a map f from a nodal curve $E \cup_p \mathbb{P}^1$ such that $f(E) = pt$ and $f(\mathbb{P}^1)$ is a nodal cubic; D_2 generically parametrize maps from a curve with two nodes $\mathbb{P}^1 \cup_q E \cup_p \mathbb{P}^1$ where f maps one of the two rational component into a quadric Q, the other into a line L and E is contracted onto one of the two intersection points $Q \cap L$.

Even if we decide to stick with genus zero:

Example 11.3.2. Let us take $X = \mathrm{Bl}_p \mathbb{P}^2$ and $\beta = 3H = 3H' + 3E$ where H' is the class of the strict transform of a line through p and E is the class of the exceptional divisor.

In the moduli space $\overline{\mathcal{M}}_0(X, 3H)$ we have one irreducible components V_1 of dimension 8 given by the clusure of the locus of rational cubics (i.e. they have at least one node) in $\mathbb{P}^2$ *not* passing through p. But we also have a second component V_2 given by the closure of the locus of maps from a nodal curve $C_1 \cup_q C_2$ with $f_*[C_1] = 3H - 2E$ and $f_*[C_2] = 2E$. So a generic $f \in V_2$ is the data of: the strict transform of a cubic in $\mathbb{P}^2$ having a node in p, and a two to one cover $f \colon C_2 \to E \cong \mathbb{P}^1$. This component has also dimension 8.

Exercise 11.3.3.
- Make sure you are happy with the examples above
- The genus one example can be generalised to all the $\mathbb{P}^r$, and all d. What are in general all the irreducible components of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d[L])$?
- The genus zero example can also be generalised in various ways, by considering blow-ups in more than one point or blow-ups of $\mathbb{P}^r$ with $r \geq 2$. Can you find at least one more example with more than one component?

Remark 11.3.4. You might not see this during this week, but logarithmic geometry can be used to *extract* and *resolve the singularities* of the main component of these moduli spaces. The philosophy is explain for example in [ACG+10, Section 4] and [vNS23, Remark 1.4].

Beutiful applications of this principle for moduli of maps in genus one are given in [RSPW19b, RSPW19a].

Message to take home; Even if you are only interested in absolute stable maps, logarithmic geometry is a key tool to study the geometry of these spaces.

11.4. **Genus zero curves in hypersurfaces of $\mathbb{P}^r$.** There are some cases where the moduli space of maps is indeed smooth and the invariants are really the one defined above capping the cohomology classes pulled back along the ev_i with the fundamental class of the moduli space.

Proposition 11.4.1. *The moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d[L])$ is a smooth Deligne-Mumford stack!*

Proof. (sketch) A sufficient condition for a moduli space to be smooth is the *vanishing* of the obstructions to the deformations.

To deform $f\colon C \to X$ we need to deform both C and the map. C we can always deform (Pierrick told us $\overline{\mathcal{M}}_{g,n}$ is smooth in the stacky sense). Once we fix a deformation of C, deforming f is equivalent to deform $\Gamma_f \subset C \times X$. Then, from what we learnt about the Hilbert scheme, the obstructions are contained in $\mathrm{H}^1(\Gamma_f, \mathcal{N}_{\Gamma_f/C\times X}) = \mathrm{H}^1(C, f^*T_X)$.

Exercise 11.4.2. You can now conclude computing $\mathrm{H}^1(C, f^*T^r_{\mathbb{P}})$ for C of genus zero. To do so, it is helpful to look at the pullback to C of the Euler sequence

$$0 \to f^*\mathcal{O}_{\mathbb{P}^r} \to f^*\mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r+1} \to f^*T^r_{\mathbb{P}} \to 0.$$

To prove the vanishing of $\mathrm{H}^1(C, f^*T_{\mathbb{P}^r})$ it is sufficient to argue that $\mathrm{H}^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(1))$.

♣

Exercise 11.4.3. Compute the dimension of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d[L])$.

Hint A stable map is the data of a marked genus zero curve C together with $r+1$ sections of $H^0(C, f^*\mathcal{O}(1))$ (up to a common scalar), up to reparametrization of the curve.

Let $X = V(s)$ be a degree l smooth hypersurface in $\mathbb{P}^r$ i.e. $s \in \mathrm{H}^0(\mathbb{P}^r, \mathcal{O}(l))$. Let us denote by

$$\overline{\mathcal{M}}_{0,n}(X, d[L]) = \bigsqcup_{\{\gamma \in \mathrm{H}_2(X,\mathbb{Z}) \mid i_*\gamma = d[L]\}} \overline{\mathcal{M}}_{0,n}(X, \gamma)$$

where i is the embedding of X in the projective space.

Proposition 11.4.4. *The space $\overline{\mathcal{M}}_{0,n}(X, d[L])$ is embedded inside the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d[L])$ as the zero locus of a section of a vector bundle $\mathcal{E}_{d,n}$*

Proof. Take $\mathcal{E}_{d,n}$ to be $\pi_*(F^*\mathcal{O}(l))$ where F is the universal map and π the projection from the universal curve. The fact that this is a vector bundle follows from the Cohomology and Base change Theorem [Har77, Theorem 12.11] combined with the observation that $\mathrm{H}^1(C, f^*\mathcal{O}(l)) = 0$ for any C of genus zero and any $l \geq 0$. At each point $[(C, p_i), f]$ of the moduli space of maps to $\mathbb{P}^r$, thr fiber of this vector bundle is nothing but $\mathrm{H}^0(C, f^*\mathcal{O}(l)) \cong \mathbb{C}^{dl+1}$. We have a natural section S of $\mathcal{E}_{d,n}$ over the moduli space, defined by

$$ev_{[(C,p_i),f]}(S) = f^*s \in \mathrm{H}^0(C, f^*\mathcal{O}(l)).$$

Notice that the image of f lies in X, i.e. the map factor trough the closed embedding if and only if $f^*s = 0$. In other words, $\overline{\mathcal{M}}_{0,n}(X, d[L]) = V(S)$. ♣

Remark 11.4.5. Notice that if consider moduli spaces of genus one stable maps, then it is no longer true that $\mathrm{H}^1(C, f^*\mathcal{O}(l)) = 0$ for any C and f (think about some examples) and $\pi_*(F^*\mathcal{O}(l))$ is no longer a vector bundle.

11.5. **Quantum Lefschetz.** If we do not ask for the section of the vector bundle to be *generic*, the singularities appearing in schemes Z defined as zero loci of a section of a vector bundle on a smooth scheme A can be quite bad.

Example 11.5.1. Let us consider $A = \mathbb{P}^3$ and $\mathcal{E}$ the vector bundle $\mathcal{O}(2)^{\oplus 2}$. Let $s = (x_1x_2, x_0x_1) \in \mathrm{H}^0(\mathbb{P}^3, \mathcal{E})$ then $Z = V(s)$ has two irreducible componets of different dimensions $V(x_1)$ and $V(x_0, x_2)$ meeting in a point.

The section we choose is not generic, as both its components correspond to singular quadrics, which furthermore do not intersect in the expected dimension.

If we *deform the sections* to get to the generic situation, e.g., $s_\epsilon = (x_1x_2 + \epsilon(x_0^2 + x_3^2), x_0x_1\epsilon(x_1^2 + x_3^2 + x_0x_2))$ then $Z_\epsilon = V(s_\epsilon)$ is a smooth degree 4 curve in $\mathbb{P}^3$ and it has a natural *fundamental class* $[Z_\epsilon] \in A_1(Z_\epsilon)$ such that

$$\iota_*[Z_\epsilon] = 4[L] = c_{\mathrm{top}}(\mathcal{E}) \cap [A] \in A_1(A).$$

Gysin pullback. Even when s is not a regular section, using Fulton's Gysin pull-back

$$0_E^!\colon A_k(A) \to A_{k-\mathrm{rk}(\mathcal{E})}(Z(s)) \tag{17}$$

it is still possible to define a class, wich we denote as

$$[Z(s)]^{\mathrm{vir}} := 0_E^![A] = c_{top}^{ref}(\mathcal{E}) \tag{18}$$

of the *expected dimension.* Furthermore it is still true that

$$\iota_* 0_E^![A] = c_{\mathrm{top}}(\mathcal{E}) \cap [A]$$

[Ful98, Section 14.1]; for this reason $= 0_E^![A]$ is called the *refined top Chern class* or *refined Euler class.*

The Gysin pull-back $0_E^!$ is given by the composition of the maps:

$$A_n(A) \xrightarrow{\sigma} A_n(C_XA) \xrightarrow{\iota_*} A_n(E|_X) \xrightarrow{s^*} A_{n-\mathrm{rk}(E)}(X)$$

where $\sigma([V]) := [C_{X\cap V}V]$ is the specialization morphism defined in [Ful98, Proposition 5.2], ι_* the proper pushforward, and s^* the inverse of the (flat) pull-back [Ful98, Section 3.3]. Here we denoted by $E|_X$ the vector bundle with sheaf of sections $\mathcal{E}$ and by C_XA the normal cone of the embedding of X in A, defined by $\mathrm{Spec}(\oplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1})$ for $\mathcal{I}$ the ideal sheaf of the embedding. Notice that the embedding $C_XA \hookrightarrow E$ comes from the fact that X cut out by a section of E mean that we have a surjection of $\mathcal{E}^\vee \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$.

We saw in Proposition 11.4.4 that $\overline{\mathcal{M}}_{0,n}(X, d[L])$ is the zero locus $V(S)$ of a section of a vector bundle inside the smooth ambient space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d[L])$, which by Exercise 11.4.3 has dimension $r - 3 + d(r+1) + n$.

If S were generic, than $\overline{\mathcal{M}}_{0,n}(X, d[L])$ would be smooth of dimension $\mathrm{vdim} = r - 3 + d(r + 1) + n - \mathrm{rk}(\mathcal{E}_{d,n})$. We call this the *virtual dimension* of $\overline{\mathcal{M}}_{0,n}(X, d[L])$. Rearraging the terms we see that

$$\mathrm{vdim} = (1-g)(\dim(X) - 3) - K_X \cdot d[L] + n.$$

As in the example 11.5.1, even if we can't check the transversality of S, we can define

$$[\overline{\mathcal{M}}_{0,n}(X, d[L])]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(\overline{\mathcal{M}}_{0,n}(X, d[L])),$$

and the class satisfies

$$\iota_*[\overline{\mathcal{M}}_{0,n}(X,d[L])]^{\mathrm{vir}} = c_{dl+1}(\mathcal{E}_{d,n}) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d[L])].$$

Then given $\gamma_1, \dots, \gamma_n \in H^*(X,\mathbb{Z})$ which are pulled back from γ_i' in $\mathbb{P}^r$ and such that $\sum_i \deg(\gamma_i) = \mathrm{vdim}$ we have

$$\begin{aligned}\mathrm{GW}^X_{0,d[L]}(\gamma_1 \cdots \gamma_n) :=& \deg([\overline{\mathcal{M}}_{0,n}(X,d[L])]^{\mathrm{vir}} \cap \mathrm{ev}_1^*(\gamma_1) \cup \dots \cup \mathrm{ev}_n^*(\gamma_n) \\ =& [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d[L])] \cap c_{dl+1}(\mathcal{E}_{d,n}) \cup \mathrm{ev}_1^*(\gamma_1') \cup \dots \cup \mathrm{ev}_n^*(\gamma_n')\end{aligned}$$

Exercise 11.5.2. Lines on the Quintic 3-fold.

In this guided exercise we use the Quanthum Lefschetz to compute the numeber of lines on a quintic 3-fold X_5 in $\mathbb{P}^4$.

The moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, [L])$ is nothing but the Grassmanian $\mathrm{Gr} := \mathrm{Grass}(2,5)$, which comes equipped with a short exact sequence of vector bundles:

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathrm{Gr}}^{\oplus 5} \to \mathcal{Q} \to 0$$

such that at each point $[W] \in \mathrm{Gr}$ we have $L = \mathbb{P}(\mathcal{S}_{[W]}) \subset \mathbb{P}^4$.

- Remember that $\mathcal{S}_{[W]} \to L$ is the tautological line bundle $\mathcal{O}_L(-1)$;
- Show that $\overline{\mathcal{M}}_{0,0}(X_5,[L]) \subseteq \mathrm{Gr}$ is cut out by a section σ_{X_5} of $\mathrm{Sym}^5(\mathcal{S}^*)$ (you have to reinterpret Proposition 11.4.4 in this context)
- Using the above, verify that the virtual dimension of $\overline{\mathcal{M}}_{0,0}(X_5,[L])$ is zero.

The last step to compute the number of lines in X_5 is to compute $\deg(c_6(\mathrm{Sym}^5(\mathcal{S}^*)) \in A_0(\mathrm{Gr})$. To do so, we need to know somenthing about the Chow ring (you can think the homology) of the Grassmanian.

Ingredients to compute the invariants. The Chow cohomology/homology ring of the Grassmanian $\mathrm{Grass}(2, n+1)$ can be explicitly described in terms of Schubert cycles and Pieri's formula, which we recall (Further details and proofs can be found in [EH16, Section 4.3].) Fix $\mathcal{V} := 0 \subset V_1 \subset V_2 \subset \dots V_n \cong K^{n+1}$ a complete flag, and $n-1 \geq a_1 \geq a_2 \geq 0$ two positive integer such that $a_1 + a_2 \leq 2(n+1) - 4 = \dim(\mathrm{Grass}(2,n))$. Then

$$\Sigma_{a_1,a_2}(\mathcal{V}) := \left\{ L \subset K^{n+1} \mid V_{n-1} \cap L \neq 0 \ \text{ and } \ L \subseteq V_{n+1-a_2} \right\}.$$

The class of $\Sigma_{a_1,a_2}(\mathcal{V})$ in the cohomology doe not depend from the choice of the flag. We denote it by

$$\sigma_{a_1,a_2} \in A^{a_1+a_2}(\mathrm{Grass}(2,n+1));$$

these are called Schubert cycles. There is a closed formula for the product of Schubert cycles in $\mathrm{Grass}(2,n+1)$[EH16, Proposition 4.11]: if $a_1 - a_2 \geq b_1 - b_2$

$$\sigma_{a_1,a_2}\sigma_{b_1,b_2} = \sigma_{a_1+b_1,b_2+a_2} + \sigma_{a_1+b_1-1,b_2+a_2+1} + \dots \sigma_{a_1+b_2,b_1+a_2}.$$

Moreover, the total Chern class of $\mathcal{S}^*$, and thus of $\mathrm{Sym}^5(\mathcal{S}^*)$, can be expressed in terms of Schubert cycle. By the splitting principle [EH16, Section 5.4], to compute Chern classes we can pretend $S^* = \mathcal{L} \oplus \mathcal{M}$ for $\mathcal{L}$ a subbundle and $\mathcal{M} = \mathcal{S}^*/\mathcal{L}$

$$c(\mathcal{S}^*) = 1 + (c_1(\mathcal{L}) + c_1(\mathcal{M})) + c_1(\mathcal{L})c_1(\mathcal{M}) = 1 + (\alpha+\beta) + \alpha\beta.$$

On the other hand, interpreting Chern classes as sections of exterior powers of vector bundles we get [EH16, Section 5.6.2]

$$c(\mathcal{S}^*) = 1 + \sigma_{1,0} + \sigma_{1,1}.$$

Now you can use: the splittimg principle for $\mathrm{Sym}^5(\mathcal{S}^*)$, the Pieri's formula and the fact that $\deg(\sigma_{3,3}) = 1$ to compute that there are 2875 lines on the quintic 3-fold.

Exercise 11.5.3. Following the same exact strategy, you can also reprove the classical statement assertying that there are 27 lines on a generic smooth cubic surface $S \subset \mathbb{P}^3$.

What do we need to define the virtual class. In the previous section we saw how to endow with a virtual class a scheme X which is cut out by a section of a vector bundle on a smooth ambient space. The construction can easily be adapted to the case where X is a scheme over some base S and it is cut out by a section of a vector bundle E on a scheme A smooth over S. Indeed, as long as S is equidimensional, it has a well defined fundamental class in $A_{\dim(S)}(S)$ whose flat pull-back give the fundamental class $[A]$ and we can once again use Fulton's Gysin pullback to define a virtual class. To emphasize that the embedding is on a scheme smooth over a base we will write

$$0^!_{E/S} \colon A_{\dim(S)}(S) \to A_{\dim(A)-\mathrm{rk}(E)}(S)$$

where $0^!_{E/S}$ is simply the Gysin pull-back defined in the previous section pre-composed with the flat pull-back from S to A. In this case we will call the normal cone $C_{X/A}$ the *relative* normal cone.

Remark 11.5.4. Let us notice that in order to run the Gysin construction we do not really X to be the zero locus of a vector bundle on A but rather we need *a vector bundle E on X in which the relative normal cone $C_{X/A}$ embeds.*

In general, we do not have an explicit (let alone a somewhat natural) global embedding of a moduli space $\mathcal{M}$, e.g. for the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$, into a smooth ambient space or a smooth ambient space over a base.

The seminal work of Behrend and Fantechi [BF97] allow us nonetheless to construct virtual classes on several moduli spaces of interest, including the Kontsevich moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The key ingredient is a so called (possibly relative)*perfect obstruction theory*.

This is (roughly speaking) a two term complex of vector bundles $E^\bullet$ (well defined as an object in $\mathrm{D}^b(\mathcal{M})$) with the following property. On a neighborhoood $U \to \mathcal{M}$ admitting an embedding $U \hookrightarrow A/S$ into a smooth ambient space (relative to a base S) we have :

$$E^\bullet|_U \cong T_{A/S}|_U \to E_{U/S}$$

and $C_{U/A}$ embeds in $E_{U/S}$ (see [BF97] for the correct definition.)

Given such $E^\bullet$, after taking care of all the subtle technicalities arising in this way more general framework, one can define a generalization of the Gysin pullback, the so called *virtual pull-back* [Man11]

$$0^!_{E^\bullet/S} \colon A_{\dim(S)}(S) \to A_{\dim(S)-\mathrm{rk}(E^\bullet)}(\mathcal{M}).$$

The stable maps case. For the Kontsevich moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ with X smooth, there is a perfect obstruction theory relative to the smooth Artin stack $\mathfrak{M}_g, n$ given by the two term complex $R^\bullet\pi_*F^*T_X$ for F the universal map and π the projection from the universal curve. So in this case $\mathrm{rk}(R^\bullet\pi_*f^*T_X) = h^1(C, f^*T_X) - h^0(C, f^*T_X)$ for any $[C, f]$ (notice that the difference is constant). The latter can be computed using Grothendieck-Riemann-Roch and the virtual pull-back construction gives as a clss

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in A_{(1-g)(\dim(X)-3)-\beta\cdot[K_X]+n}(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

Remark 11.5.5. Notice that the *virtual dimension* is the one suggested by deformation theory, i.e. by the count of parameters. Indeed we have $3g-3+n$ parameterS for the deformation of the marked pre-stable curve; the infinitesimal deformationd of the map once the deformation of the curve has been fixed are parametrized by $H^0(C, f^*T_X)$ and the obstructions to deform the map to a given deformation of the curve are in $H^1(C, f^*T_X)$, so we expect $-h^1(C, f^*T_X) + h^0(C, f^*T_X)$ extra parameters.

11.6. **Gromov–Witten invariants.** We can now define numerical invariants as in the naive approach; we have *primitive Gromov Witten invariants*

$$\begin{aligned}\mathrm{GX}^X_{g,\beta}(\gamma_1\cdots\gamma_n) &:= deg([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \cap \mathrm{ev}_1^*(\gamma_1)\cup\cdots\cup\mathrm{ev}_n^*(\gamma_n)) \\ &= \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \mathrm{ev}_1^*(\gamma_1)\cup\cdots\cup\mathrm{ev}_n^*(\gamma_n) \in \mathbb{Q}\end{aligned}$$

for γ_i cohomology class on X (we get rational number because $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a Deligne-Mumford stack, i.e. there are autotphisms to take into account), but we can also cap $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ with classes (e.g ψ_i classes and λ-classes) pulled-back from the cohomology of $\overline{\mathcal{M}}_{g,n}$ via the stabilization morphism $\mathrm{st}\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$. We thus get a richer set of Gromov-Witten invariants:

$$deg([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \cap \mathrm{ev}_1^*(\gamma_1)\cup\cdots\cup\mathrm{ev}_n^*(\gamma_n)) \cap \mathrm{st}^*(\bigcup \zeta_j)$$

for ζ_j some interesting classes on $\overline{\mathcal{M}}_{g,n}$; these are usually called *descendant Gromov-Witten invaraints* and they appear in computations even if we restrict our interests to the primary case.

12. Gromov-Witten theory II– properties and structures of the invariants

Other properties of the virtual class. The class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ also satisfy all the *expected* compatibility with the tautological maps:

pull-back: We can take the flat pull-back along the forgetful map and get

$$ft^*[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = [\overline{\mathcal{M}}_{g,n+1}(X,\beta)]^{vir}$$

cutting edge: The recursive structure of the moduli space is compatible with wirtual classes in the following sense. The *virtual divisor*

$$\overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1)\times_X \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) = \overline{\mathfrak{M}}_{g_1,n_1+1}\times\overline{\mathfrak{M}}_{g_2,n_2+1}\times_{\overline{\mathfrak{M}}_{g,n}}\overline{\mathcal{M}}_{g,n}(X,\beta)$$

has a natural class given by

$$0^!_{R^\bullet\pi_*F^*T_X}(\iota_*[\overline{\mathfrak{M}}_{g_1,n_1+1}\times\overline{\mathfrak{M}}_{g_2,n_2+1}])$$

for ι the natural map to $\overline{\mathfrak{M}}_{g,n}$. This class coincide with the one obtained intersecting the *split class* $[\overline{\mathcal{M}}_{g_1,n_1+1}](X,\beta_1)]^{vir} \boxtimes [\overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2)]^{vir}$ with the diagonal, i.e. expressing the virtual divisor as the fiber product:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times_X \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) & \longrightarrow & \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \\ \downarrow{\scriptstyle ev_q} & & \downarrow{\scriptstyle ev_p \times ev_s} \\ X & \xrightarrow{\quad\Delta\quad} & X \times X \end{array}$$

we have

$$[\overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times_X \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2)]^{vir} = \Delta^!([\overline{\mathcal{M}}_{g_1,n_1+1}](X,\beta_1)]^{vir} \boxtimes [\overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2)]^{vir})$$

where $\Delta^!$ is the Fulton Gysin pull-back along the regular embedding Δ.

We will use the following notation:

$$D(n_1,n_2;\beta_1,\beta_2) := [\overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times_X \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2)]^{vir}$$

Exercise 12.0.1. Recall that the class of the diagonal $[\Delta]$ in $H^*(X \times X)$ is given by $[\Delta] = \sum_{e,f} g^{ef} T_e \otimes T_f$ for T_e, T_f both running through a base of $H^*(X)$ and $(g^i j)$ the inverse of the intersection form.

Prove that the cutting edge axiom for the virtual class translate into the following equations for the invariants:

$$\int_{[D(n_1,n_2;\beta_1,\beta_2)} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)) =$$
$$\sum_{e,f} g^{ef}\, \mathrm{GX}^X_{g_1,\beta_1}(\gamma_1 \cdots \gamma_{n_1} \cdot T_e)\, \mathrm{GX}^X_{g_2,\beta_2}(\gamma_{n_1+1} \cdots \gamma_n \cdot T_f)$$

mapping to a point: The classes defined from a perfect obstruction theory might not always be what we would expect naively. Consider for example $\overline{\mathcal{M}}_{g,n}(X,0)$; a point in this moduli space is determined by a stable curve and the point in X to which the latter is contracted, i.e $\overline{\mathcal{M}}_{g,n}(X,0) \cong \overline{\mathcal{M}}_{g,n} \times X$. In particular this moduli space is smooth of dimension $3g-3+n+\mathrm{dim}X$. Notice however that the *virtual dimension* is $3g-3+n+\mathrm{dim}X - g\mathrm{dim}X$.

In this case, we can compute explicitly

$$R^\bullet \pi_* F^* T_X = T_{\overline{\mathcal{M}}_{g,n} \times X/\overline{\mathcal{M}}_{g,n}} \to R^1\pi_* \mathcal{O}_{C_{g,n}} \otimes T_X \cong \mathbb{E}^\vee \otimes T_X$$

where $\mathbb{E} = \pi_* \omega_\pi$ is the Hodge bundle. In this case none of the technicalities and complications of the general case arise, and running Fulton's Gysin construction we see that:

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = [\overline{\mathcal{M}}_{g,n}(X,\beta)] \cap c_{\mathrm{top}}(\mathbb{E}^\vee \otimes T_X)$$

Deformation invariance: An important consequence of the definition of the intersection theoretic definition of the Gromov-Witten invariants is their so called *deformation invaraince*. This means that if $\mathcal{X} \to B$ is a smooth family of projective varieties, say over B a smooth curve, then $\mathrm{GX}^{X_b}_{g,\beta}(\alpha_1,\ldots,\alpha_n)$ does not depend on b. If the family of moduli spaces $\overline{\mathcal{M}}_{g,n}(\mathcal{X}/B,\beta) \to B$ were smooth, the deformation invariance would simply reeduce to the constance of intersection numbers [Ful98, Chapter 20]. The virtual class technology ensures the possibility of extending the result to families of moduli of maps.

We will come back to this later when we will mention the degeneration formula.

The properties of the virtual class impose relations among the Gromov-Witten invariants which are crucial to compute them. We make some of these relations explicit and we restrict to the genus zero case.

mapping to a point for $g=0$ **:** When $\beta = 0$ and $\overline{\mathcal{M}}_{0,n}(X,0) = \overline{\mathcal{M}}_{0,n} \times X$ the Gromov-Witten invariants are zero unless $n = 3$. Indeed in this case all the evaluation are equal to the projection p onto X and, by projection formula

$$[\overline{\mathcal{M}}_{0,n} \times X] \cap p^*(\cup\gamma_i) = p_*[\overline{\mathcal{M}}_{0,n} \times X] \cap (\cup\gamma_i).$$

The class $p_*[\overline{\mathcal{M}}_{0,n} \times X]$ is zero unless $n = 3$ and simply $[X]$ in the latter case with $\mathrm{GW}^X_{0,0}(\gamma_1,\gamma_2,\gamma_3) = [X] \cap \gamma_1 \cup \gamma_2 \cup \gamma_3$.

trivial class insertion: If one of the insertion, say $\gamma_1 = 1$, then the invariant vanish as soon as $\beta \neq 0$ and, from the above observation, we have that for $\beta = 0$ only 3-pointed invariant do not vanish and are given by $\mathrm{GW}^X_{0,0}(1,\gamma_2,\gamma_3) = [X] \cap \cup\gamma_2 \cup \gamma_3$.

To see that the invariant vanish for $\beta \neq 0$ we notice that if $\gamma_1 = 1$ then

$$[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir} \cap (\bigcup ev_i^*\gamma_i)$$

is a class pulled-back along $\mathrm{ft}_1 \colon \overline{\mathcal{M}}_{0,n}(X,\beta) \to \overline{\mathcal{M}}_{0,n-1}(X,\beta)$. Since ft_1 has positive dimensional fiber the degree of a class pulled-back from the target must be zero.

divisor axiom: Let the first insertion be the (dual) class of a divisor in X, $\gamma_1 \in A^1(X)$. Then for $\beta \neq 0$

$$\mathrm{GW}^X_{0,\beta}(\gamma_1 \cdots \gamma_n) = (\gamma_1 \cap \beta)\,\mathrm{GW}^X_{0,\beta}(\gamma_1 \cdots \gamma_n).$$

We do not give full details, but this comes from the fact that in an ideal situation (the moduli spaces are smooth and we can choose representatives of the cohomology classes so that the intersections are transverse) we have that

$$[\overline{\mathcal{M}}_{0,n}(X,\beta)] \cap \mathrm{ev}_1^*(\gamma)_1 = [ev_1^{-1}(D_1])]$$

and the forgetful map

$$ev_1^{-1}(D_1) \xrightarrow{ft_1} \overline{\mathcal{M}}_{0,n}(X,\beta)$$

is generically finite of degree $\gamma_1 \cap \beta$ which are the possible ways to map the first point into β. Then the result simply follows by projection formula.

12.1. **WDVV and associativity of the Quantum cohomology.** From the description of the moduli spaces and the properties of the virtual classes we see that Gromov-Witten invariants are related by certain *recursive relations*.

In order to exploit this recursive structure is then often useful to assembly the invariants into *generating series*, rather than consider them one by one. This approach to study Gromov-Witten invariants is particularly convenient to study genus zero invariant for *homogeneus varieties* (e.g projective space, Grassmanian and Flag varieties.)

Let consider $T_0, \dots, T_m$ a basis for $\mathrm{H}^*(X)$; for example, if $X = \mathbb{P}^r$ we can consider the standard basis $T_0 = 1$ is the (dual) of the fundamental class, $T_1 = h$ is the hyperplane class, and $T_r = h^r$ is the class of a point. More in general, for homogeneus varieties basis are giver by Schubert cycles. We will denote by (g_{ij}) the matrix with entries

$$g_{ij} = [X] \cap T_i \cup T_j$$

and by (g^{ij}) its inverse.

We consider the formal power series in $\mathbb{Q}[[\underline{x}]] = \mathbb{Q}[[x_0, \dots, x_m]]$ defined by:

$$\Phi(x_0, \dots, x_m) = \sum_{(n_0,\dots,n_m)} \sum_{\beta} \mathrm{GW}^X_{0,\beta}(T_0^{n_0} \dots T_m^{n_m}) \frac{x_0^{n_0}}{n_0!} \dots \frac{x_m^{n_m}}{n_m!}.$$

Writing $\gamma = \sum x_i T_i$ we can rewrite

$$\Phi(x_0, \dots, x_m) = \sum_{\beta} \mathrm{GW}_{0,\beta}(\exp(\gamma)) = \sum_{n \geq 0} \sum_{\beta} \frac{1}{n!} \mathrm{GW}_{0,\beta}(\gamma^n) \tag{19}$$

Exercise 12.1.1. If you never did it before, look at the (easy) formal computation showing that

$$\exp(\gamma) = \sum_{(n_0,\dots,n_m)} \frac{\underline{x}^n}{n!} \underline{T}^n$$

where $\underline{x}^n = x_0^{n_0} \cdots x_m^{n_m}$ and $n! = n_0! \cdots n_m!$

Denote by

$$I(\gamma^n) = \sum_{\beta} \mathrm{GW}^X_{0,\beta}(\gamma^n)$$

so that

$$\Phi = \sum_{n \geq 0} \frac{1}{n!} I(\gamma^n)$$

Denote by $\Phi_i = \frac{\partial}{\partial x_i} \Phi$ the i-th partial derivative of the genus zero Gromov Witten potential. The usual derivative rules imply that the latter is the generating function of genus zero Gromov-Witten invariants with an extra input class T^i:

$$\Phi_i = \sum_{n \geq 0} \frac{1}{n!} I(\gamma^n \cdot T_i);$$

and thus

$$\Phi_{ijk} = \sum_{n \geq 0} \frac{1}{n!} I(\gamma^n \cdot T_i \cdot T_j \cdot T_k) \tag{20}$$

Remarkably, the potential Φ satisfy the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) differential equations:

$$\sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl} = \sum_{e,f} \Phi_{jke} g^{ef} \Phi_{ifl}, \qquad \text{for all} i, j, l, k$$

Φ being a solution of the equations above is equivalent to the associativity of the so calles *quantum product*, which we know define. On the $\mathbb{Q}[[\underline{x}]]$ module $\mathrm{H}^*(X)\otimes\mathbb{Q}[[\underline{x}]]$ we define the quantum product by the rule:

$$T_i \star T_j = \sum_{e,f} \Phi_{ije} g^{ef} T_f \tag{21}$$

and then extend by $\mathbb{Q}[[\underline{x}]]$-linearly.

Theorem 12.1.2. *With the above definition, $H^*(X)\otimes\mathbb{Q}[[\underline{x}]]$ is a commutative associative $\mathbb{Q}[[\underline{x}]]$ algebra with unit T_0*

Proof. (sketch) The commutativity simply follows drom the fact that Φ_{ijk} is symmetric in the indices.

Seeing that T_0 is the unit for this product is a little bit more work, but in the end it boils down to the mapping to a point and trivial insertion axiom we saw before.

Exercise 12.1.3. Using the mapping to a point axiom and the trivial insertion axiom complete the proof of the fact that T_0 is the unit for the product $\star$.

The associativity of $\star$ is the core of the statement.

We do not provide full details, but we sketch how this follow from the following simple geometric fact: the boundary divisors

$$D(1,2|3,4),\quad D(2,3|1,4)\in\overline{\mathcal{M}}_{0,4}\cong\mathbb{P}^1$$

are linear equivalent, together with the *cutting edge* property for the virtual class discussed above.

For simplicty, we restrict to the case $X=\mathbb{P}^r$ and refer the reader to [FP96, Theorem 4] for the proof of the general case.

For $X=\mathbb{P}^r$ we have that $g^{ef}=1$ if $e+f=r$ and 0 otherwise, thus the associativity relation becomes

$$\sum_{e+f=r}\Phi_{ije}\Phi_{fkl}=\sum_{e+f=r}\Phi_{jke}\Phi_{ifl} \tag{22}$$

Exercise 12.1.4. (1) Verify that

$$(\sum_{i\geq 0}\frac{x^i}{i!}f_k)(\sum_{k\geq 0}\frac{x^j}{j!}g_j)=\sum_{k\geq 0}\frac{x^k}{k!}h_k$$

where $h_k=\sum_{i+j=k}\binom{k}{i}f_i g_j$

(2) Using the product rule above and the expression for Φ_{ijk} given in (20) verify that

$$\sum_{e+f=r}\Phi_{ije}\Phi_{fkl}=\sum_{e+f=r}\sum_{n_1+n_2=n}\frac{n!}{n_1!n_2!}I(\gamma^{n_1}\cdot T_i\cdot T_j\cdot T_e)I(\gamma^{n_2}\cdot T_f\cdot T_k\cdot T_l)$$

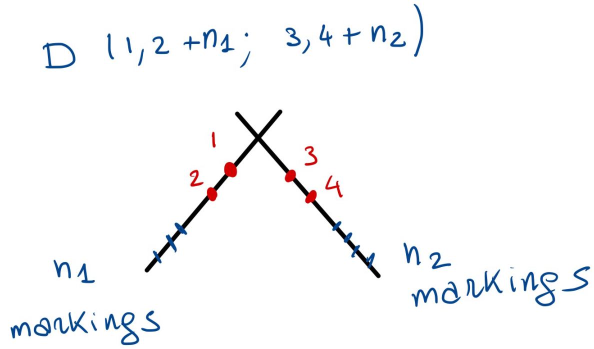
For any $n \geq 0$, $\beta = d[L]$ curve class in $\mathbb{P}^r$ we consider the morphism

$$\overline{\mathcal{M}}_{0,n+4}(\mathbb{P}^r, \beta) \xrightarrow{st} \overline{\mathcal{M}}_{0,n+4} \xrightarrow{ft} \overline{\mathcal{M}}_{0,4}$$

given forgetting the map and all the markings except the last 4 and stabilizing. Since $D(1,2|3,4)$ and $D(2,3|1,4)$ are linear equivalent in $\overline{\mathcal{M}}_{0,4}$, so are their pull-back to $\overline{\mathcal{M}}_{0,n+4}(\mathbb{P}^r, \beta)$, which, by a slight abuse of notation, we keep detoniting with $D(1,2|3,4)$ and $D(2,3|1,4)$.

Notice that the pull-back $ft^*D(1,2|3,4)$ of these divisors to $\overline{\mathcal{M}}_{0,n+4}$ is the sum of $\frac{n!}{n_1!n_2!}$ irreducible components, corresponding to the ways of distributing the remaning $n = n_1 + n_2$ markings between the two component of the curve (See Figure 12.1). Further pulling back to

FIGURE 5. Generic point of a component in $D(1,2|3,4)$



$\overline{\mathcal{M}}_{0,n+4}(\mathbb{P}^r, \beta)$, each irreducible component of $ft^*D(1,2|3,4)$ splits accordingly to the possible distributions of the degree of the map.

Exercise 12.1.5. (1) Use the cutting edge axiom to verify that

$$\int_{st^* ft^* D(i,j|k,l)} ev_i^*T_i \cdot ev_j^*T_j \cdot ev_j^*T_k \cdot ev_k^*T_k \cdot \underline{ev}^*(\gamma^n) =$$

$$\sum_{\substack{\beta_1+\beta_2=\beta \\ n_1+n_2=n}} \frac{n!}{n_1!n_2!} \Big(\sum_{e+f=r} \mathrm{GM}_{0,\beta_1}(\gamma^{n_1} \cdot T_i \cdot T_j \cdot T_e) \, \mathrm{GM}_{0,\beta_2}(\gamma^{n_2} \cdot T_l \cdot T_k \cdot T_f) \Big)$$

(2) Taking the sum of the integral above over all β and $n > 0$ idetify it with the right hand side of (22) and thus conclude the proof of the associativity from the fact that $D(i,j|k,l), \;\; D(j,k|i,l) \in \overline{\mathcal{M}}_{0,4}$ are linearly equivalent.

♣

12.2. **Kontsevich Recursive formula.** Although hystorically Kontsevich recursion formula pre-dated the definition of quathum cohomolohy, (the main idea used to prove associativity is in fact taken from Kontsevich's work) we will go backwords and use the associativity to prove the recursion formula.

To do so, it is convenient to decompose the potential as

$$\Phi = \Phi^{cl} + \Gamma = \sum_{n\geq 0} \frac{1}{n!} \mathrm{GW}_{0,0}(\gamma^n) + \sum_{n\geq 0} \frac{1}{n!} \sum_{d>0} \mathrm{GW}_{0,d[L]}(\gamma^n).$$

We concentrate on $X = \mathbb{P}^2$ and choose as a bases for the cohomology $T_0 = 1,$, $T_1 = h$ the class of a line and $T_2 = h^2$ the class of a point.

Step 1: Using the mapping to a point deduce that the

$$\Phi^{cl}_{ijk} = \mathrm{GW}_{0,0}(T_i \cdot T_j \cdot T_k)$$

are the structure constants of the usual cup product in cohomology, i.e.

$$T_i \cup T_j = \sum_{e+f=2} \Phi^{cl}_{ije} T_f$$

Step 2: Using Step 1, the mapping to a point axiom, and separating the classical from the quantum part verify that the quantum table multiplication for $\mathbb{P}^2$ is as follows:

$$\begin{aligned} T_1 \star T_1 &= T_2 + \Gamma_{111} T_1 + \Gamma_{112} T_0 \\ T_1 \star T_2 &= \Gamma_{121} T_1 + \Gamma_{122} T_0 \\ T_2 \star T_2 &= \Gamma_{221} T_1 + \Gamma_{222} T_0 \end{aligned}$$

Step 3: From the associativity relations $(T_1 \star T_1) \star T_2 = T_1 \star (T_1 \star T_2)$ and $(T_1 \star T_2) \star T_2 = T_1 \star (T_2 \star T_2)$ extract the following relation

$$(\dagger) \Gamma_{222} + \Gamma_{111}\Gamma_{122} = \Gamma_{112}\Gamma_{112}$$

which comes from equating the T_0 terms in the first equation (or the T_1 terms in the second)

Step 4: We know want to write down the series Γ_{ijk} more explicitly.

To do so, it is convenient to go back to the first expression we had for the potential $\Phi = \Phi(x_0, x_1, x_2)$ (19) and notice that: by the m apping to a point axiom there is no contribution every time that there is a T_0 factor; moreover, by the divisor axiom the Gromov–Witten invariant with a T_1 axiom are completely determined by the ones without. This allows to reduce to

$$\Gamma_{ijk} = \sum_{n\geq 0} \frac{x^n}{n!} \sum_{d>0} \mathrm{GW}_{0,d}(T_2^n \cdot T_i \cdot T_j \cdot T_k)$$

Use the product rule for generating series to show that the relation (†) from step 3 correspond to

$$\begin{aligned} \sum_{d>0} \mathrm{GW}_{0,d}(T_2^n \cdot T_2 \cdot T_2 \cdot T_2) + \sum_{n_1+n_2=n} \frac{n!}{n_1! n_2!} (\sum_{d_1>0} \mathrm{GW}_{0,d_1}(T_2^{n_1} \cdot T_1 \cdot T_1 \cdot T_1))(\sum_{d_2>0} \mathrm{GW}_{0,d_2}(T_2^{n_2} \cdot T_1 \cdot T_2 \cdot T_2)) \\ = \sum_{n_1+n_2=n} \frac{n!}{n_1! n_2!} (\sum_{d_1>0} \mathrm{GW}_{0,d_1}(T_2^{n_1} \cdot T_1 \cdot T_1 \cdot T_2))(\sum_{d_2>0} \mathrm{GW}_{0,d_2}(T_2^{n_2} \cdot T_1 \cdot T_1 \cdot T_2)) \end{aligned}$$

Step 5: Use the computation of the dimension for $\overline{M}_{n+3}(\mathbb{P}^2, d)$ and the fact that the invariant is non zero only if the degree of the insertion match the condimension to see that the only non trivial contribution are for

$$n = 3d + 2 - i - j - k$$

Step 6: Recall that by definition the number N_d of rational degree d curves in $\mathbb{P}^2$ through $3d-1$ points in general position is

$$N_d = \mathrm{GW}_{0,d}(T_2^{3d-1}).$$

Then substituting in the recursion formula of Step 4 the relevant values of n computed in Step 5 and applying the divisor axiom, verify that we find precisely Kontsevich's recursion formula

$$N_d + \sum_{d_1+d_2=d} \frac{(3d-4)!}{(3d_1-1)!(3d_2-3)!} d_1^3 d_2 N_{d_1} N_{d_2} =$$
$$\sum_{d_1+d_2=d} \frac{(3d-4)!}{(3d_1-2)!(3d_2-2)!} d_1^2 d_2^2 N_{d_1} N_{d_2}$$

Exercise 12.2.1. Following the same exact strategy find the Kontsevich's recursion formula for $\mathbb{P}^1 \times \mathbb{P}^1$

12.3. **Virtual Localization.** We have now seen some tools to compute invariants in the case of genus zero. There is another special geometric situation where we have a sharp computation tool at our disposal, namely when the target variety X admits a torus action. It was proved by Graber-Pandaripandhe [GP99] that the classical Atiyah-Bott localization formula admits a generalization for virtual classes. This has found vast applications in enumerative geometry and it would be difficult to give an exhaustive account. Already in [GP99] the authors apply the localization to compute higher genus Gromov-Witten invariants of $\mathbb{P}^n$ and to perform various multiple covers calculations for local curves.

12.3.1. *Equivariant cohomology and Atiyah-Bott.* Given $T = \mathbb{G}_m$ an algebraic torus, let us denote by BT its classifying stack and by $ET \to BT$ the universal T-torsor. In more classical algebraic topology terms we can think of $ET = \mathbb{C}^\infty \setminus 0$ and $BT = \mathbb{P}^\infty$.

For X a variety with a T- action, the T-equivariant cohomology $H_T^*(X; R)$ is defined to be the cohomology of the space $X_T = ET \times X/T$ where T acts anti-diagonally on $ET \times X$; notice that the action is free. The structure map $X_T \to BT$ induces a

$$H_T^*(pt; R) = H^*(BT; R) = R[u]$$

module structure on X_T.

Example 12.3.2. Consider the action of $\mathbb{G}_m$ on $\mathbb{P}^r$ given by

$$t \cdot [x_0 : \cdots : x_r] = [t^{w_0} x_0 : \cdots : t^{w_r} x_r]$$

for some integers w_i. Then $\mathbb{P}^r_T \to BT$ is the $\mathbb{P}^r$ bundle $\mathbb{P}(\mathcal{O}(w_0) \oplus \ldots \mathcal{O}(w_r)) \to \mathbb{P}^\infty$. It is well know that for $\mathbb{P}(E) \to X$ we have that $H^*(\mathbb{P}(E))$ is the $H^*(X)$ algebra generated by the class H of the relative hyperplane with the relation

$$H^{r+1} + c_1(E)H^r + \dots c_{r+1(E)} = 0.$$

For $E = \oplus_{i=0}^r \mathcal{O}_{\mathbb{P}^\infty}(w_i)$ we have that the total Chern class is $c(E) = \prod_{i=1}^{r+1}(1 + w_i u)$. We obtain a complete description of the equivariant cohomology.

Given $E \to X$ is a T-equivariant vector bundle we get a vector bundle $E_T \to X_T$; one can thus define equivariant Chern classes as follows

$$c_{k,T}(E) = c_k(E_T) \in H^{2k}(X_T) = H^{2k}_T(X)$$

and similarly for the equivariant Chern characters and Euler class.

Example 12.3.3. A one dimensional rapresentation of T, i.e. $E = \mathbb{C}$ with action $v \to t^a v$ in an equivariant vector bunsle on a point. In this case one gets that $E_T = \mathcal{O}_{\mathbb{P}^\infty}(-a)$ (recall that the action is antidiagonal) and thus $c_1(E_T) = -au$.

Before stating Atiyah-Bott localization formula, we need a couple more observation. The map $X_T \to BT$ is a fibration with fiber X and for X proper we can (and there is a way to do it carefully) define an *integration along the fibers* map

$$\int_{X_T} : H^*_T(X) \to H^*_T(pt)$$

which should be thought as capping with the class of the fibers. We do not define this properly since Atiyah-Bott localization formula (stated below) allow us to replace this integral with integrals on the fixed loci, which admit a very explicit description.

12.3.4. Let us suppose that X is a smooth variety and the fixed locus X^T of the torus action decompose as a disjoint union of smooth subvarieties $X_1, \dots X_N$.

Notice that since the action of T on the X_i is trivial $H^*_T(X_i) = H^*(X_i; R) \otimes R[u]$. This allow us to express each cohomology class as a sum of simple tensors. Moreover, since $X_{i,T} = X_i \times BT$, by Kunneth the integration along the fibers reduces to the usual cap product on the X_i factor.

Notice furthermore that the normal bundles N_i are naturally T-equivariant vector bundles on the X_i.

We can finally state Atiyah-Bott localization

Theorem 12.3.5. *In the situation above we*

$$\int_{X_T} \alpha = \sum_{j=1}^{N} \int_{X_j} \frac{i_j^* \alpha}{e_T(N_j)}$$

for $i_j \colon X_j \to X$ the closed embedding, where $e_T(N_j)$ is invertible in the localised ring $H^*_T(X; \mathbb{Q}) \otimes_{\mathbb{Q}[u]} \mathbb{Q}(u)$.

12.3.6. *Virtual localization (in the baby case).* Remarkably, in [GP99] the authors show that the result of Atiyah-Bott can be extended to the case of virtual classes in the following setting:

Setting. X is a scheme or a DM stack which comes with a global embedding $X \subseteq Y$ in a smooth ambient space (this hypothesis can now be removed [?]), equivariant with respect the action of a torus T. Suppose furthermore that X has a perfect obstruction theory $\phi\colon E^\bullet \to \mathbb{L}_X^\bullet$ [BF97] and that the T- action admits a lifting to an action on $E^\bullet$ making ϕ a T-equivariant morphism in the derived category.

For each component X^i of the fixed locus X^T let $E_i^\bullet$ denote the restriction of the obstruction theory. The T-action induces a $\mathbb{Z}$-grading on $E_i^\bullet$, i.e. a eigendecomposition. Denote by $E_i^{\bullet,f}$ and $E_i^{\bullet,m}$ the fixed part (degree 0) and the moving part (degree $\neq 0$) respectevely. In [GP99] the authors show that $E_i^{\bullet,f}$ is a perfect obstruction theory for X_i and that defining $N_i^{vir} = E_i^{\bullet,m}$ the following localization formula holds in $H_{*,T}(X;\mathbb{Q}) \otimes_{\mathbb{Q}[u]} \mathbb{Q}(u)$.

Theorem 12.3.7.

$$[X]^{vir} = \sum_{j=1}^{N} \frac{i_{j,*}[X_j]^{vir}}{e_T(N_j^{vir})}$$

for $i_j\colon X_j \to X$ the closed embedding.

In the baby case where X is the zero locus of an equivariant section of a vector bundle V on a smooth scheme Y we can express the terms of the localization formula in an elementary way.

Let Y_i the components of the fixed locus Y^T and $X_i = X \cap Y_i$ The restriction $V_i = Y|_{Y_i}$ decomposes in eigenbundles and we can define as above the fixable and movable part. We have the following equalities in $H_{*,T}(X;\mathbb{Q}) \otimes_{\mathbb{Q}[u]} \mathbb{Q}(u)$

$$[X_i]^{vir} = e^{ref}(V_i^f)$$

$$N_i^{vir} = \frac{e(N_{Y_i/i})}{e(V_i^m)}$$

$$e^{ref}(V) = \iota_* \sum \frac{e^{ref}(V_i^f) \cap e(V_i^m)}{e(N_{Y_i/i})}$$

where the refined Euler class is the one defined in (18). In this baby case, the virtual localisation formula is simply a consequence of the Atiyah-Bott localization formula on the smooth ambient Y, together with the naturality of the refined Euler class.

12.3.8. *Fixed loci in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.* We will not say how to prove the virtual localization formula in the general case. However, giving it for granted, let us sketch how to use in order to compute higher genus Gromov-Witten invariants on $\mathbb{P}^r$.

Let $W = \mathbb{C}^{r+1}$, $\mathbb{P}^r = \mathbb{P}(W)$ and $T = \mathbb{C}^*$ acting on W with generic weights $w_0, \dots, w_r$, so that the fixed points of the action coincide with the fixed points $p_0, \dots, p_r$ of the natura $(\mathbb{C}^*)^r$ action.

This induces, by post-composition, an action on the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.

For a map $f\colon C \to \mathbb{P}^r$ to be fixed by the T-action, one needs that: the image of C is a T-fixed curve in $\mathbb{P}^r$; the image of all marked points, nodes, contracted components and ramification points are T-fixed points.

Since the only fixed points are $p_0, \dots, p_r$ and the only T-invariant curves are the lines joining two of this points, each non contracted component $C_{d_i} \subseteq C$ of a T-invariant stable maps is a cover of a T-invariant line ramified over its two torus fixed points.

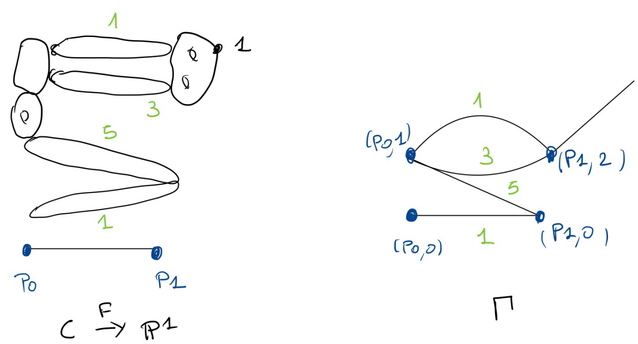
This implies that C_{d_i} has to be rational and totally ramified over the torus fixed points; in particular the map $f|_{C_i}$ is completed determined by the degree.

12.3.9. *Marked graphs.* The above discussion of T-fixed maps allows to identify components of the fixed locus with marked graphs.

Let f be a T-fixed map; define a marked graph Γ as follows (see Figure 12.3.9):

- Γ has an edge e for each non contracted, labelled by the degree d_e
- Γ has a vertvex v for each connected component of $f^{-1}(\{p_0, \dots, p_r\})$ labelled by $i(v) = p_{f(v)}$ $i\colon V(\Gamma) \to \{p_0, \dots, p_r\}$ and by the genus $g\colon V(\Gamma) \to \mathbb{Z}_{\geq 0}$. Notice that there might be connected components in $f^{-1}(\{p_0, \dots, p_r\})$ consisting of a single point; in this case $g(v) = 0$
- Γ has a leg for each marking
- an edge (or a leg) and a vertex are incident if the corresponding subschemes are incident in C.

FIGURE 6. The marked graph associated to a T-fixed point



Warning. Γ *is not the dual graph of the source curve.*

Let denote by

$$\overline{M}_\Gamma = \prod_{v \in V(\Gamma)} \overline{M}_{g(v), \mathrm{val}(v)}$$

where $\overline{M}_{0,n}$ for $n = 1, 2$ is interpreted as a point. Consider the quotient stack $\overline{M}_\Gamma$ by $A_\Gamma \cong \mathrm{Aut}(\Gamma) \rtimes \prod_e \mathbb{Z}/d_e\mathbb{Z}$ where $\prod_e \mathbb{Z}/d_e$ acts trivially. Then

Proposition 12.3.10. [GP99] *There are natural maps $\overline{M}_\Gamma \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ factoring through the closed embeddings $[\overline{M}_\Gamma / A_\Gamma] \hookrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.*

The T fixed loci are supported on such substacks.

In order to apply the localization formula it is then necessary to analize the fixed and movable part of the perferct obstruction theory $R\pi_* F^* T_{\mathbb{P}^r}$.

The analysis is carried out in [GP99, Section 4] and we do not get into it here. In particular the authors show that the fixed loci are unobstructed.

We just conclude by saying that the upshot is the following explicit formula for the higher genus Gromov-Witten invariants of $\mathbb{P}^r$

$$\mathrm{GW}^{\mathbb{P}^r}_{g,d}(h^{l_1}\dots h^{l_n}) = \sum_{\Gamma} \frac{1}{|A_\Gamma|}\int_{\overline{M}_\Gamma} \frac{\prod_{m=1}^n w_{i(m)}^{l_m}}{e(N_\Gamma^{vir})}$$

and $e(N_\Gamma^{vir})$ itself admits an explicit expression in terms of equivariant Chern classes of the Hodge bundle and powers of weights of the T-action.

The one message to take home: *localization allows to reduce the computation of the invariants to integrals on (smooth!) moduli spaces of curves which are, at least in principle, more computable.* [20]

←20

12.4. **Degenerations– a step logarithmic and orbifold Gromov-Witten theory.** The last computational technique we want to mention, which will naturally lead us towards logarithmic Gromov-Witten invariants, is the celebrated *Degeneration–Formula*.

We saw before that the Gromov-Witten invariants are unchanged for deformations of the target variety X. Suppose then to consider a one parameter family $\mathcal{X} \to \mathbb{A}^1_t$ such that $\mathcal{X}_t$ is smooth projective for $t \neq 0$ and $\mathcal{X}_0 = Y \cup_D Z$ is the union of two smooth varieties along a smooth divisor D.

It would be useful if we could compare the Gromov-Witten theory of $\mathcal{X}_t$ and $\mathcal{X}_0 = Y_1 \cup_D Y_2$, which hopefully can be expressed in terms of some Gromov- Witten theories for pairs (Y_i, D), which ideally can be in turn read off from the Gromov-Witten theory of Y_i andd D. The advantage is in the fact that one expect that Y_i, D will be simpler targets; e.g. we can let an elliptic curve degenerate to a union of two $\mathbb{P}^1$ along $D = 0 + \infty$.

Making the above vague idea into a mathematically honest and fruitful technique required a great deal of work:

- There is the need to define invariants for the singular space $\mathcal{X}_0$ in such a way that they agree with invariants of $\mathcal{X}_t$ and to define invariants for *relative theories* [Li01, AF16, AC14, GS13].
- One need a degeneration formula relating the invariants for $\mathcal{X}_0$ to those of (Y_i, D) [Li02, AF16, Che14a] as well as techniques to compute the relative invariants.
- More broadly, one might have to consider more general degenerations which require more sophisticated relative theories and more sophisticated degeneration formulae.. This is a more recent and exciting story and you will hear more about it from Hulya and Dan!

12.4.1. *Moduli of pre-deformable stable maps to expanded targets.* Let $\mathcal{X} \to \mathbb{A}^1_t$ a proper one parameter family degeneration as above, with $\mathcal{X}_0 = Y_1 \cup_D Y_2$.

12.4.1.1. **Firts Issue.** One can consider Kontsevich's moduli space of stable maps to

$$\overline{\mathcal{M}}_{g,n}(\mathcal{X}/\mathbb{A}^1;\beta) \to \mathbb{A}^1.$$

We however do not have a perfect obstruction theory (relative to $\mathbb{A}^1$) allowing the definition of the invariants for $\overline{\mathcal{M}}_{g,n}(\mathcal{X}_0;\beta)$ and allowing the comparison.

Problems arise at those points parametrizing *degenerate maps*, i.e. maps $f: C \to \mathcal{X}_0$ where some irreducible component of the curve C falls into the singular locus D.

[20] (Francesca) might be worth to add a very easy explicit example

A priori there is no way to avoid these maps if we want a moduli space proper over $\mathbb{A}^1$ as one can easily have a family of maps

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ S \cong A^1_t & \longrightarrow & \mathbb{A}^1_t \end{array}$$

where for $t = 0$ the map is degenerate.

Exercise 12.4.2. Find an example of a degenerate map.

Hint Take $\mathcal{X} \to \mathbb{A}^1$ a family where for $t \neq 0$ we have $\mathbb{P}^2$ and for $t = 0$ we have the union of $\mathbb{P}^2$ and $\mathbb{F}^1$ along a toric line D. Take $C = \mathbb{P}^1 \times \mathbb{A}^1$ and a map f_t of degree 2 which is an embedding for $t \neq 0$ and becomes the $2:1$ cover of D when $t = 0$.

12.4.2.1. **Solution.** Jun Li's idea was to allow *expansions* of the target, i.e. different degenerations $\widetilde{\mathcal{X}} \to \mathbb{A}^1$ of $\mathcal{X}_t \to \mathbb{C}^*$ such that the stable maps limit in the modify target $f\colon C \to \widetilde{\mathcal{X}}$ would be no longer degenerate.

Exercise 12.4.3. Consider the Example of the previous exercise. You can take as an explicit model for the target for example $\mathcal{X} = \mathrm{Bl}_{t=x_0=0}\, \mathbb{P}^2 \times \mathbb{A}^1$ (then the family of maps you can choose accordingly.)

- Modify $\mathcal{X}$ by first considering the base change $\mathcal{X}'$ along $\mathbb{A}^1_s \to \mathbb{A}^1_t, t = s^2$ and then taking $\widetilde{\mathcal{X}}$ a resolution of the singularities in the total space.
- Verify that the central fiber of $\widetilde{\mathcal{X}} \to \mathbb{A}^1$ has an extra componend $\mathbb{P}(N_{D_1/Y_1} \oplus \mathcal{O})$ in between $(Y_1, D_1), (Y_2, Y_2)$
- Compute the limit of $\mathbb{P}^1 \times \mathbb{A}^1_s \xrightarrow{f_s} \widetilde{\mathcal{X}}$ and verify that this is no longer degenerate.

Remark 12.4.4. To construct the expansion of the target the idea is always to take a ramified base change $t = s^r$ and then resolve the singularities of the total space, which will have the effect of inserting $r - 1$ $\mathbb{P}^1$-bundles between the components (Y_1, D_1), (Y_2, D_2) of the central fiber. These extra components are called *accordions*.

These are the easisest example of *tropical expansions* or *expanded degenerations*[Ran22, MR24], nowadays ubiquitus in logarithmic enumerative geometry.

Jun Li [Li01] then shows that, given a one parameter family of maps $f\colon C/\operatorname{Spec}(K) \to \mathcal{X}/\mathbb{A}^1 \setminus 0$ there exist a minimal r such that the limit $\tilde{f}\colon \widetilde{C}/\operatorname{Spec}(R) \to \mathcal{X}[r]/\mathbb{A}^1$ is not degenerate. Moreover such r only depends on the family of maps on the puncture disc, suggesting that a moduli space parametrizing non degenerate maps to expanded degenerations will have the desired property of being proper.

The first step to construct such a moduli space, completed in [Li01, Section 1], is to constrcut a space parametrizing all the *expanded degenerations*.

This has the structure of a smooth Artin stack over $\mathbb{A}^1$, let us dnote it by $\mathrm{Exp}(\mathcal{X}, D)$[21] and comes with a universal family $\mathfrak{X} \to \mathrm{Exp}(\mathcal{X}, D)$ or target expansion.

12.4.4.1. **Second Issue.** A further complication arise from the fact that if one consider all non degenerate maps to $\mathfrak{X}/\mathrm{Exp}(\mathcal{X}, D)$ the fiber over 0 will include morphism that cannot arise as a degeneration of a map $f\colon C \to \mathcal{X}_t$. This happen when smooth points of the curve are mapped into the singular locus.

Since one is interested in building a degeneration of the moduli space of maps to the smooth target, these morphisms should be excluded.

12.4.4.2. **Solution.** To achieve that, [Li01] introduce the notion of *pre-deformable maps.*

Definition 12.4.5. A map $f\colon C \to \mathcal{X}[r]$ is called *pre deformable* if: it is *non degenerate*, $f^{-1}(\mathcal{D}) \subseteq C^{sing}$, i.e. only nodes are mapped into the singular locus $D[r] = \sqcup_{l=1}^{r} D_l$, and satisfies the *kissing condition*. This means that if a node p maps into D_l joining the components Y_l and Y_{l+1}, then the order of f along p computed on the branch C^- mapping to (Y_l, D_l) or computed on the branch C^+ mapping to (Y_{l+1}, D_l) coincide.

Finally, to get a moduli space which is a proper Deligne-Mumford stack one needs to impose a notion of stability for pre-deformable maps. As usual the notion of stability is to ensure that $\mathrm{Aut}(f)$ is finite.

Here one should be a litthe careful since a map is the data of a diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Exp}(\mathcal{X}, D) \end{array}$$

and $\mathrm{Exp}(\mathcal{X}, D)$ is an Artin stack.

Stability. Unravelling what this means, one sees that the stability condition says the following: a map $f\colon C \to \mathcal{X}[r]$ is stable if every rational components which is either contracted or mapped into a fiber of an *accordion* has at least three special points.

Theorem 12.4.6. [Li01] *The moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{X}/\mathbb{A}^1; \beta)$ of predeformable stable maps to expanded degeneration of $\mathcal{X}/\mathbb{A}^1$ is a proper DM stack over $\mathbb{A}^1$.*

Remark 12.4.7. The fiber over 0 is the moduli space of predeformable stable maps to expansion of $\mathcal{X}_0$, which we denote by $\overline{\mathcal{K}}_{g,n}(\mathcal{X}_0; \beta)$.

The fiber over $t \neq 0$ is the usual moduli space of Kontsevich stable maps to the smooth target $\mathcal{X}_t$.

As for the usual case of stable maps, in order to then compute invariants one need to define a virtual class for these moduli spaces.

This is done, via a delicate deformation theory nalysis in [Li01]. The main issue is that the condition for a morphism to be predeformable is *closed* inside the space of maps. This causes trouble and does not allow to use directly the Behrend-Fantechi [BF97] obstruction theory.

[21] The notation is borrowed from [Ran22] where the stack of expanded degenerations for more general families $\mathcal{X} \to \mathbb{A}^1$ is constructed

These difficulties lead to the deevelopment of new techniques to deal with degeneration of maps, chief among them the theory of stable maps to orbifold [AF16] and logarithmic stable maps [ACG+10, AC14, GS13]. Dan and Hulya will explain more on these topics.

We conclude by stating the degeneration formula in the Jun Li's setting without discussing further the issue related to defining a suitable obstruction theory.

12.4.8. *Relative stable maps.* We want to describe maps to $\mathcal{X}_0$ as the data of a map to Y_1 and a map to Y_2 which glue along the joining divisor D. This lead tp the problem of constructed moduli spaces of *relative stable maps.*

Let Y be a smooth projective variety and D a smooth divisor. Consider $\mathcal{M}_{g,n+m,(c_i)_{i=1}^n}(Y|D;\beta)$ the space of stable maps satisfying

$$f^*\mathcal{O}_Y(D) = \mathcal{O}_C(\sum_{i=1}^n c_i x_i)$$

for x_i marked point of C. Geometrically, these are maps whose image intersect D with prescribed tangency. This moduli space is clearly not proper: in a one parameter family it can happen that some components of the curve falls into the divisor D and we cannot longer talk about contact order, or that the different point of tangencies come together.

We will denote by Γ the choice of the discrete data for the relative map, i.e.

$$\Gamma = (g, n+m, (c_i)_{i=1}^n, \beta)$$

is the data of the genus, the marked points–both usual and with non trivial contact order along the divisor–the contact orders and the curve class.

We will write $\mathcal{M}_\Gamma(Y|D)$ for the moduli space.

Exercise 12.4.9. Find Examples of these phenomena. It suffices to take $Y = \mathbb{P}^2$ and $D = L$ a coordinate line.

Once again though, one can consider *expansions* $Y[r]$ of (Y, D), which consists of Y union a lenght $r-1$ accordion, i.e. $Y[r] = Y \cup_D \mathbb{P}_D(\mathcal{O}(D)+\mathcal{O}) \cup_{D_2} \mathbb{P}_{D_2}(\mathcal{O}(D_2)+\mathcal{O}) \cdots \cup_{D_{r-1}} \mathbb{P}_{D_{r-1}}(\mathcal{O}(D_{r-1})+\mathcal{O})$.

Then consider predeformable stable maps to expansion which have the prescribed order of tangency at the markings with $D[r] \subseteq Y[r]$ [Li01].

Again one can construct an Artin stack $\mathfrak{M}$ with a universal family $(\mathfrak{Y}, \mathfrak{D})$ parametrizing expansions of Y, D

Theorem 12.4.10. [Li01] *There is a proper DM stack $\overline{\mathcal{M}}_\Gamma(Y|D)$ parametrizing relative stable maps to expansions and containing $\mathcal{M}_\Gamma(Y|D)$ as a open.*

Remark 12.4.11. As for the case of maps to degeneration, since teh condition for a map to be pre-deformable is closed, it is challenging to construct a virtual class on these proper moduli space. The theory of maps to orbifold and the theory of logarithmic stable maps are once again the way to proceed.

This comes equipped with evaluation maps for the *markings with non trivial contact order*

$$\overline{\mathcal{M}}_\Gamma(Y|D) \xrightarrow{ev_{x_i}} D$$

12.4.12. *Degeneration formula–statement only.*

Theorem 12.4.13. *The moduli space $\overline{\mathcal{K}}_{g,n}(\mathcal{X}_0;\beta)$ comes equipped with a virtual class which is homologous to the virtual class of the generic fiber in the homology of $\overline{\mathcal{M}}_{g,n}(\mathcal{X}/\mathbb{A}^1;\beta)$.*

Furthermore, this class admits an expression in terms of the virtual cycles of relative stable maps to (expansions of) $(Y_1,D),(Y_2,D)$.

Theorem 12.4.14. [Li02] *The following degeneration formula holds:*

$$[\overline{\mathcal{K}}_{g,n}(\mathcal{X}_0;\beta)]^{vir} = \\ = \sum_{\eta\in\Omega}\frac{m(\eta)}{|Eq(\eta)|}\Phi_{\eta,*}\Delta^!([\overline{\mathcal{M}}_{\Gamma_1}(Y_1|D)]^{vir}\times[\overline{\mathcal{M}}_{\Gamma_2}(Y_2|D)]^{vir})$$

We conclude by explaining the notation appearing in the formula and pass the ball to Dan for further explanations.

- The sum is over the set Ω of equivalence classes of *admissible triples* $\eta=(\Gamma_1,\Gamma_2,I)$. These consist of: Γ_1 a combinatorial type of relative stable maps to (Y_1,D); Γ_2 a combinatorial type of relative stable maps to (Y_2,D) such that the sets $(n_1 = r,(c_i)_{i=1}^r),(n_2=r,(c_i)_{i=1}^r)$ of markings carrying non trivial contact order–let as call these *roots*– and the contact order data in Γ_1 and Γ_2 are identified; I is a ordering choice for the union of the sets of standard markings.

  Two triples are equivalent if they differ by a permutation $\sigma\in S_r$ of the roots.

  The identification of the roots is what allows to glue a relative map to (Y_1,D) and a relative map to (Y_2,D) to obtain a map to $\mathcal{X}_0$.
- $\Delta^!$ denotes the Gysin pull-back along the regular immersion $\Delta\colon D^r\to D^r\times D^r$ where the map
$$\overline{\mathcal{M}}_{\Gamma_1}(Y_1|D)\times\overline{\mathcal{M}}_{\Gamma_2}(Y_2|D)\to D^r\times D^r$$
  is given by the evaluations at the *roots*
- The morphism
$$\phi_\eta\colon\overline{\mathcal{M}}_{\Gamma_1}(Y_1|D)\times_{D^r}\overline{\mathcal{M}}_{\Gamma_2}(Y_2|D)\to\overline{\mathcal{K}}_{g,n}(\mathcal{X}_0;\beta)$$
  defined by gluing a pair (f_1,f_2) of relative maps of admissible type is a colavl immesion and $|Eq(\eta)|$ is its degree;
- $m(\eta)=\prod_{i=1}^r c_i$ is the product of the contact orders of the roots

13. Preamble to Log GW: Gluing and degeneration in GW theory

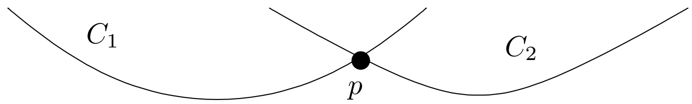
By Dan

I want to tell two parallel stories here — one is about gluing, and one is about degeneration. The stories are intertwined, but I feel the end of the story, where they are neatly combined, is not yet written. Mark's paper [Gro23] goes in remarkable detail into how much can already be done and how much we woudl like to understand better.

13.1. **Gluing.** The study of gluing start with the very beginning of GW theory — indeed the WDVV formula and associativity of quantum cohomology is all about gluing.

13.2. **Boundary of moduli.** Understanding the subspace of maps with degenerate source curve C is key to Gromov–Witten theory.

13.2.1. *Fixed degenerate curve.* Suppose we have a degenerate curve



$$C = C_1 \overset{p}{\cup} C_2.$$

So C is a fibered coproduct of two curves. By the universal property of coproducts

$$\begin{aligned} Hom(C, X) &= Hom(C_1, X) \underset{Hom(p,X)}{\times} Hom(C_2, X) \\ &= Hom(C_1, X) \underset{X}{\times} Hom(C_2, X) \end{aligned}$$

13.2.2. *Varying degenerate curve: the boundary of moduli.* We can work this out in the fibers of universal the families. If we set $g = g_1 + g_2$, $n = n_1 + n_2$ and $\beta = \beta_1 + \beta_2$ we get a morphism

$$\overline{\mathcal{M}}_{g_1, n_1+1}(X, \beta_1) \times_X \overline{\mathcal{M}}_{g_2, n_2+1}(X, \beta_2) \longrightarrow \overline{\mathcal{M}}_{g,n}(X, \beta),$$

with the fibered product over e_{n_1+1} on the left and e_{n_2+1} on the right. On the level of points this is obtained by gluing curves C_1 at point $n_1 + 1$ with C_2 at point $n_2 + 1$ and matching the maps f_1, f_2. This is a finite unramified map, and we can think of the product on the left as a space of stable n-pointed maps of genus g and class β with a distinguished marked node.

13.2.3. *Combinatorial picture.* Since we now know about dual graphs, we know an even better way to encode things: we can associate a space of stable maps to any decorated dual graph τ, where vertices are decorated with both genus $g(v)$ and a curve class $\beta(v)$. The space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ corresponds to the simplest possible genus-g graph τ_0- one vertex decorated with genus g, with n legs, and curve class β. The space on the right hand side is $\overline{\mathcal{M}}(X, \tau)$ where Γ is a two-vertex graph with one edge, with the genus, legs, and classes split as indicated. It has a nodal evaluation map to X. If we denote by τ' the graph obtained by splitting τ at its edge, creating a disconnected graph with two legs replacing the edge, the moduli space has two corresponding evaluations to X. We obtain a cartesian diagram as follows:

$$\begin{array}{ccccc} \overline{\mathcal{M}}(X, \tau_0) & \longleftarrow & \overline{\mathcal{M}}(X, \tau) & \longrightarrow & \overline{\mathcal{M}}(X, \tau') \\ & & \downarrow & & \downarrow \\ & & X & \xrightarrow{\Delta} & X^2 \end{array}$$

Half of the WDVV formula is just an interpretation of this diagram, along with virtual fundamental classes. You have seen the beauty of this in the GW lectures. I want to use it to leverage further understanding in the logarithmic case.

But first, a long digression.

13.3. **A digression to orbifold stable maps.** One big compromise we had to make in this lecture series is that we have not discussed orbifold targets. This digression is about that - and about what it teaches us when we look at the logarithmic theory.

13.3.1. *Stable maps to a stack.* Consider a semistable elliptic surface, with base B and a section. We can naturally view this as a map $B \to \overline{\mathcal{M}}_{1,1}$. Angelo Vistoli, when he was on sabbatical at Harvard in 1996, asked the following beautiful, and to me very inspiring, question: what's a good way to compactify the moduli of elliptic surfaces? can one use stable maps to get a good compactification?

Now consider in general:

$$\begin{array}{cl} \mathcal{X} & \quad \text{Deligne–Mumford stack with} \\ \Big\downarrow & \\ X & \quad \text{projective coarse moduli space} \end{array}$$

In analogy to $\overline{\mathcal{M}}_{g,n}(X, \beta)$, we want a compact moduli space of maps $C \to \mathcal{X}$.

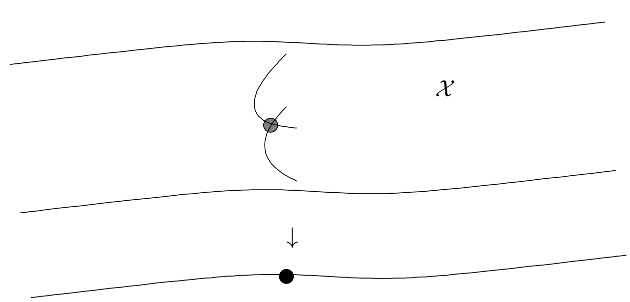
One can define stable maps as in the scheme case, but there is a problem: the result is not compact. As Angelo Vistoli likes to put it, trying to work with a non-compact moduli space is like trying to keep your coins when you have holes in your pockets. The solution that comes naturally is that

> **the source curve $\mathcal{C}$ must acquire a stack structure as well as it degenerates!**

Both problem and solution are clearly present in the following example, which is "universal" in the sense that we take $\mathcal{X}$ to be a one parameter family of curves itself:

Consider $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates x, s near the origin and the projection with coordinate s onto $\mathbb{P}^1$. Blowing up the origin we get a family of curves, with general fiber $\mathbb{P}^1$ and special fiber a nodal curve, with local equation $xy = t$ at the node. Taking base change $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 with equation $t^2 = s$ we get a singular scheme X with a map $X \to \mathbb{P}^1$ given by coordinate s. This is again a family of $\mathbb{P}^1$'s with nodal special fiber, but local equation $xy = s^2$.

This is a quotient singularity, and using the chart $[\mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})]$ with coordinates u, v satisfying $u^2 = x, v^2 = y$ we get a smooth orbifold $\mathcal{X}$, with coarse moduli space X and a map $\mathcal{X} \to \mathbb{P}^1$. It is a family of $\mathbb{P}^1$'s parametrized by $\mathbb{P}^1$, degenerating to an orbifold curve.

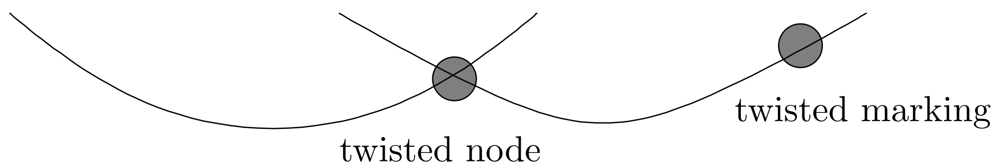


If you think about the family of stable maps $\mathbb{P}^1 \to X$ parametrized by $\mathbb{P}^1 \smallsetminus \{0\}$ given by the embedding of $\mathbb{P}^1$ in the corresponding fiber, there simply isn't any stable map from a nodal curve that can be fit over the missing point $\{0\}$! The only reasonable thing to fit in there is the fiber itself, which is an orbifold nodal curve. We call these *twisted curves.*

13.3.2. *Twisted curves.* (detailed discussion - to be distilled - add sections at markings) This is what happens in general: degenerations force us to allow stacky (or twisted) structure at the nodes. Thinking ahead about gluing curves we see that we had better allow these structures at markings as well.

A *twisted curve* is a gadget as follows:

$$\begin{array}{ccc} \Sigma_i & \subset & \mathcal{C} \\ & & \downarrow \\ & & C. \end{array}$$



- C is a nodal curve.
- $\mathcal{C}$ is a Deligne–Mumford stack with C as its coarse moduli space.
- Over a node $xy = 0$ of C, the twisted curve $\mathcal{C}$ has a chart
$$[\{uv = 0\}/\boldsymbol{\mu}_r]$$
where the action of the cyclotomic group $\boldsymbol{\mu}_r$ is described by
$$(u, v) \mapsto (\zeta u, \zeta^{-1} v).$$
We call this kind of action, with two inverse weights ζ, ζ^{-1}, a *balanced action*. It is necessary for the existence of smoothing of $\mathcal{C}$! In this chart, the map $\mathcal{C} \to C$ is given by $x = u^r, y = v^r$.
- At a marking, $\mathcal{C}$ has a chart $[\mathbb{A}^1/\mu_r]$, with standard action $u \mapsto \zeta u$, and the map is $x = u^r$.
- The substack Σ_i at the i-th marking is locally defined by $u = 0$. This stack Σ_i is canonically an étale gerbe banded by $\boldsymbol{\mu}_r$.

Note that we introduce stacky structure only at isolated points of C and never on whole components. Had we added stack structures along components, we would get in an essential manner a 2-stack, and I don't really know how to handle these.

As defined, twisted curves form a 2-category, but it is not too hard to show it is equivalent to a category, so we are on safe grounds.

The automorphism group of a twisted curve is a fascinating object - I'll revisit it later.

This notion of twisted curves was developed in [AV02]. As we discovered later, a similar idea appeared in Ekedahl's [Eke95].

13.3.3. *Twisted stable maps.* (detailed discussion - to be distilled. sections at markings)

Definition 13.3.4. A twisted stable map consists of
$$(f : \mathcal{C} \to \mathcal{X}, \Sigma_1, \dots, \Sigma_n),$$
where

- $\Sigma_i \subset \mathcal{C}$ gives a pointed twisted curve.
- $\mathcal{C} \xrightarrow{f} \mathcal{X}$ is a representable morphism.
- The automorphism group $\mathrm{Aut}_{\mathcal{X}}(f, \Sigma_i)$ of f fixing Σ_i is finite.

I need to say something about the last two *stability condition*, necessary for the moduli problem being separated.

Representability of $f : \mathcal{C} \to \mathcal{X}$ means that for any point x of $\mathcal{C}$ the associated map

$$\mathrm{Aut}(x) \to \mathrm{Aut}(f(x))$$

on automorphisms is injective. So the orbifold structure on $\mathcal{C}$ is the "most economical" possible, in that we do not add unnecessary automorphisms.

The second condition is in analogy with the usual stable map case, and indeed it can be replaced by conditions on ampleness of a suitable sheaf or number of special points on rational and elliptic components. Most conveniently, it is equivalent to the following schematic condition: the map of course moduli spaces

$$f : C \to X$$

is stable.

But as I defined things I have not told you what an element of $\mathrm{Aut}_{\mathcal{X}}(f, \Sigma_i)$ is! In fact, to make this into a stack I need a category of families of such twisted stable maps.

Definition 13.3.5. A map from $(f : \mathcal{C} \to \mathcal{X}, \Sigma_1, \ldots, \Sigma_n)$ over S to $(f' : \mathcal{C}' \to \mathcal{X}', \Sigma'_1, \ldots, \Sigma'_n)$ over S' is the following:

$$\begin{array}{ccccc} & & \overset{f}{\overbrace{\qquad\qquad\qquad\qquad}} & & \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}' & \xrightarrow[f']{} & \mathcal{X} \\ \downarrow & & \downarrow & & \\ S & \longrightarrow & S', & & \end{array}$$

consisting of

- a fiber diagram with morphism F as above, and
- a 2-isomorphism $\alpha : f \to f' \circ F$.

Note that the notion of automorphisms is more subtle than the case of stable maps to a scheme, even if $\mathcal{C}$ is a scheme. For instance, in the case $\mathcal{X} = \overline{\mathcal{M}}_g$, a map $\mathcal{C} \to \mathcal{X}$ is equivalent to a fibered surface $S \to C$ with fibers of genus g, and S can easily have automorphisms acting on the fibers and keeping C fixed, for instance if the fibers are hyperelliptic!

We write $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ for the resulting category. The main result is:

Theorem 13.3.6. *The category $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ is a proper Deligne–Mumford stack with projective coarse moduli space.*

13.3.7. *Gluing and rigidified inertia.* Much more subtle is the issue of gluing, and the related evaluation maps. To understand it we consider a nodal twisted curve with a separating node:

$$\mathcal{C} = \mathcal{C}_1 \overset{\Sigma}{\sqcup} \mathcal{C}_2.$$

As expected, one can prove that $\mathcal{C}$ is a coproduct in a suitable stack-theoretic sense, and therefore

$$\mathrm{Hom}(\mathcal{C}, \mathcal{X}) \quad = \quad \mathrm{Hom}(\mathcal{C}_1, \mathcal{X}) \underset{\mathrm{Hom}(\Sigma,\mathcal{X})}{\times} \mathrm{Hom}(\mathcal{C}_2, \mathcal{X}).$$

but Σ is no longer a point but a gerbe! We must ask

- How can we understand $\mathrm{Hom}(\Sigma, \mathcal{X})$?

- What is the universal picture?

We right answer is that legs of the dual graph must be marked by inertia components, and more generally oriented nodes as well. The evaluation at opposite sides of an edge have *inverse* inertia elements.

Proposition 13.3.8. *The evaluation maps* $\overline{\mathcal{M}}(\mathcal{X},\tau) \to \mathcal{I}\mathcal{X}^m$ *are virtually smooth.*
Given an edge of τ *with splitting* τ' *we have a cartesian splitting diagram*

$$\begin{array}{ccc} \overline{\mathcal{M}}(\mathcal{X},\tau) & \longrightarrow & \overline{\mathcal{M}}(\mathcal{X},\tau') \\ \downarrow & & \downarrow \\ \mathcal{I}\mathcal{X} & \longrightarrow & \mathcal{I}\mathcal{X} \times \mathcal{I}\mathcal{X} \end{array}$$

of stacks with compatible virtual fundamental classes.

This leads to a wonderful WDVV for orbifolds, leading to Chen–Ruan cohomology, quantum cohomology, Tseng's orbifold upgrade of the Givental formalism, Coates–Corti–Iritani–Tseng, the crepant resolution conjecture, etc.

13.3.9. *Lessons learned.* In any generalization of GW theory,

(1) The structure of curves should reflect the structure of targets, and vice versa.
(2) The structure of points should say something about where one evaluates.
(3) Gluing should be a fibered diagram, with compatible virtual structure, as above.

Indeed, for orbifold maps curves are necessarily orbifold curves. Evaluation must take into account the gerbe structure, hence lands in $\mathcal{X}^{ev} = \mathcal{I}(\mathcal{X})$. The fibered product is over the fancy diagonal $\mathcal{I}\mathcal{X} \to \mathcal{I}\mathcal{X} \times \mathcal{I}\mathcal{X}$.

13.4. **The degeneration formula - comments back and forward.**

13.4.1. *A first case.* In 1998 [LR01] and in 2001 [Li01, Li02] An-Min Li and Ruan, and Jun Li, showed how to compute Gromov Witten invariants using a simple degeneration.

One wants to access the GW invariants of smooth X, which lies in a family $\mathcal{X} \to B$, with $\mathcal{X}$ and B smooth, B a curve, and special fiber $X_0 = Y_1 \cup^D Y_2$, again with Y_i, D smooth, a simple normal crossings degeneration.

The key to this work is

(1) A new relative GW theory of Y_i relative to the divisor D,
(2) A fiberwise GW theory of $\mathcal{X}/B$, which is close enough to the above, and
(3) A degeneration formula of the following form:

$$GW(X_0/0) = GW(Y_1, D) * GW(Y_2, D).$$

Deformation invariance guarantees that much of $GW(X)$ is encoded in $GW(X_0/0)$. Techniques for recovering the hidden parts of $GW(X)$ have recently been developed.

We have seen this in some detail in the GW lecture series. It is a major tool of GW theory and a major success!

13.4.2. *Approaches in the first case.* Nowadays the degeneration formula in this case has several proofs. I will not address the Li–Ruan paper as it relies on symplectic techniques (and does apply in that generality - a fact that has ben useful even in the algebraic case).

Jun Li's approach is to construct the family and relative moduli spaces using expanded degenerations. Within that work he uses logarithmic deformation theory in analyzing virtual fundamental classes, but his use of logarithmic geometry is limited to that case. His use of log deformation theory was inspired by Siebert's legendary lecture on the subject.

The paper [AF16] still uses expanded degenerations, and addresses relative moduli spaces and invariants of (Y, D) and of X/B using orbifold geometry. The idea is, when one takes root stacks of sufficiently divisible order, contact orders of expanded maps become transverse, circumventing the hardest step in Jun Li's computations. So thorny issues are circumvented using the magic powder of stacks.

The paper [Kim10] again uses expanded degenerations, but directly applying logarithmic structures. Chen [Che14a] uses it to provide a degeneration formula. One views $\mathcal{X}/B$ or (Y, D) as a log smooth scheme, one defines expanded log maps to such targets, and, similarly to [AF16], the issue of non-transversality is circumvented using the magic powder of log.

Finally the paper [KLR23] provides a treatment through logarithmic stable maps, the subject of the next series of lectures.

The fact that all these approaches compute the same objects is proved in [AMW14].

Exercise 13.4.3. Compute the number $N_3 = 12$ of cubic plane curves through 8 points by a degeneration.

(1) Degenerate the plane to the union of a plane and an F_1 by blopwing up $\mathbb{P}^2 \times \mathbb{A}^1$ along a line over 0.
(2) Let three points degenerate to the plane and 5 to the F_1 component.
(3) Show that the only contribution is from curves formed by conics in the plane and curves of class $\ell + 2f$.
(4) The total count 12 comes out as $5 \times 1 + 3 \times 1 + 2 \times 2$.
    - The first term has to do with breaking $\ell + 2f$ as $(\ell + f) + f$, where f passes through one of the 5 points.
    - The second with breaking the conic in two lines.
    - The third with irreducible curves tangent to the intersection D. An important point is that these count with multiplicity 2, and there are 2 of these.

Exercise 13.4.4. Compute the number 12 of rational plane sections of a cubic surface in the pencil of planes through two general points, by degenerating the cubic into a union of a plane and a quadric, and one of the points landing on each component.

(1) Perform the necessary small blowup to make the family log smooth. This involves 6 points on the conic D where the components intersect.
(2) The count is of the form $6 \times 1 + 2 \times 1 + 2 \times 2$.
    - The first term has to do with lines in the plane through one of the 6 points.
    - The second with breaking the conic section of the quadric in two lines.

- The third with irreducible curves tangent to the intersection D. An important point is that these again count with multiplicity 2, and there are 2 of these.

13.4.5. *The general case.* This was the original purpose of logarithmic GW theory, and the story is not as clean as I would wish to tell you. In an ideal world, Siebert's dream would have been realized directly, with no caveats.

Exercise 13.4.6. Not a reasonable exercise! Compute the number 12 of rational plane sections of a cubic surface in the pencil of planes through two general points, by degenerating the cubic into a union of three general planes, with the points landing on different components.

(1) Perform the necessary small blowup to make the family log smooth. This involves 9 points on the three lines.
(2) The count is of the form $9 \times 1 + 3 \times 1$.
    - The first term has to do with lines in the plane through one of the 9 points. That's quite natural.
    - The second with the unique plane through the origin.
    - It is a highly nontrivial fact that the plane section (with the triple point replaced by a $\mathbb{P}^1$) counts as 3!

I'll revisit the question in the closing lecture.

13.4.7. *Here there are again several approaches.*

(1) Brett Parker in a series of papers (see exposition in [Par12]) has an approach for a full degeneration formula using his theory of exploded manifolds.
(2) Dhruv Ranganathan [Ran22] and Maulik–Ranganathan [MR24] have an approach for a full degeneration formula using expanded degenerations controlled by combunatorics.
(3) there is also an orbifold approach by Fan Wu and You [FWY20]. Unlike the simplest case, there is a difference between this approach and the logarithmic approach.
(4) The approach through logarithmic geometry [GS13, Che14b, AC14, ACGS20, ACGS24, Gro23] provides a gluing mechanism, which works in many important cases but not all. I feel that we are still missing a simplifying component of the theory, and would welcome ideas. I also feel that it is a rather natural approach. My understnding is that this approach agrees with Tehrani's [FT22].

[22]

22→

13.4.8. *Decomposition.* State the decomposition formula here? It can also go in Hulya's lecture or in the punctured lectures under "advanced"

[22](Dan) Forgotten anyone?

14. Logarithmic Gromov–Witten theory

Gromov-Witten theory has a generalisation to the setting of logarithmic geometry [GS13, AC14]. In log Gromov-Witten theory, to study *log curves* in a log scheme $(X, \mathcal{M}_X)$ one works over a base log scheme $(S, \mathcal{M}_S)$. The scheme S in practice could be the spectrum of a discrete valuation ring with the log structure induced by the closed point (one-parameter degeneration), or it could be $\operatorname{Spec} \mathbf{k}$, for $\mathbf{k}$ an algebraically closed field of characteristic zero, endowed with the trivial log structure (absolute situation), or $\operatorname{Spec} \mathbf{k}$ endowed with *the standard log structure* (special fibre of one-parameter degeneration). Throughout this section, we focus in the absolute situation, and assume $(X, \mathcal{M}_X)$ is a log scheme over the trivial log point.

One generalises the notion of a stable map to the log setting as follows. Consider an ordinary stable map with a number, say ℓ, of marked points. Thus we have a proper curve C with at most nodes as singularities, a regular map $f : C \to X$, a tuple $\mathbf{x} = (x_1, \ldots, x_\ell)$ of closed points in the non-singular locus of C. Moreover the triple $(C, \mathbf{x}, f)$ is supposed to fulfill the stability condition of finiteness of the group of automorphisms of $(C, \mathbf{x})$ commuting with f. To promote such a stable map to a stable log map amounts to endow all spaces with (fine, saturated) log structures and lift all morphisms to *morphisms of log schemes*, defined as follows.

Definition 14.0.1. A *morphism of log schemes* $f\colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a morphism of schemes $f\colon X \to Y$ along with a homomorphism of sheaves of monoids $f^\sharp\colon f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ such that the diagram

$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \xrightarrow{f^\sharp} & \mathcal{M}_X \\ {\scriptstyle \alpha_Y}\downarrow & & \downarrow{\scriptstyle \alpha_X} \\ f^{-1}\mathcal{O}_Y & \xrightarrow{f^*} & \mathcal{O}_X. \end{array}$$

is commutative. Here, f^* is the usual pull-back of regular functions defined by the morphism f. Given a morphism of log spaces $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, we denote by $\underline{f}\colon \underline{X} \to \underline{Y}$ the underlying morphism of schemes.

A stable log map is obtained by promoting $C \to W$ to a smooth morphism of log schemes

$$\pi : (C, \mathcal{M}_C) \longrightarrow (W, \mathcal{M}_W).$$

and also $f : C \to X$ to a morphism of log schemes, $f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$. We demand that the regular points of C where π is not strict are exactly the marked points, and the morphism $\pi : (C, \mathcal{M}_C) \to (W, \mathcal{M}_W)$ to be *log smooth* – we define the notion of log smoothness in the §14.1, and then expand the discussion on log smooth curves in §14.2.

14.1. Smoothness in log geometry.

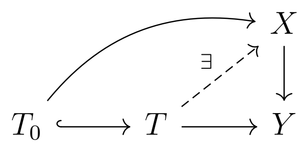
In this section we define log smooth morphisms, and give criteria on how to characterize them, after shortly recalling smooth morphisms of schemes.

Definition 14.1.1. A morphism of schemes $f : X \to Y$ is called *smooth* if the following holds (see Infinitesimal Lifting Property [Har77, §II.8]):

- f is "locally of finite presentation": this means if Y is covered by affine open sets $V = \operatorname{Spec} A$, and $f^{-1}(V)$ is covered by affine open sets $U = \operatorname{Spec} B$, then the ring

map $A \to B$ corresponding to $U \to V$ is a finitely presented ring homomorphism (i.e. B is the quotient of some polynomial ring $A[x_1, \dots, x_n]$ by a finitely generated ideal.)
- f is "formally smooth": this means for any affine scheme T_0 with a square-zero embedding $T_0 \subset T$ fitting into the diagram (23), there exists a lifting $T \to X$ as indicated.

(23)



Furthermore, the morphism $f : X \to Y$ is called *étale* if the above lifting is unique.

Remark 14.1.2. The assumption that T_0 is affine ensures the existence of a lifting "locally". If the lifting is unique, then "existence locally" implies "existence", as unique liftings must coincide on overlaps and maps which coincide on overlaps glue together.

We define logarithmically smooth and étale morphisms similarly in a moment, after we reacll the notion of a *strict* morphism which will be needed.

Definition 14.1.3. Let $f\colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a log morphism and let x be a point of X. We say that f is *strict* at the point x if $f^\sharp$ induces an isomorphism $f^{-1}\mathcal{M}_{Y,f(p)} \simeq \mathcal{M}_{X,p}$.

Definition 14.1.4. A morphism $f : X \to Y$ of "fine" logarithmic schemes is called *log smooth* if the following holds:

i) The underlying morphism of schemes $\underline{X} \to \underline{Y}$ is locally of finite presentation, and
ii) For any fine log scheme T_0 which is affine and $T_0 \subset T$ a strict square-zero embedding fitting into a diagram as in (23), viewed in the category of log schemes, there exists a lifting $T \to X$ as indicated. Furthermore, the morphism $f : X \to Y$ is called *étale* if this lifting is unique.

The following proposition which can be dound in [Kat89, Prop.3.4] provides a natural example of a log smooth or étale morphism.

Proposition 14.1.5. *Let R be a ring, P and Q finitely generated integral monoids, and $Q \to P$ a monoid homomorphism. Let $X = \operatorname{Spec} R[P]$ and $Y = \operatorname{Spec} R[Q]$ be endowed with log structures $\mathcal{M}_X$ and $\mathcal{M}_Y$ associated to the pre-log structures $P \to R[P]$ and $Q \to R[Q]$ respectively. Assume*

i) *The kernel of $Q^{gp} \to P^{gp}$ is finite and with order invertible in R,*
ii) *The torsion part of the cokernel of $Q^{gp} \to P^{gp}$ has order invertible in R.*

Then, $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is log smooth. Furthermore, if the cokernel of $Q^{gp} \to P^{gp}$ is finite then $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is log étale.

Proof. Condition i) in Definition 14.1.4 is automatic, since P, Q are finitely generated. To check condition ii), let $T_0 \subset T$ be a strict square zero embedding fitting into a diagram as in (23), viewed in the category of log schemes. We will show that there exists a lifting $(T, \mathcal{M}_T) \to (X, \mathcal{M}_X)$. To do this, it suffices to define a map $P \to \mathcal{M}_X$, since the log structure $\mathcal{M}_X$ on $X = \operatorname{Spec} R[P]$ is induced from $P \to R[P]$.

Let $I \subset \mathcal{O}_T$ be the ideal defining $T_0 \subset T$, such that $I^2 = 0$. Note that we have an embedding

$$\begin{aligned} \phi : I &\hookrightarrow \mathcal{O}_T^\times \subset \mathcal{M}_T \\ x &\longmapsto 1 + x \end{aligned}$$

To check that that ϕ is a homomorhism, note that $\phi(x+y) = \phi(x)\cdot\phi(y)$, since $(1+x)(1+y) = 1 + x + y + xy = 1 + (x + y)$ as $I^2 = 0$. Therefore, we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{M}_T & \longrightarrow & \mathcal{M}_T/I = \mathcal{M}_{T_0} \\ \downarrow & & \downarrow \\ \mathcal{M}_T^{gp} & \longrightarrow & \mathcal{M}_T^{gp}/I = \mathcal{M}_{T_0}^{gp}. \end{array} \tag{24}$$

Then, by the assumption on $Q^{gp} \to P^{gp}$, we obtain the following commutative diagram, where the dotted arrow is defined étale locally

$$\begin{array}{ccc} \mathcal{M}_{T_0}^{gp} & \longrightarrow & P^{gp} \\ \downarrow & \swarrow & \downarrow \\ \mathcal{M}_T^{gp} & \longrightarrow & Q^{gp} \end{array} \tag{25}$$

From the cartesian diagram in (24) the diagram in (25), we obtain the desired map $P \to \mathcal{M}_X$, hence the result follows. ♣

Example 14.1.6. Dominant toric morphisms are log smooth.

Exercise 14.1.7. Verify using Proposition 14.1.5, that the morphism $\mathbb{A}^2 \to \mathbb{A}^1$ defined by

$$\begin{aligned} \operatorname{Spec} \mathbb{C}[x, y] &\longrightarrow \operatorname{Spec} \mathbb{C}[t] \\ (x, y) &\longmapsto x \cdot y = t\,, \end{aligned} \tag{26}$$

as illustrated in Figure 7 is log smooth.

Note that log smooth morphisms are not necessarily smooth – for instance the morphism in (26) is not smooth, as the fiber over $t = 0$ is not smooth.

Exercise 14.1.8. Verify that the following morphisms are log smooth (where we consider the tpric divisorial log structures on each scheme), but not smooth

1) $\operatorname{Spec} \mathbb{C}[t] \to \operatorname{Spec} \mathbb{C}[s]$ given by $s = t^2$
2) $\operatorname{Spec} \mathbb{C}[x, y] \to \operatorname{Spec} \mathbb{C}[x, z]$ given by $z = xy$

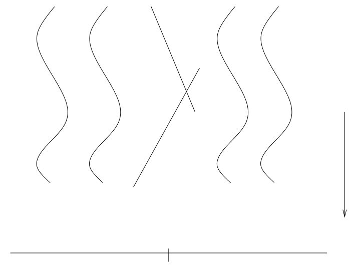


FIGURE 7. The map $(x, y) \longmapsto xy$

Note that in Exercise 14.1 the morphism $\operatorname{Spec}\mathbb{C}[x, y] \to \operatorname{Spec}\mathbb{C}[x, z]$ given by $z = xy$ descibes an affine chart of the blow-up of $\mathbb{A}^2$ at the origin, which is not flat. Unfortunately, unlike smoothness, log smoothness does not imply flatness. However, if a log smooth morphism is *integral* defined as below, then flatness of the underlying morphism follows.

Definition 14.1.9. A monoid homomorphism $Q \to P$ is called *integral* if $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat. A morphism $f : X \to Y$ of logarithmic schemes is *integral* if for every geometric point x of X the homomorphism $(f^{-1}\overline{M}_Y)_x \to (\overline{M}_X)_x$ between characteristic sheaves on stalk level is integral.

The following statement can be found in [Kat89, Corollary 4.5].

Proposition 14.1.10. *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism between fine log schemes. If f is log smooth and integral, then the underlying morphism of schemes is flat.*

Generally, to check if a morphism of log schemes $f : X \to Y$, not necessarily of the form as in Proposition 14.1.5, is log smooth we study *charts* for f. Recall, that a chart for a fine log structure $\mathcal{O}_X$ on a scheme X is given by a morphism of monoid sheaves $P_X \to \mathcal{M}_X$, where P_X the constant sheaf associated to a finitely generated integral monoid P, such that the associated log structure is isomorphic to $\mathcal{M}_X$. For morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of schemes with fine log structures a *chart for f* is a triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ where $P_X \to \mathcal{M}$, $Q_Y \to \mathcal{M}_Y$ are charts of $\mathcal{M}$ and $\mathcal{M}_Y$ respectively and $Q \to P$ is a homomorphism for which the following diagram commutes.

$$\begin{array}{ccc} Q_X & \longrightarrow & f^{-1}\mathcal{M}_Y \\ \downarrow & & \downarrow \\ P_X & \longrightarrow & \mathcal{M}_X \end{array}$$

The following theorem can be found in [Kat89, Theorem 3.5].

Theorem 14.1.11. *A morphism $f : X \to Y$ of fine log schemes is* log smooth *if and only if étale locally there exists a chart $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ such that*

i) *$Q \to P$ satisfies conditions i) and ii) in Proposition 14.1.5, and*
ii) *$\underline{X} \to \underline{Y} \times \operatorname{Spec}\mathbb{Z}[Q] \operatorname{Spec}\mathbb{Z}[P]$ is smooth in the usual sense.*

14.2. **Log smooth curves.**

Definition 14.2.1. A map of fine log schemes $\pi : (C, \mathcal{M}_C) \to (W, \mathcal{M}_W)$ is called a *log curve* over W if π is a proper, log smooth, integral morphism of relative dimension 1 such that every fibre is a reduced and connected curve.

The following theorem is due to Fumiharu Kato [Kat00].

Theorem 14.2.2. *Assume $\pi : C \to W$ is a log curve, $0 \in \underline{W}$ be a closed point and $Q := \overline{M}_{W,0}$. Then étale locally on the fiber over 0,*

1) *All singularities are nodal,*
2) *We can choose disjoint sections $s_i : \underline{W} \to \underline{C}$ in the nonsingular locus $\underline{C}_0$ of $\underline{C}/\underline{W}$ such that*
   i) *Away from s_i we have the log structure on the smooth locus is given by $C_0 = \underline{C}_0 \times_{\underline{W}} W$, so π is strict away from s_i.*
   ii) *Near each s_i we have a strict étale morphism*
   $$C_0 \longrightarrow W \times \mathbb{A}^1,$$
   with the standard divisorial log structure on $\mathbb{A}^1$.
   iii) *At a node $p \in C$, we have $\mathcal{M}_{C,p} = \mathbb{N}^2 \oplus_{\mathbb{N}} Q \oplus \mathcal{O}^*_{C,p}$ where $Q \to \mathbb{N}^2$ is the diagonal morphism and $\mathbb{N} \to Q$ is given by $1 \mapsto \rho$ for some non-zero $\rho \in Q$.*

In Theorem 14.2.2 cases (i), (ii), (iii) correspond to neighbourhoods of general points, marked points and nodes of C respectively.

Definition 14.2.3. A *prestable marked log curve* over $(W, \mathcal{M}_W)$ is a log curve $\pi : C \to W$ together with a tuple of sections $\mathbf{s} = (s_1, \dots, s_\ell)$ in the non-singular locus $\underline{C}_0$ of $\underline{C}/\underline{W}$ such that away from the images of s_i's we have $C_0 = \underline{C}_0 \times_{\underline{W}} W$. A pre-stable log curve is *stable* if forgetting the log structure leads to an ordinary stable curve.

We investigate the moduli space of stable log curves in the following section.

14.3. **Moduli spaces of stable log curves.** We define a category $\overline{\mathcal{M}}_{g,n}^{\log}$ of stable log curves: the objects are genus g log curves $C \to W$ with n marked points and the morphisms are fiber diagrams

$$\begin{array}{ccc} C_1 & \longrightarrow & C_2 \\ \downarrow & & \downarrow \\ W_1 & \longrightarrow & W_2 \end{array}$$

There is a forgetful functor from $\overline{\mathcal{M}}_{g,n}^{\log}$ to the category of fine, saturated log schemes:

$$\begin{aligned} \overline{\mathcal{M}}_{g,n}^{\log} &\longrightarrow \mathfrak{LogSch}^{\mathrm{fs}} \\ (C \to W) &\longmapsto W \end{aligned}$$

So, $\overline{\mathcal{M}}_{g,n}^{\log}$ is a category fibered in groupoids over $\mathfrak{LogSch}^{\mathrm{fs}}$ as discussed in §2.2.1.

Let $\overline{\mathcal{M}}_{g,n}$ denote moduli space of n-marked, genus g stable curves as in §11. Recall that the union of singular curves defines a boundary divisor $\partial\overline{\mathcal{M}}_{g,n}$ in $\overline{\mathcal{M}}_{g,n}$. The following result can be found in [Kat00, Theorem 4.5].

Theorem 14.3.1. *There exists a natural isomorphism*

$$\overline{\mathcal{M}}_{g,n}^{\log} \cong (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n}) \,,$$

as categories fibered in groupoids over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Note that in Theorem 14.3.1 both $\overline{\mathcal{M}}_{g,n}^{\log}$ and $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ are categories fibered in groupoids over the category of fs log schemes $\mathfrak{LogSch}^{\mathrm{fs}}$. In particular, we view here $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ as the category fibered in groupoids over fs log schemes whose fiber over an fs log scheme W is the groupoid of log morphisms from W to the log scheme $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$.

Now, by definition $\overline{\mathcal{M}}_{g,n}^{\log}$ parametrizes stable log curves over the category of fs log schemes $\mathfrak{LogSch}^{\mathrm{fs}}$. On the other hand, $\overline{\mathcal{M}}_{g,n}$ parametrizes so called *minimal* [AC14] or *basic* [GS13] stable log curves over the category of schemes – we discuss the notion of minimality in the remaining part of this subsection.

14.3.2. *Minimality.* Given a family of stable curves $\underline{C} \to \underline{W}$, we explain below that there is a natural way to impose a log structure both on $\underline{C}$ and on $\underline{W}$ and to lift the map $\underline{C} \to \underline{W}$ to a log morphism $C \to W$.

By the universal property of the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$, there exists a unique morphism $\underline{W} \to \overline{\mathcal{M}}_{g,n}$ such that $\underline{C} \to \underline{W}$ is the pull-back of the universal curve $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. The *minimal* log structure on $\underline{W}$ is then obtained by the pull-back of the divisorial log structure on $\overline{\mathcal{M}}_{g,n}$ defined by the divisor $\partial\overline{\mathcal{M}}_{g,n}$. The minimal log structure on $\underline{C}$ is obtained similarly, by the pull-back of the divisorial log structure on $\overline{\mathcal{M}}_{g,n+1}$, defined by the boundary divisor $\partial\overline{\mathcal{M}}_{g,n+1}$.

The fact that there is a natural lift of the map $\underline{C} \to \underline{W}$ to a log morphism $C \to W$ follows from the fact that the universal curve $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ naturally lifts to a log morphism

$$(\overline{\mathcal{M}}_{g,n+1}, \partial\overline{\mathcal{M}}_{g,n+1}) \longrightarrow (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n}) \,.$$

From now on we will denote by C^{min} the log scheme with underlying scheme $\underline{C}$, endowed with the minimal log structure discussed above, and we will use similar notation for W^{min}. Note that the reason why these log structures are called "minimal" is that for any log curve $C \to W$ with underlying scheme theoretic map $\underline{C} \to \underline{W}$, there exists a unique map $W \to W^{min}$ such that $C \to W$ is a pull-back of $C^{min} \to W^{min}$ – this is a direct consequence of Theorem 14.3.1.

From the discussion above for any stable curve $\underline{C} \to \underline{W} := \operatorname{Spec}\mathbb{C}$, there is an associated log morphism

$$C^{min} \to W^{min} := (\operatorname{Spec}\mathbb{C}, \mathbb{C}^* \oplus Q) \,,$$

where we will refer to Q as the *minimal monoid* or *basic monoid*. It follows from the definition of the minimal log structure that

$$Q = \mathbb{N}^r \,, \tag{27}$$

where r is the number of nodes of $\underline{C}$.

Recall that the nodes of $\underline{C}$ are in one-to-one correspondence with the bounded edges of the dual graph of $\underline{C}$. So, the set of monoid homomorphisms $\operatorname{Hom}(Q, \mathbb{R}_{\geq 0}) \cong \mathbb{R}_{\geq 0}^r$ can be interpreted as the moduli space of tropical curves with underlying graph the dual graph of $\underline{C}$, where $\mathbb{R}_{\geq 0}^r$ parametrizes the lengths of the bounded edges.

14.4. **Stable log maps.** Recall from §11 that an n-marked stable map to a scheme X is a map $f : C \to X$, from a proper nodal curve C with n marked points $(p_1, \dots, p_n)$, such that $\mathrm{Aut}(C, p_1, \dots, p_n)$ is finite. In the log geometric setup, the generalization of a stable map is given as follows.

Definition 14.4.1. Let X be a log scheme. An *n-marked stable log map with target X* is a diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle \pi}\big\downarrow & & \\ W & & \end{array} \tag{28}$$

together with a tuple of sections $\mathbf{x} = (x_1, \dots, x_n)$ of π, where π is a proper, log smooth, integral morphism of log schemes, such that, for every geometric point s of W, the restriction of $\underline{f}$ to s with the marked points $\underline{\mathbf{x}}(s)$ is an ordinary stable map, and furthermore, if $U \subset C$ is the non-critical locus of $\underline{\pi}$, we have $\overline{\mathcal{M}}_C|_U \simeq \underline{\pi}^* \overline{\mathcal{M}}_W \oplus \bigoplus_{j=1}^n (x_j)_* \mathbb{N}_W$.

In what follows we will use the notation $f : C/W \to X$ to denote a stable log map as in (28).

Recall from §11, one fixes discrete data $\underline{\Gamma} = (g, \beta, n)$ to define the moduli space of stable maps to $\underline{X}$, of genus g with n marked points, such that the image is of class $\beta \in H_2(\underline{X}, \mathbb{Z})$. To describe the moduli space of stable log maps to X, in addition we fix *contact orders* c_i at each marked point p_i, for $1 \le i \le n$, defined as follows. If $p_i \in$ is a marked point of a geometric fibre of π, we have

$$c_i : \overline{\mathcal{M}}_{X, f(p_i)} \longrightarrow \overline{\mathcal{M}}_{C, p_i} = \overline{\mathcal{M}}_{W, \pi(p_i)} \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N},$$

The element $c_i \in \mathrm{Hom}(\overline{\mathcal{M}}_{X, f(p_i)}, \mathbb{N})$, called the *contact order at p_i*. In what follows, we collect the numerical data under the umbrella

$$\Gamma = (g, \beta, c_i)\,.$$

We will refer to an n-marked stable log map as a map of type Γ, if the domain curve is of genus g, the image is of class $\beta \in H_2(\underline{X}, \mathbb{Z})$ and each marked point p_i has contact order c_i, for $1 \le i \le n$.

We define a category $\overline{\mathcal{M}}^{\log}_{\Gamma}(X)$ of stable log maps to X of type Γ: the objects are stable log maps $f : C/W \to X$ of type Γ and the morphisms are given by fiber diagrams of stable log maps. Similarly as for the moduli space of curves, we have a forgetful functor from $\overline{\mathcal{M}}^{\log}_{\Gamma}(X)$ to the category of fine, saturated log schemes. Hence, $\overline{\mathcal{M}}^{\log}_{\Gamma}(X)$ is also a category fibered in groupoids over $\mathfrak{LogSch}^{\mathrm{fs}}$.

The following theorem is due to Abramovich–Chen [AC14] and Gross–Siebert [GS13] under some assumptions on global generation of the ghost sheaf – these assumptions were removed in Abramovich–Chen–Marcus–Wise [ACMW17].

Theorem 14.4.2. *Let X be projective log scheme. There exists a proper log Deligne-Mumford stack $\overline{\mathcal{M}}_{\Gamma}(X)$ such that*

$$\overline{\mathcal{M}}^{\log}_{\Gamma}(X) \cong \overline{\mathcal{M}}_{\Gamma}(X)\,,$$

as categories fibered in groupoids over $\mathfrak{LogSch}^{\mathrm{fs}}$.

In a moment we explain that $\overline{\mathcal{M}}_\Gamma(X)$ parametrizes so called *minimal* or *basic* stable log maps to X over the category of schemes.

14.5. **Minimal stable log maps.** Let $f : C/W \to X$ be s stable log map of type Γ. By Theorem 14.4.2 there exists a unique log morphism $W \to \overline{\mathcal{M}}_\Gamma(X)$ such that $f : C/W \to X$ is the pull-back of the universal stable map over $\overline{\mathcal{M}}_\Gamma(X)$.

Denote by W^{min} the log scheme with underlying scheme $\underline{W}$, endowed with the log structure obtained by pulling back the log structure on $\overline{\mathcal{M}}_\Gamma(X)$ along the map $W \to \overline{\mathcal{M}}_\Gamma(X)$. We call the stable log map $f : C/W \to X$ *minimal* or *basic* if the natural log morphism $W \to W^{min}$ is an isomorphism.

It follows by the definition of a minimal stable log map that for any scheme $\underline{W}$, the data of a scheme theoretic morphism $\underline{W} \to \overline{\mathcal{M}}_\Gamma(X)$ is equivalent to the data of a basic stable log map $f : C/W \to X$.

We provide below an explicit characterization of basic stable log maps over a log point $W = (\operatorname{Spec}\mathbb{C}, \mathbb{C}^* \oplus Q)$, where Q is a sharp monoid, that is, the only invertible element is zero.

Let $f : C/W \to X$ be a stable log map. By the discussion above, there exists a log scheme W^{min} necessarily of the form

$$W^{min} = (\operatorname{Spec}\mathbb{C}, \mathbb{C}^* \oplus Q^{min}),$$

where Q^{min} is a sharp monoid, referred to as the *minimal monoid* or *basic monoid*. It is shown in [GS13, Remark 1.18] that one can describe the basic monoid Q, by first defining its dual as

$$Q^{min,\vee} = \left\{ \big((V_\eta)_\eta, (e_q)_q\big) \in \bigoplus_\eta \overline{M}^\vee_{X,f(\eta)} \oplus \bigoplus_q \mathbb{N} \,\middle|\, \forall q : V_{\eta_2} - V_{\eta_1} = e_q u_q \right\}$$

where the sum is over generic points η of C and nodes q of C and u_q denotes the contact order of the node q (see [GS13, (1.8)]). One then obtains the minimal, or basic, monoid as

$$Q^{min} := \operatorname{Hom}(Q^{min,\vee}, \mathbb{N}).$$

The basic monoid Q^{min} has a natural tropical interpretation. To elaborate on this we first need the following definition.

Definition 14.5.1. The *combinatorial type* of a stable log map $f \colon C/W \to X/S$ consists of:

- The dual intersection graph $G = G_C$ of C, with set of vertices $V(G)$, set of edges $E(G)$, and set of legs $L(G)$.
- The genus function $g : V(G) \to \mathbb{N}$ associating to $v \in V(G)$ the genus of the irreducible component $C(v) \subset C$.
- The map $\sigma : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$ mapping $x \in C$ to $\big(\overline{M}_{X,f(x)}\big)^\vee_{\mathbb{R}}$, where $\Sigma(X)$ denotes the tropicalisation of X.
- The contact data $u_p \in \overline{M}^\vee_{X,f(p)} = \operatorname{Hom}(\overline{M}_{X,f(p)}, \mathbb{N})$ and $u_q \in \operatorname{Hom}(\overline{M}_{X,f(q)}, \mathbb{Z})$ at marked points p and nodes q of C.

The combinatorial type of a stable log map naturally determines the combinatorial type of the associated tropical stable map. It follows by the definition of the basic monoid that $\operatorname{Hom}(Q^{min}, \mathbb{R}_{\geq 0})$ is the moduli cone of tropical stable maps of this combinatorial type.

Example 14.5.2. Consider the target $X = \mathbb{P}^1$ with divisorial log structure along a point $\infty \in \mathbb{P}^1$. Let $f : C \to X$ be a minimal stable log map with combinatorial type given by the tropical stable map to $\Sigma(X) = \mathbb{R}_{\geq 0}$ as in Figure 8. The weights 1, 1, 2 attached to the edges E_1, E_2, E_3 encode the contact orders. The corresponding minimal monoid is $Q^{min} = \mathbb{N}$. Indeed, the corresponding moduli space of tropical stable maps $\mathrm{Hom}(Q^{min}, \mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0}$ is parametrized by the position of the image of the vertex V_3. Note that if we were considering the domain curve only, independently of the map, then one could vary independently the lengths of the two bounded edges E_1 and E_2, and so the basic monoid would be $\mathbb{N}^2$, as in Equation 27. However, the existence of the tropical map to $\Sigma(X)$ forces these two lenghts to be equal and determined by the position of the image of the vertex V_3.

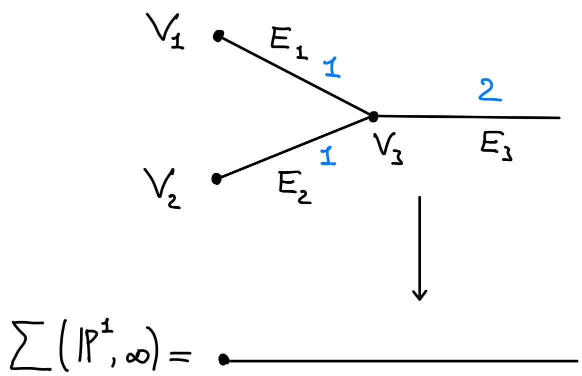


FIGURE 8. A tropical stable map to the tropicalisation of $(\mathbb{P}^1, \infty)$

Exercise 14.5.3. As in Example 14.5.2, consider the target $X = \mathbb{P}^1$ with divisorial log structure along a point $\infty \in \mathbb{P}^1$. Let $f : C \to X$ be a minimal stable log map with combinatorial type given by the tropical stable map to $\Sigma(X) = \mathbb{R}_{\geq 0}$ in Figure 9. Determine the corresponding basic monoid Q^{min}.

Finally, we are at a position to define log Gromov–Witten invariants.

14.6. **Log Gromov–Witten invariants.** Recall the moduli space $\overline{\mathcal{M}}_\Gamma(X)$ from Theorem 14.4.2 parametrizes basic/minimal stable log maps to X. We have the following result, due to Abramovich–Chen [AC14] and Gross–Siebert [GS13]:

Theorem 14.6.1. *Let X be a log smooth, projective log scheme. Then, the moduli space $\overline{\mathcal{M}}_\Gamma(X)$ carries a natural virtual fundamental class $[\overline{\mathcal{M}}_\Gamma(X)]^{vir}$.*

We briefly review in what follows, how to construct the virtual fundamental class $[\overline{\mathcal{M}}_\Gamma(X)]^{vir}$. To do this, we will consider the Artin fan associated to X as defined in §8, denoted by $\mathcal{A}_{X,\mathcal{M}_X}$ – in what follows we will abbreviate the notation and denote it just by $\mathcal{A}_X$. Denote by $\mathfrak{M}_\Gamma(\mathcal{A}_X)$ the moduli stack of basic/minimal pre-stable log maps to $\mathcal{A}_X$. Recall that there

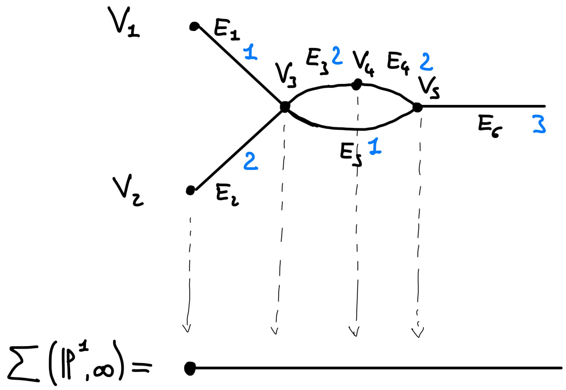


FIGURE 9. Another tropical stable map to the tropicalisation of $(\mathbb{P}^1, \infty)$

is a natural log morphism $X \to \mathcal{A}_X$, which by composition with $f : C/W \to X$ defines a map $C/W \to \mathcal{A}_X$. Hence, we obtain a log morphism

$$\epsilon : \overline{\mathcal{M}}_\Gamma(X) \longrightarrow \mathfrak{M}_\Gamma(\mathcal{A}_X)\,.$$

There are two key points to emphasize: First, there is a natural perfect obstruction theory on $\overline{\mathcal{M}}_\Gamma(X)$ relative to ϵ – for the general notion of a perfect obstruction theory see [BF97], and for the existence of it in our context see [GS13, §6]. Hence, there is a corresponding virtual pull-back

$$\epsilon^! : A_*(\mathfrak{M}_\Gamma(\mathcal{A}_X)) \longrightarrow A_*(\overline{\mathcal{M}}_\Gamma(X))\,.$$

Second, the stack $\mathfrak{M}_\Gamma(\mathcal{A}_X$ is equi-dimensional and therefore has a well-defined fundamental class $[\mathfrak{M}_\Gamma(\mathcal{A}_X)] \in A_*(\overline{\mathcal{M}}_\Gamma(X))$ [AW18, Prop 1.6.1]. Thus, the virtual fundamental class on $\overline{\mathcal{M}}_\Gamma(X)$ is defined by

$$[\overline{\mathcal{M}}_\Gamma(X)]^{vir} = \epsilon^!([\mathfrak{M}_\Gamma(\mathcal{A}_X)])\,.$$

Log Gromov–Witten invariants of X are then obtained by integration over this virtual fundamental class – to concretely write this integral, one needs to discuss the technically subtle issue of "evaluation spaces" which we skip in these notes (for a detailed discussion see [ACGM10]).

Exercise 14.6.2. Let $X = \mathbb{P}^2$ with the divisorial log structure defined by a smooth cubic curve $E \subset \mathbb{P}^2$. Let Γ be the type of $g = 0$ curves of degree $\beta = 1$, with a single marked point with contact order $c = 3$ with E. Show that the moduli space $\overline{\mathcal{M}}_\Gamma(X)$ consists of 9 reduced points. Hint: think about the flex points of E.

One can show that in the situation of Exercise 14.6.2, the virtual fundamental class coincides with the usual fundamental class, and so this number 9 is an example of log Gromov–Witten invariant.

Example 14.6.3. Let X be the blow-up of $\mathbb{P}^2$ at a point $p \in \mathbb{P}^2$. Denote by E the exceptional curve and by D the strict transform of a line in $\mathbb{P}^2$ passing through p – see Figure 10. We view X as a log scheme for the divisorial log structure defined by D. For every $k \geq 1$, let Γ_k be the type of $g = 0$ curves of degree $\beta = k[E]$, with a single marked point with contact order $c = k$ with E. For $k = 1$, the moduli space $\overline{\mathcal{M}}_{\Gamma_1}(X)$ is a single reduced point corresponding to the exceptional curve. For $k > 1$, $\overline{\mathcal{M}}_{\Gamma_k}(X)$ is a complicated moduli space of degree k covers of E fully ramified over the intersection point $E \cap D$. However, one can show that the virtual dimension of $\overline{\mathcal{M}}_{\Gamma_k}(X)$ is equal to zero for every $k \geq 1$. The corresponding log Gromov–Witten invariant is shown in [GPS10, Proposition 6.1] to be given by

$$\deg[\overline{\mathcal{M}}_{\Gamma_k}(X)]^{vir} = \frac{(-1)^{k-1}}{k^2}\,. \tag{29}$$

Note that this log Gromov–Witten invariant is only a *virtual* count of curves if $k > 1$: in particular, it is non-integer and can be negative. The formula (29) plays an essential role in the application of log Gromov–Witten theory to mirror symmetry – see [GHK15] in dimension two and [AG22] in higher dimension.

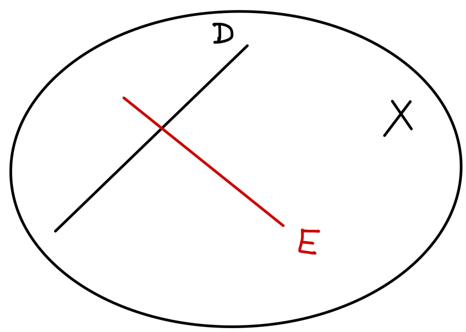


FIGURE 10. Exceptional curve E and strict transform D of a line.

ADVANCED TOPICS START HERE

15. Resolution of singularities using stacks and logarithmic geometry

15.1. **Hidden smoothness and resolution.** By Dan. This should be something between the Taipei lectures and the James60 unfulfilled promise

The early origin of stacks included the idea that stacks present hidden smoothness: the quotient of a smooth variety by a group action retains properties akin to a smooth variety, and the formalism of stacks makes this rigorous - the stack itself is smooth.

Similarly, the origin of logarithmic geometry is tightly connected with hidden smoothness. In logarithmic geometrym, any toric variety is logarithmically smooth, immediately enlarging the scope of smooth-like object. Also, in logarithmic geometry, and family of nodal curves is logarithmically smooth, immediately justifying their choice for compact moduli spaces.

Historically, these hidden smoothness properties appear sporadically also in resolution of singularities: many explicit varieties are naturally embedded in weighted projective spaces, which are orbifolds. And in many cases their singularities can be treated using weighted blowing up. On the logarithmic side, a variety with toroidal singularities is easily resolvable. Exceptional divisors can be treated using logarithmic techniques, for instance logarithmic derivatives and logarithmic differentials.

Surprisingly though, it took some time before stacks and logarithmic geometry were brought into general resolution algorithms. Ours is this story.

15.2. **Resolving singularities in families and the necessity of logarithmic geometry.** Algebraic geometry puts a high value on working with schemes in families. If one has a resolution of singularities of schemes, what should one do to create resolution of singularities for a family of schemes?

We know, for instance through the moduli spaces of stable curves, that there is no way we could take a family of schemes $X \to B$ and resolve all its fibers. Some compromise must be made. The classical approach is to replace a resolution by *semistable reduction* which will be discussed later, and, in its original form, this is only pursued over one-dimensional bases.

Logarithmic geometry provides us with a different approach for a compromise. We note, following Fumiharu Kato, that a family of stable curves $X \to B$ is log smooth, and in fact the morphism is integral and *saturated*.

We are thus presented with the following problem:

Problem 15.2.1. given a projective surjective morphism $X \to B$, with X, B normal, find a logarithmic modification $X' \to B'$ which is logarithmically smooth. This procedure shoudl be functorial for log smooth morphisms on X.

This problem is again solved in characteristic 0, which is a new concept even when B is a point, since functoriality is different!

15.3. **Functoriality in logarithmic resolution and the necessity of stacks.** Functoriality in log resolution does present a difficulty: suppose you want to logarithmically resolve the nodal curve $x^2 = u^2$ on the affine plane endowed with the logarithmic structure associated to $\{x = 0\}$. It is of course natural to blow up the origin, and that in fact does provide a logarithmic resolution, where the exceptional is included in the log structure.

But this picture is the pullback of $y = u^2$ via the logarithmically smooth map $x = y^2$. In other words, the blowup on the y, u plane must pull back to the blowup of (x, u). Since $x = \sqrt{y}$ this m,eans we must blow up the object $(\sqrt{y}, u)$.

The only way we found to understand this is the *weighted blowup* associated with $(y^{1/2}, u)$. This actually works, and is an essential feature of log resolution.

In other words, both logarithmic structures and algebraic stacks are an essential ingredient of a central problem in resolution of singularities.

15.4. **Weighted resolution of singularities.** The story takes a surprising turn here: now that we agreed that weighted blowups are needed for log resoolution, maybe they have something to say about resolution?

tell the story of weighted resolution

15.5. **Logarithmic weighted resolution of singularities.** Tell the story of Quek toroidal resolution and NCD resolution

15.6. **A view to the future.** Questions in characteristic 0, questions in positive characteristics

What follows is copied almost verbatim from the McKernan submission – with coauthors Belotto da Silva, Quek, Temkin, Włodarczyk.

15.7. **Weighted resolution of singularities.** We continue to work exclusively in characteristic 0.

The main result of [ATW19] provides a resolution of singularities $X' \to X$ where X' is a Deligne–Mumford stack. Some may see this as a drawback, though we do not. In any case, standard techniques of combinatorial nature allow one to replace a resolution using Deligne–Mumford stacks by a resolution using varieties. See [ATW19, precise statememnt].

In analogy with Hironaka's proof, the transition in [ATW19] from Y to Y' involves *weighted blowups*, again allowing one to track the change of geometry (e.g the Chow ring, see [AO23]), and guaranteeing projectivity on coarse moduli spaces.

15.8. **Toroidal weighted resolution of singularities.** In [Que20], Quek extended the techniques of [ATW19] and provided a *toroidal resolution* $X' \to X$, in which X' is a toroidal Deligne–Mumford stack, and the exceptional divisor is subsumed in the toroidal structure. In particular X' may have toroidal singularities.

Once again, toroidal singularities are easily resolved. In particular Quek deduces a result equivalent in many ways to Hironaka's, [Que20, precise statement]. On the other hand, one no longer tightly controls the geometric changes of the ambient variety through $Y' \to Y$, as the transitions involve *toroidal weighted blowups*, which are more intricate than weighted blowups.

We will recall Quek's methods and intermediate results in Section 15.14 below, as they are relevant to the present exposition.

15.9. **Logarithmic weighted resolution of singularities.** The main result here shows how to combine the results of [ATW19] with the methods of [Que20] to give a functorial logarithmic resolution of singularties $X' \to X$, adhering to McKernan's principle of a natural, fully motivated and understandable proof.

In brief, to a singular point p of a Deligne–Mumford stack X embedded with pure codimension in a smooth deligne–Mumford stack Y, meeting a simple normal crossings divisor D properly, we attach an upper-semicontinuous singularity invariant $\mathrm{loginv}^*_X(p)$ taking values in a well-ordered set Γ. The formation of loginv^* and J^* is functorial for *smooth* base change on Y. The maximal locus of loginv^* is the support of a weighted blowup center J^*, which is also functorial.

Theorem 15.9.1 (Functorial logarithmic resolution of singularities)**.** *The weighted blowup $Y' \to Y$ of the reduced center associated to J^* is a smooth stack with transformed simple normal crossings divisor D', formed as union of the pre-image of D and the exceptional. The proper transform $\mathcal{X}'$ of $\mathcal{X}$ satisfies*

$$\mathrm{loginv}^*_{\mathcal{X}'}(p') < \mathrm{loginv}^*_{\mathcal{X}}(p)$$

for any point p in the support of J^* and any $p' \in X'$ above it.

After finitely many iterations, the proper transform $X^{(n)}$ is a smooth locus on a smooth stack $Y^{(n)}$ carrying a simple normal crossings divisor $D^{(n)}$.

Remark 15.9.2. We come close to proving that "sufficiently strong resolution implies equally strong logarithmic resolution", but in some steps we go into the techniques of [Que20] to complete the proof.

15.10. **Resolution via embedded resolution and principalization.** We follow standard techniques for resolution of singularities which reduce the geometric problem to more algebraic ones.

First, the procedure we devise requires X to be embedded in a smooth variety Y. This can always be achieved locally, but to globalize it one needs to verify that the procedure is independent of choices, what we call the *re-embedding principle*.

Second, instead of working with $X \subset Y$ one works with improving the ideal $\mathcal{I}_X \subset \mathcal{O}_Y$. The problem is *principlalization* of an ideal $\mathcal{I} \subset \mathcal{O}_Y$, which in our case boils down to having the total transform of $\mathcal{I}$ become exceptional for $Y^{(n)} \to Y$. A simple observation, sometime
23→ known as *accidental resolution*, guarantees that X is resolved along the way.[23]

15.11. **Where this comes from.** In [ABdSTW24] one proves resolution and principalization in the presence of foliations. In that situation working with extended invariants similar to loginv^* introduced here becomes essential, and the logarithmic case occurs as a natural byproduct. On the other hand, Quek worked on simplifying his presentation of results of [Que20] for a course at Stanford and a workshop in Heidelberg. We decided to combine
24→ forces and attempt to describe as natural an argument as we could produce.[24]

15.12. **Principalization of monomial ideals.** We revisit weighted principalization of monomial ideals (see [Que20, AQ21, Wł22]) with an approach which feeds into of the paper. Assume given a variety Y with a normal crossings divisor D, a non-unit monomial ideal $\mathcal{I}$, and point $p \in Y$ where $\mathcal{I}$ vanishes. The paper [ATW19] provides a functorially defined upper-semicontinuous invariant $\mathrm{inv}_{\mathcal{I}}(p) = (b_1, \ldots, b_m)$, taking values in a well-ordered set, with rational terms $b_1 \leq \cdots \leq b_m$, and center J, locally of the form $(y_1^{b_1}, \ldots, y_m^{b_m})$. Taking

[23](Dan) recall these at the end

[24](Dan) Remarks on other approaches - Jarek and A-Quek

the weighted blow up $Y' \to Y$ of the associated reduced center $\bar{J}$, with transformed ideal $\mathcal{I}'$, the main result of [ATW19] shows

Theorem 15.12.1. *For every point $p' \in V(\mathcal{I}) \subset Y'$ over p, one has*

$$\mathrm{inv}_{\mathcal{I}'}(p') < \mathrm{inv}_{\mathcal{I}}(p).$$

One deduces from this that after finitely many repetition the transformed ideal no longer vanishes, giving a principalization result which in turn implies resolution.[25]

Note that the divisor D does not figure in the above results. Here we note the following consequence of functoriality:

Proposition 15.12.2 (Principalization of monomial ideals)**.** *The center J is toroidal, in particular the parameters z_i may be taken to be monomials.*

In particular $\mathcal{Y}' \to Y$ is a toroidal morphism with associated normal crossings divisor $D' \subset Y$, and $\mathcal{I}'$ is a monomial ideal. After finitely many iterations, the pullback of the ideal $\mathcal{I}$ is a locally principal monomial ideal on a smooth variety $Y^{(n)}$ carrying a normal crossings divisor $D^{(n)}$.

Proof. Passing to a local chart we may assume there is an étale morphism $\phi : Y \to \mathbb{A}^n := \operatorname{Spec} k[t_1, \dots, t_n]$ such that $D = \phi^*(V(t_1 \cdots t_n))$ and a monomial ideal $\mathcal{I}_0 \subset \mathcal{O}_{\mathbb{A}_n}$ such that $\mathcal{I} = \phi^*\mathcal{I}_0$. It follows that $\mathcal{I}_0$ is invariant under the action of the torus $T := \operatorname{Spec} k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Functoriality of centers implies that J_0 associated to $\mathcal{I}_0$ is T-invariant, hence its generators may be chosen to be T-eigenvectors, namely monomials. Functoriality for étale morphisms provides that $J = \phi^* J_0$, which enjoys the same properties, as needed. ♣

15.13. **Principalization of ideals.** The proof of the main result of [ATW19], already used in section 15.12, relies on the notions and ideas we now discuss.

15.13.1. *Derivatives, order, and maximal contact.*

Definition 15.13.2. Consider an ideal $\mathcal{I} \subset \mathcal{O}_Y$ on a smooth variety Y and the sheaf of differential operators $\mathcal{D}_Y^{\leq a}$ of order $\leq a$ on Y. We define $\mathcal{D}^{\leq a}(\mathcal{I}) \subset \mathcal{O}_Y$ to be the ideal generated by the collection of $\nabla(f)$, with ∇ a local section of $\mathcal{D}_Y^{\leq a}$ and f a local section of $\mathcal{I}$.

It is clear that $\mathcal{I} \subset \mathcal{D}_Y^{\leq 1}(\mathcal{I}) \subset \cdots$ is an increasing chain of ideals. If $\mathcal{I} \neq 0$ it eventually stabilizes at the trivial ideal (1).

Since we work in characteristic 0 we have that the composition $\mathcal{D}_Y^{\leq i}\mathcal{D}_Y^{\leq j} = \mathcal{D}_Y^{\leq i+j}$ hence $\mathcal{D}_Y^{\leq i}\mathcal{D}_Y^{\leq j}(\mathcal{I}) = \mathcal{D}_Y^{\leq i+j}(\mathcal{I})$.

Derivative ideals are compatible with localization: $\mathcal{D}^{\leq a}(\mathcal{I}_p) = \left(\mathcal{D}^{\leq a}(\mathcal{I})\right)_p$.

Definition 15.13.3. We define the *order of $\mathcal{I}$ at p* as follows:

$$\mathrm{ord}_p(\mathcal{I}) = \min\left\{a : \mathcal{D}^{\leq a}(\mathcal{I}_p) = \mathcal{O}_{Y,p}\right\}.$$

We set $\mathrm{maxord}(\mathcal{I}) = \max_p \mathrm{ord}_p(\mathcal{I})$. The order of the zero ideal is set to ∞.

The locus of order $> a$ is thus the closed vanishing locus of $\mathcal{D}^{\leq a}(\mathcal{I})$, hence order is upper-semicontinuous.

[25] In [ATW19] the notation a_i, x_i is used instead of y_i, b_i, but we replace the notation in anticipation of discussion below: b_i are the second stage in our invariant.

Definition 15.13.4. If $\operatorname{ord}_p(\mathcal{I}) = a$, the *maximal contact ideal* of $\mathcal{I}$ at p is $\mathcal{D}^{\leq a-1}(\mathcal{I})_p$. A *maximal contact element* of $\mathcal{I}$ at p is a section $x \in \mathcal{D}^{\leq a-1}(\mathcal{I})_p$ with $\mathcal{D}_Y((x))_p = (1)$, that is, some derivative of x is a unit.

Since we are working in characteristic 0, maximal contact elements always exist *locally*: if $1 \in \mathcal{D}^{\leq a}(\mathcal{I})$ then there is $\nabla \in \mathcal{D}^{\leq 1}_{Y,p}$ and $x \in \mathcal{D}^{\leq a-1}(\mathcal{I})_p$ such that $\nabla x = 1$.

Note that the formation of derivative ideals is functorial for smooth morphisms: given a smooth morphism $f : Y_1 \to Y$ with $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$ and $f(p_1) = p$ we have $\mathcal{D}^{\leq a}_{Y}(\mathcal{I}_p)\mathcal{O}_{Y_1} = \mathcal{D}^{\leq a}_{Y_1}(\mathcal{I}\mathcal{O}_{Y_1,p_1})$. It follows that

Proposition 15.13.5. *For a smooth morphism $f : Y_1 \to Y$ with notation as above we have* $\operatorname{ord}_{p_1}(\mathcal{I}_1) = \operatorname{ord}_p(\mathcal{I})$. *Also $x \in \mathcal{O}_{Y,p}$ is a maximal contact element if and only if $f^*x \in \mathcal{O}_{Y_1,p_1}$ is a maximal contact element.*

If, moreover, f is surjective, then $\operatorname{maxord}(\mathcal{I}_1) = \operatorname{maxord}(\mathcal{I})$.

We note, however, that the *choice* of a maximal contact x is not unique, and is therefore not functorial.

15.13.6. *Coefficient ideals.* To go further we need some inductive process, and the standard approach involves induction on dimension by restriction to $V(x)$. Since x is only known to exist locally, functoriality is required to glue the results.[26]

26→ This inductive process requires defining an ideal on $V(x)$ which remembers much of what $\mathcal{I}$ is. The standard approach involves a coefficient ideal, which is already defined on Y.[27]

27→ **Definition 15.13.7.** Say $\mathcal{I}$ has maximal order a. Consider $D^{a-i}(\mathcal{I})$ as having weight i, for $i < a$. We take the ideal $C(\mathcal{I}, a)$ generated by all the monomials in $D^{a-1}(\mathcal{I}), \ldots, D(\mathcal{I}), \mathcal{I}$ of weighted degree $\geq a!$. Concretely,

$$C(\mathcal{I}, a) = \sum_{\sum i \cdot b_i \geq a!} D^{a-1}(\mathcal{I})^{b_1} \cdots D(\mathcal{I})^{b_{a-1}} \cdot \mathcal{I}^{b_a}.$$

Proposition 15.13.8. *Formation of coefficient ideals is functorial in smooth morphisms: with notation as above, $C(\mathcal{I}_1, a) = C(\mathcal{I}, a)\mathcal{O}_{Y_1}$.*

15.13.9. *Centers and their properties.* Assume $\operatorname{ord}_{\mathcal{I}}(p) = a_1$ and x_1 is a maximal contact element. If $\mathcal{I}_p = (x_1^{a_1})$ we are basically done: we define the invariant of the ideal at p to be $\operatorname{inv}_p(\mathcal{I}) = (a_1)$, and we define the center as $J = (x_1^{a_1})$. Otherwise define $\mathcal{I}[2] = C(\mathcal{I}, a)|_{V(x_1)}$, the restriction. By assumption it is not the zero ideal. We may now invoke induction, so that invariants and centers are already defined on $V(x_1)$.

Definition 15.13.10. Say $\operatorname{inv}_p(\mathcal{I}[2]) = (b_2, \ldots, b_k)$, with center $(\bar{x}_2^{b_2}, \ldots, \bar{x}_k^{b_k})$. Choose *arbitrary* lifts x_i of $\bar{x}_i$. Define $a_i = b_i/(a_1 - 1)!$ for $i = 2, \ldots, k$. Set

$$\operatorname{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k)$$

and

$$J = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}).$$

28→ **Proposition 15.13.11** (Functoriality, ℵ-Të-Wł precise reference). [28]

[26](Dan) Mention alternative approaches such as Jarek's process of increasing Rees algebras, or Kawanoue–Matsuki

[27](Dan) Remark on the fractionally graded Rees algebra approach

[28](Dan) sync with previous text

- *The invariants and centers are independent of choices.*
- *If $Y_1 \to Y$ is smooth, $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, and $p_1 \mapsto p$ then*
$$\mathrm{inv}_{\mathcal{I}}(p) = \mathrm{inv}_{\mathcal{I}_1}(p_1).$$
- *If $Y_1 \to Y$ is smooth, $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, and $p_1 \mapsto p$ then*
$$J_1 = J\mathcal{O}_{Y_1}.$$

This implies *gluing* as well as *equivariance.*

Coefficient ideals and centers possess remarkable properties, formalized by Włodarczyk and Kollár [Kol07], showing it faithfully retains information in $\mathcal{I}$ yet is more homogeneous. In particular:

Proposition 15.13.12. *Assume* $\mathrm{maxord}(\mathcal{I}) = a_1$ *with maximal contact x_1. Let $J_{\mathcal{I}}$ be the local invariant of $\mathcal{I}$, and let $C_{x_1}(\mathcal{I}, a_1) = Gr_{x_1}(\mathcal{I}, a_1)$, the graded ideal with respect to x_1.*

(1) $\mathrm{maxinv}(C(\mathcal{I}, a_1)) = \mathrm{maxinv}(C_{x_1}(\mathcal{I}, a_1)) = (a_1 - 1)!\mathrm{maxinv}(\mathcal{I})$,
(2) x_1 is maximal contact for $C(\mathcal{I}, a_1)$ and $C_{x_1}(\mathcal{I}, a_1)$, and
(3) $J_{C(\mathcal{I},a_1)} = J_{\mathcal{I}}^{(a_1-1)!}$ and $J_{C_{x_1}(\mathcal{I},a_1)} = Gr_{x_1} J_{\mathcal{I}}^{(a_1-1)!}$.

15.13.13. *Blowing up.* We recall briefly a presentation of the stack theoretic blowup associated to $J = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k})$, see [QR]. First, we define
$$(a_1, \ldots, a_k) = \ell(w_1^{-1}, \ldots, w_k^{-1}),$$
where ℓ, w_i all integers and $\gcd(w_1, \ldots, w_k) = 1$, and
$$\overline{J} = (x_1^{1/w_1}, x_2^{1/w_2}, \ldots, x_k^{1/w_k}).$$
Considering $\overline{J}$ as a valuative $\mathbb{Q}$-ideal, for any $k \in \mathbb{Z}$ we let $\mathcal{J}_k = \overline{J}^k \cap \mathcal{O}_Y$ and write
$$\mathcal{A}_J := \bigoplus_{k \in ZZ} \mathcal{J}_k,$$
with its "irrelevant" or "vertex" ideal
$$\mathcal{A}_{J+} := \left(\bigoplus_{k>0} \mathcal{J}_k \right)$$
generated by terms of positive degree, and morphism
$$B_J := \mathrm{Spec}_Y \mathcal{A}_J \to \mathbb{A}^1$$
induced by the grading. Writing $s \in \mathcal{I}_{-1} \simeq \mathcal{O}_Y$ for the section corresponding to $1 \in \mathcal{O}_Y$, the locus
$$C_J := V(s) = \mathrm{Spec}_Y \bigoplus_{k \geq 0} \mathcal{J}_k / \mathcal{J}_{k+1}$$
is the weighted normal cone of $\overline{J}$, and B_J is the degeneration of Y to the weighted normal cone. We note that $V(\mathcal{A}_{J+}) \simeq \mathrm{supp}(\overline{J}) \times \mathbb{A}^1$. Set $B_{J+} := B_J \smallsetminus V(\mathcal{A}_{\overline{J}+})$.

We define
$$Y' = Bl_{\overline{J}}(Y) := [B_{J+}/\mathbb{G}_m].$$
The exceptional divisor is $[C_J \smallsetminus V(\mathcal{A}_{\overline{J}+})/\mathbb{G}_m]$.

15.13.14. *Local equations.* A local presentation of this construction is desirable, for which we use an open set where $(x_1, \ldots, x_k)$ can be completed to local parameters $(x_1, \ldots, x_n)$ so
29→ that $Y \to \mathbb{A}^n$ is étale. We continue to follow [QR].[29]

In this case

$$\mathcal{A}_J = \mathcal{O}_Y[s, x_1' \ldots, x_k']/(x_1 - s^{w_1}x_1', \ldots, x_1 - s^{w_k}x_k'),$$

on which $\mathbb{G}_m$ with parameter t acts via

$$t \cdot (s, x_1' \ldots, x_k') = (t^{-1}s, t^{w_1}x_1' \ldots, t^{w_k}x_k').$$

In other words s appears in degree (-1), and x_i' in degree w_i, as the presentation above suggests.

The exceptional locus $V(s)$ in $B_J = \mathrm{Spec}_Y(\mathcal{A}_J)$ is

$$\mathrm{Spec}_Y\left(\mathcal{O}_Y/(x_1, \ldots, x_k)\,[x_1' \ldots, x_k']\right),$$

which is an $\mathbb{A}^k$-bundle over the center $V(J) = \mathrm{Spec}_Y\left(\mathcal{O}_Y/(x_1, \ldots, x_k)\right)$. We note the resulting presentation of the vertex ideal $\mathcal{A}_{\overline{J}+} = (x_1', \ldots, x_k')$. Its vanishing locus is simply

$$\mathrm{Spec}_Y\left(\mathcal{O}_Y/(x_1, \ldots, x_k)\,[s]\right) \quad \simeq \quad V(J) \times \mathbb{A}^1.$$

15.13.15. *The invariant drops.* We find it useful to illuminate the fact that the invariant drops already in this established case.

On B_J one considers $\mathcal{I}\mathcal{O}_{B_J} = \mathcal{I}_{C_J}^{\ell}\mathcal{I}'$, with $\mathcal{I}'$ the *weak transform* of $\mathcal{I}$ on B_J. In the local presentation above, we have that $\mathcal{I}\mathcal{A}_J = (s)^l\mathcal{I}'$.

We note that $B_J \smallsetminus C_J = \mathbb{G}_m \times Y$, so by functoriality $\mathrm{maxinv}_{B_J \smallsetminus C_J}(\mathcal{I}') = \mathrm{maxinv}_Y(\mathcal{I}) = (a_1, \ldots, a_k)$. Upper semicontinuity implies that $\mathrm{maxinv}_{B_J}(\mathcal{I}') \geq (a_1, \ldots, a_k)$. The fundamental reason invariants drop is the following stronger fact:

Proposition 15.13.16. *We have*

$$\mathrm{maxinv}_{B_J}(\mathcal{I}') = \mathrm{maxinv}_Y(\mathcal{I}) = (a_1, \ldots, a_k),$$

with maximal locus precisely $V(A_{J+}) = \mathrm{supp}(\overline{J}) \times \mathbb{A}^1$.

Proof of Theorem 15.12.1 given Proposition 15.13.16. We slightly abuse notation, using $\mathcal{I}'$ for the weak transform on the bloaup Y' as well. This should not cause confustion since it
30→ pulls back to $\mathcal{I}'$ on B_{J+}.[30]

Since the maximality locus on B_J is $V(A_{J+})$ and $B_{J+} = B_J \smallsetminus V(A_{J+})$, we have $\mathrm{maxinv}_{B_{J+}}(\mathcal{I}') < \mathrm{maxinv}_{B_J}(\mathcal{I}') = \mathrm{maxinv}_Y(\mathcal{I}) = (a_1, \ldots, a_k)$. Observe that the quotient map $B_{J+} \to Y'$ is, by definition, smooth and surjective. By functoriality we have $\mathrm{maxinv}_{B_{J+}}(\mathcal{I}') = \mathrm{maxinv}_{Y'}(\mathcal{I}')$, hence $\mathrm{maxinv}_{Y'}(\mathcal{I}') < \mathrm{maxinv}_Y(\mathcal{I})$ as needed. ♣

Lemma 15.13.17. *We have*

$$(\mathcal{D}(\mathcal{I}))' \subset \mathcal{D}(\mathcal{I}')$$

Proof of Lemma. We work this out on B_J using the local presentation above. We note that if $\mathcal{I}$ is transformed using $(s)^{-\ell}$ then $\mathcal{D}(\mathcal{I})$ is transformed using $(s)^{-\ell+w_1}$, where again $w_i a_i = \ell$.

We prove a slightly stronger result, which will go some way towards results below: we show that $(\mathcal{D}(\mathcal{I}))' \subset \mathcal{D}_{B_J/\mathbb{A}^1}(\mathcal{I}')$. This suffices since $\mathcal{D}_{B_J/\mathbb{A}^1}(\mathcal{I}') \subset \mathcal{D}(\mathcal{I}')$. It also makes computations easier since we need not take s-derivarives.

[29](Dan) Draw picture of the deformation to the weighted normal cone

[30](Dan) improve notation?

Plugging in $x_i = s^{w_i} x_i'$ we obtain $\mathcal{I}' = (s^{-\ell} f(s^{w_i} x_i') \mid f \in \mathcal{I})$, so that, by the chain rule,

$$\begin{aligned}\mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}') &= \left(\tfrac{\partial}{\partial x_i'}(s^{-\ell} f(s^{w_i} x_i')) \mid f \in \mathcal{I}, i = 1, \ldots, n\right) \\ &= \left(s^{-\ell} \tfrac{\partial f}{\partial x_i}(s^{w_i} x_i') \cdot s^{w_i} \mid f \in \mathcal{I}, i = 1, \ldots, n\right) \\ &= \left(s^{-\ell + w_i} \tfrac{\partial f}{\partial x_i}(s^{w_i} x_i') \mid f \in \mathcal{I}, i = 1, \ldots, n\right)\end{aligned}$$

On the other hand $\mathcal{D}(\mathcal{I}) = \left(\frac{\partial f}{\partial x_i}(x_i) \mid f \in \mathcal{I}, i = 1, \ldots, n\right)$ so that

$$\mathcal{D}(\mathcal{I})' = \left(s^{-\ell + w_1} \frac{\partial f}{\partial x_i}(s^{w_i} x_i') \mid f \in \mathcal{I}, i = 1, \ldots, n\right).$$

Note that $w_1 \geq w_i$ for all i, so

$$\mathcal{D}(\mathcal{I})' = \left(s^{-\ell + w_1} \frac{\partial f}{\partial x_i}(s^{w_i} x_i')\right) \subset \left(s^{-\ell + w_i} \frac{\partial f}{\partial x_i}(s^{w_i} x_i'))\right) = \mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}'),$$

as needed. ♣

Proof of Proposition 15.13.16. [31] ←31

The lemma implies inductively that $(\mathcal{D}^i(\mathcal{I}))' \subset \mathcal{D}^i(\mathcal{I}')$.

Two particular outcomes are that

$$1 \in (\mathcal{D}^{a_1}(\mathcal{I}))' \subset \mathcal{D}^{a_1}(\mathcal{I}') \qquad \text{and} \qquad x_1' \in (\mathcal{D}^{a_1-1}(\mathcal{I}))' \subset \mathcal{D}^{a_1-1}(\mathcal{I}'),$$

in particular $\operatorname{maxord}(\mathcal{I}') = a_1$ with maximal contact x_1'.

The homogeneity property of $C(\mathcal{I}', a_1)$ with respect to x_1' implies that

$$\operatorname{maxinv}(C(\mathcal{I}', a_1)) = \operatorname{maxinv}(C_{x_1'}(\mathcal{I}', a_1)).$$

By induction applied to $V(x_1')$, the graded ideal has graded center

$$({x_1'}^{a_1}, {x_2'}^{a_2}, \ldots, {x_k'}^{a_k})^{(a_1-1)!}$$

By semicontinuity this is the center of $C(\mathcal{I}', a_1))$, as needed. ♣

Remark 15.13.18. The proof in fact shows that the invariant is compatible with grading with respect to $J_{\mathcal{I}}$, and further, compatible with multigrading with respect to all of $x_1, \ldots, x_k$. This flexible compatibility with grading is useful when using more refined centers, as we do below.

15.14. **The toroidal analogues.** We now replace everything by logarithmic analoques, giving first Quek's toroidal results. The idea is trhat adding the adjective "logarithmic" or "toroidal" in every step works as stated!

- We now assume Y is provided with a simple normal crossings divisor D, giving rise to a logarithmically smooth structure sometimes denoted $(Y|D)$, and sheaves of logarithmic differential operators $\mathcal{D}^{\leq a}_{(Y|D)}$.
- We define $\mathcal{D}^{\leq a}_{(Y|D)}(\mathcal{I}) \subset \mathcal{O}_Y$ to be the ideal generated by the collection of $\nabla(f)$, with ∇ a local section of $\mathcal{D}^{\leq a}_{(Y|D)}$ and f a local section of $\mathcal{I}$. This time the ideals $\mathcal{I} \subset \mathcal{D}^{\leq 1}_Y(\mathcal{I}) \subset \cdots$ stabilize at the monomial saturation $M(\mathcal{I})$ [Kol07, ATW20a]

[31] (Dan) reorganize. Must have some lifting argument from cone to B to replace formal lifting. The lemma above is probably overdone

We define the *logarithmic order of* $\mathcal{I}$ *at* p as follows:

$$\operatorname{logord}_p(\mathcal{I}) = \min\left\{a : \mathcal{D}^{\leq a}_{(Y|D)}(\mathcal{I}_p) = \mathcal{O}_{Y,p}\right\}.$$

We set $\operatorname{maxlogord}(\mathcal{I}) = \max_p \operatorname{logord}_p(\mathcal{I})$. If $M(\mathcal{I}) \neq (1)$ the logarithmic order is set to ∞.

• If $\operatorname{logord}_p(\mathcal{I}) = a < \infty$, the *logarithmic maximal contact ideal* of $\mathcal{I}$ at p is $\mathcal{D}^{\leq a-1}_{(Y|D)}(\mathcal{I})_p$. A *logarithmic maximal contact element* of $\mathcal{I}$ at p is a section $x \in \mathcal{D}^{\leq a-1}_{(Y|D)}(\mathcal{I})_p$ with $\mathcal{D}_{(Y|D)}((x)) = (1)$, that is, some logarithmic derivative of x is a unit.

The formation of logarithmic derivative ideals is functorial for smooth morphisms, and even for logarithmically smooth morphisms. It follows that for a logarithmically smooth morphism $f : Y_1 \to Y$ with notation as above we have $\operatorname{logord}_{p_1}(\mathcal{I}_1) = \operatorname{logord}_p(\mathcal{I})$ and $x \in \mathcal{O}_{Y,p}$ is a logarithmic maximal contact if element and only if $f^*x \in \mathcal{O}_{Y_1,p_1}$ is a logarithmic maximal contact element.

• Say $\mathcal{I}$ has maximal logarithmic order $a < \infty$. Consider $D^{a-i}_{(Y|D)}(\mathcal{I})$ as having weight i, for $i < a$. We take the logarithmic coefficient ideal $C_{(Y|D)}(\mathcal{I}, a)$ generated by all the monomials in $D^{a-1}_{(Y|D)}(\mathcal{I}), \ldots, D(\mathcal{I})_{(Y|D)}, \mathcal{I}$ of weighted degree $\geq a!$. Concretely

$$C_{(Y|D)}(\mathcal{I}, a) = \sum_{\sum i \cdot b_i \geq a!} D^{a-1}_{(Y|D)}(\mathcal{I})^{b_1} \cdots D(\mathcal{I})^{b_{a-1}}_{(Y|D)} \cdot \mathcal{I}^{b_a}.$$

The formation of these ideals is functorial for logarithmically smooth morphisms.

• One may have $\operatorname{logord}_{\mathcal{I}}(p) = \infty$, which happens precisely when $M(\mathcal{I})_p \neq 1$. We then define $\operatorname{loginv}_{\mathcal{I}}(p) = \infty$ and $J_{\log} = Q := M(\mathcal{I})$. This means that in the induction we must allow invariants to be infinite and centers to involve monomial ideals and their fractional powers.

Assume $\operatorname{logord}_{\mathcal{I}}(p) = a_1 < \infty$ and x_1 is a logarithmic maximal contact element. If $\mathcal{I}_p = (x_1)$ we define the logarithmic invariant of the ideal at p to be $\operatorname{loginv}_p(\mathcal{I}) = (a_1)$, and we define the logarithmic center as $J_{\log} = (x_1^{a_1})$. Otherwise define $\mathcal{I}[2] = C_{(Y|D)}(\mathcal{I}, a)|_{V(x_1)}$, the restriction.

We may again invoque induction, so that invariants and centers are already defined on $V(x_1)$.

If $\operatorname{loginv}_p(\mathcal{I}[2]) = (b_1, \ldots, b_k)$, with center $(\bar{x}_2^{b_2}, \ldots, \bar{x}_k^{b_k})$, we define again $a_i = b_i/(a_1 - 1)!$ for $i = 2, \ldots, k$, with

$$\operatorname{loginv}_p(\mathcal{I}) = (a_1, \ldots, a_k)$$

and, using arbitrary lifts x_i,

$$J_{\log} = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}).$$

If instead $\operatorname{loginv}_p(\mathcal{I}[2]) = (b_1, \ldots, b_k, \infty)$, with center $(\bar{x}_2^{b_2}, \ldots, \bar{x}_k^{b_k}, \bar{Q}^{1/e})$, we set $d = (a - 1)!e$ and we have similarly $\operatorname{loginv}_p(\mathcal{I}) = (a_1, \ldots, a_k, \infty)$ and

$$J_{\log} = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d}).$$

This time Q is the *unique monomial lift* of $\bar{Q}$.

• It is shown in [Que20] that the logarmithmic weighted blowup $\pi : Y' \to Y$ of the reduced center associated to J, with transformed divisor $D' = \pi^*D \cup Exc(\pi)$, and transformed ideal $\mathcal{I}'$, satisfies again

$$\operatorname{loginv}_{\mathcal{I}'}(p') < \operatorname{loginv}_{\mathcal{I}}(p).$$

The blowup $Y' \to Y$ is only *log smooth*, but it satisfies stronger functoriality for *log smooth base change.*

We will not reprove that result here, but revisit its underlying arguments in the next section.

15.15. **Logarithmic principalization.** Consider again a smooth variety Y and a normal crossings divisor D. Let $\mathcal{I}$ be an ideal. Fix a point p where $\mathcal{I}$ vanishes. In the previous section we defined the logarithmic invariant and logarithmic center associated to $\mathcal{I}$.

Consider the invariant $\operatorname{inv}_Q(p) = (c_1, \ldots, c_m)$ introduced in Section 15.12, and the associated center $(y_1^{c_1}, \ldots, y_m^{c_m})$. Denoting $b_i = c_i/d$, we set $\operatorname{inv}_{Q^{1/d}}(p) = (b_1, \ldots, b_m)$ with associated center $(y_1^{b_1}, \ldots, y_m^{b_m})$. Define a new invariant

$$\operatorname{loginv}^*_{\mathcal{I}}(p) = \begin{cases} \operatorname{loginv}_{\mathcal{I}}(p) = (a_1, \ldots, a_l) & \text{if } Q^{1/d} \text{ is not present in } J \\ & \text{and} \\ (a_1, \ldots, a_l,\ \omega + b_1\ , \ldots ,\ \omega + b_m\,) & \text{if } Q^{1/d} \text{ is present in } J. \end{cases}$$

Correspondingly define

$$J^* = \begin{cases} (x_1^{a_1}, \ldots, x_l^{a_l}) & \text{if } Q^{1/d} \text{ is not present in } J \\ & \text{and} \\ (x_1^{a_1}, \ldots, x_l^{a_l},\ y_1^{b_1},\ \ldots,\ y_m^{b_m}) & \text{if } Q^{1/d} \text{ is present in } J. \end{cases}$$

The quantity loginv^* is ordered lexicographically, with the terms $\omega + c_i/d$ declared infinitely larger than the rational numbers b_j; the notation ω is here to suggest the first infinite ordinal. Note that the rational numbers c_i/d might be smaller than any b_j.

Theorem 15.15.1 (Functorial logarithmic principalization of ideals)**.** *The formation of* loginv^* *and* J^* *is functorial for* smooth *base change on* Y*. The quantity* loginv^* *is upper semicontinuous and takes values in a well-ordered set. Its maximal locus is the support of* J^**. The weighted blowup* $Y' \to Y$ *of the reduced center associated to* J^* *is a smooth stack with transformed simple normal crossings divisor* D'*, and transformed ideal* $\mathcal{I}'$ *satisfying*

$$\operatorname{loginv}^*_{\mathcal{I}'}(p') < \operatorname{loginv}^*_{\mathcal{I}}(p).$$

After finitely many iterations, the pullback of the ideal $\mathcal{I}$ *is a locally principal monomial ideal on a smooth variety* $Y^{(n)}$ *carrying a normal crossings divisor* $D^{(n)}$*.*

There is no way I will get this far...

15.16. **Exercises from slides.** ***

Exercise 15.16.1. Describe the weighted blowup of $(x^{1/3}, y^{1/2}, z^{1/2})$ in affine space using the cox construction. Sdescribe what happens to the whitney umbrella $x^2 = y^2 z$. In characteristic zero show this is an improvement: the result has NC singularity with invariant $(2, 2)$. Complete the process to resolution and principalization.

Exercise 15.16.2. Repeat for $x^2 - y_1 y_2 y_3 = 0$.

Exercise 15.16.3. Compute $\mathcal{D}(\mathcal{I})$ and $\mathcal{D}^2(\mathcal{I})$ when $\mathcal{I} = (x^2 + y^2 z)$ and when $\mathcal{I} = (x^2 + y_1 y_2 y_3)$.

Exercise 15.16.4. Compute $\mathcal{D}(\mathcal{I})$ and $\mathcal{D}^2(\mathcal{I})$ ... when $\mathcal{I} = (x^5 + x^3 y^3 + y^8)$.

Exercise 15.16.5. What is the right notion in positive characteristics?

Exercise 15.16.6. If $Y_1 \to Y$ is smooth and $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, show that $\mathcal{D}(\mathcal{I}_1) = \mathcal{D}(\mathcal{I})\mathcal{O}_{Y_1}$.

Exercise 15.16.7. Convince yourself that ord is upper-semicontinuous.

Exercise 15.16.8. Convince yourself that the locus $\{p : \mathrm{maxord}(\mathcal{I}) = \mathrm{ord}_{\mathcal{I}}(p)\}$ is closed.

Exercise 15.16.9. Compute maxord($\mathcal{I}$) in the three examples.

Exercise 15.16.10. If $g : Y_1 \to Y$ is smooth and $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, show that $\mathrm{ord}_{\mathcal{I}_1} = \mathrm{ord}_{\mathcal{I}} \circ g$.

Exercise 15.16.11. Find *two* maximal contacts when $\mathcal{I} = (x^2 + y^2 z)$, when $\mathcal{I} = (x^2 + y_1 y_2 y_3)$, and when $\mathcal{I} = (x^5 + x^3 y^3 + y^8)$.

Exercise 15.16.12. If $g : Y_1 \to Y$ is smooth, $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, and x maximal contact, show that $g^* x$ is maximal contact.

Exercise 15.16.13. Compute $C(\mathcal{I}, 2)$ when $\mathcal{I} = (x^2 + y^2 z)$ and when $\mathcal{I} = (x^2 + y_1 y_2 y_3)$.

Exercise 15.16.14. Compute $C(\mathcal{I}, 5)$ when $\mathcal{I} = (x^5 + x^3y^3 + y^8)$.

Exercise 15.16.15. If $Y_1 \to Y$ is smooth and $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1}$, show that $C(\mathcal{I}_1, a) = C(\mathcal{I}, a)\mathcal{O}_{Y_1}$.

Exercise 15.16.16. Verify the invariants and centers in our examples: $x^2 - y^2z$, $x^2 + y_1y_2y_3$, $x^5 + x^3y^3 + y^8$, $x^5 + x^3y^3 + y^7$,

16. Punctured logarithmic maps

References: [ACGS24, Gro23]

16.1. **Degeneration setup.** Consider again a log smooth family $\mathcal{X} \to B$ of target log schemes, and assume the fiber X_0 is simple normal crossings. As we learned in previous lectures, one can decompose he logarithmic GW invariants of X_0 in combinatorial terms, summing up contributions associated to rigid tropical curves.

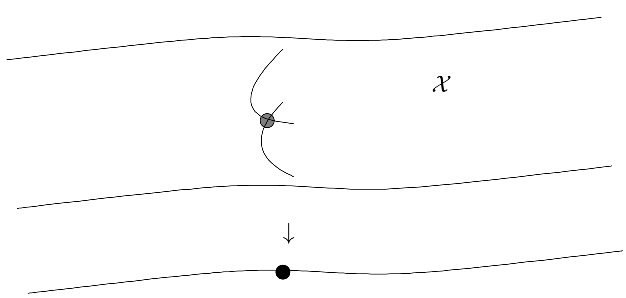
(Quote decomposition formula)

One wishes to write these in terms of invariants associated to individual strata. When following this through using logarithmic geometry, there are interesting phenomena one has to address.

16.2. **Punctured curves and idealized smoothness.** To identify these, it is useful to recall the first lesson learned:

The structure of curves should reflect the structure of targets, and vice versa.

As in the case of orbifold targets and orbifold curves, the key case is "universal":



Here $\mathcal{X} \to B$ is a log smooth family of curves over a smooth log curve B with special point $b = 0$.

We have $X_0 = Y_1 \cup^D Y_2$, where D is the node, which for simlicity has coordinates $xy = t$. The logarithmic structure $M_B$on the base is generated by $\log t$ and the logarithmic structure M_x on $\mathcal{X}$ at the node by $\log x, \log y$ with $\log x + \log y = \log t$.

Remark 16.2.1. If we chose a logarithmic section at the node, this amounts to choosing a projection of M_x to M_b. This becomes possible after base change $t = s_1 s_2$, by sending $x \mapsto s_1$ and $y \mapsto s_2$. (For instance one can take $s_1 = s^i, s_2 = s^j$.) Chosing a section is often a valuable step, and sometimes critical. See also the work of Holmes and Spelier [HS23].

What is the structure of the component $Y_1 = \{y = 0\}$ of X_0? it is a curve with a logarithmic structure, but *it is not a log curve* since we insisted that log curves are log smooth. We call these *punctured curves.*

Importantly, the element $\log y$ has the property $\alpha(\log y) = 0 \in \mathcal{O}_{Y_1}$.

Is there a redeeming property for this curve?

Definition 16.2.2 (Ogus)**.** An *idealized log structure* is a log structure $\alpha : M \to \mathcal{O}_X$ along with a monoid ideal $K \subset M$, such that $\alpha(K) = 0 \in \mathcal{O}_X$.

Locally one obtains a morphism $X \to \operatorname{Spec} \mathbb{C}[\bar{M}]/(\bar{K})$. An idealized log scheme is *idealized log smooth* if this morphism is smooth.

In the example, the monoid ideal is generated by $\log y$.

Exercise 16.2.3. Show that the log point $\{0\}$ with its induced structure from S is idealized log smooth. Show that Y_1 is idealized log smooth. For that matter, any closed toric stratum in a toric variety is idealized log smooth.

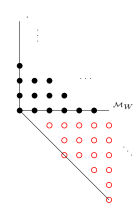
Recall that a *log curve* is a reduced 1-dimensional fiber of a flat *log smooth morphism.* F. Kato showed that these are the same as nodal marked curves, with "the natural" log structure. A *punctured curve* is the idealized version of the above.

Definition 16.2.4. A *puncturing* of a marked curve is a log structure M at a marked point with

$$M_S + \mathbb{N} \log x \quad \subseteq \quad M \quad \subsetneq \quad M_S + \mathbb{Z} \log x.$$

It is an instance of an *idealized log smooth* scheme. In particular the *splitting of a node is a punctrured curve.*

The picture describes the monoid (unfortunately $W = S$).



The embedding of M_S comes from pullback along the projection. The vertical projection is *generization* to the generic point of Y_1. The horizontal map to $\mathbb{Z}$ is the *contact order*, which here can be negative!

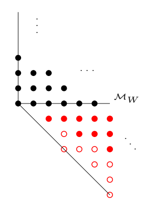
Note that any element $m \in M$ *of negative contact order* must satisfy $\alpha(m) = 0$. In paerticular any element in the ideal generated satisfies $\alpha(m) = 0$. This includes some elements of M_S — this is the so called *puncturing log ideal.* This puts a big restriction on what S can be! in particular S is also naturally an idealized log scheme!

A section would result in another projection, which sends M_S to itself and the vertical generator $\log x$ to some nonzero point of M_S. Do not confuse it with the generization map!

Families of punctured curves $C \to T$ are built in to the definition, and there is a natural notion of pullback. A *punctured log map* is thus a morphism $C \to X$ of a punctured curve to a log scheme X.

Just like with log maps, we would like to somehow limit the choices of puncturing by the given data. A punctured log map is said to be *prestable* M as above is generated by the marked structure $M_S + \mathbb{N} \log x$ and $f^\flat M_X$.

The next example shows a punctured curve where $f^\flat M_X$ is generated by $(2, -1)$. The map is not prestable, but had we taken just the solid circles we get a prestable map



A punctured log map map is *stable* if it is prestable (a log condition) and the underlying map is stable (a schematic condition).

As with log maps, we distinguish *minimal* punctured maps — those where the logarithmic structure on the base is universal. The magical theorem of Gross and Siebert — [AC14, Remark 1.2.1] — holds as stated, since neither markings nor punctures intervene. [32] ←32

16.3. **The tropical picture.** A punctured map has its tropicalization just as a log map does. However, the legs of a punctured map look a bit differently: while the leg of a log smooth curve extends indefinitely, the leg associated to a puncture, thought of as a element of the dual cone $\mathrm{Hom}(M, \mathbb{R})$ of the monoid of the puncture, must compose to a positive element on $f^\flat(M_X)$. In other words, such an elemet maps to the corresponding cone $\sigma \in \Sigma(X)$. We have come to express these as arrows within σ. It is not hard to see that, if the map is pre-stable, the arrow extends exactly as far as σ allows. [33] ←33

The *type* τ of a punctured map is define in exact analogy to the type of a log map. It records the graph marked by genus, the strata σ of each vertex, edge or leg, and the contact order at each edge or leg. It can be *decorated* by the curve class associated to each vertex.

There is also a balancing condition for the tropicalization of punctured maps, which turns out to be identical to the balancing condition of log maps. [34] ←34

16.4. **The space of punctured maps.** In analogy with the case of log stable maps we have:

Theorem 16.4.1 ([ACGS]). *$\mathcal{M}(X, \tau)$, the stack of minimal stable punctured log maps of* decorated type *τ, is a Deligne–Mumford stack which is finite and representable over $\mathcal{M}(\underline{X}, \underline{\tau})$.*

[32](Dan) Write explicitly again

[33](Dan) include picture

[34](Dan) Give exercise comparing a line in $\mathbb{P}^2$ relative $x = 0$ with marking of contact order 1, with a -1 curve with a puncture of contact order -1

There is also a space of prestable punctured maps $\mathfrak{M}(\mathcal{X}, \tau)$ in the Artin fan $\mathcal{X}$ of X of *undecorated* type τ. These serve as a tropical version of $\mathcal{M}(X, \tau)$. In the case of log stable maps they are log smooth. Here they are only *idealized log smooth*, which we will see is a source of pain.

We denote by $\widetilde{\mathcal{M}}(X, \tau)$ the space of punctured maps with a logarithmic section at each marking and labelled node. There are natural evaluation maps $\widetilde{\mathcal{M}}(X, \tau) \to X^n$ This map is not smooth, nor is it virtually log smooth, but it is *ideally virtually log smooth*: it has a perfect logarithmic obstruction theory relative to an appropriate Artin fan version $\widetilde{\mathfrak{M}}(\mathcal{X}, \tau)$.

In analogy with the orbifold picture, splitting a nodal curve at a node with a section gives
35→ in essense a formula[35]

$$Hom(C, X) = Hom(C_1^\circ, X) \times_{Hom(W,X)} Hom(C_2^\circ, X).$$

This gives:

Theorem 16.4.2 (ACGS 2020)**.** *Suppose the splitting of type τ along a set of n edges results in type τ'. The following is cartesian:*

$$\begin{array}{ccc}
\widetilde{\mathcal{M}}(X/B, \tau) & \longrightarrow & \widetilde{\mathcal{M}}(X, \tau') \\
\downarrow & & \downarrow \\
X^n & \longrightarrow & X^n \times X^n
\end{array}$$

16.5. **What this gives.** Idealized log smooth schemes do not in general possess a natural virtual fundamental class - they are not even pure dimensional. As a consequence, spaces of punctured stable maps may fail to possess a natural virtual fundamental class. Even though Manolache's theorem applies, we are not always handed a class to virtually pull back. Examples are given in [ACGS24].

The paper [Gro23] delineates a number of situations where these issues do not arise. There are several conditions imposed. First, one only considers *tropically realizable types.* Second, one requires the gluing situation to be *tropically transverse*. Third, one requires that the evaluation maps are *tropically flat*.

There is further current work addressing this issue. See in paeticular the classes defined
36→ in [BNR24, Joh24] [36]

[35](Dan) I am taking a shortcut here

[36](Dan) There is a ton of stuff uncited

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