Logarithmic geometry and stacks in resolution of singularities and moduli: Kontsevich's formula

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LMS lecture series

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Rational plane curves

Definition

$$N_d = \#\left\{ \begin{array}{l} C \subset \mathbb{P}^2 \text{ a rational curve,} \\ \deg C = d, \text{ and} \\ p_1, \ldots p_{3d-1} \in C \end{array} \right\}.$$

Theorem (Kontsevich)

For $d > 1$ we have

$$N_d = \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 > 0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

The first few numbers are

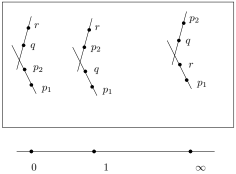
$$N_1 = 1, \; N_2 = 1, \; N_3 = 12, \; N_4 = 620, \; N_5 = 87304.$$

$\overline{\mathcal{M}}_{0,4}$

$$\overline{\mathcal{M}}_{0,4} = \overline{\left\{ p_1, p_2, q, r \in L \;\middle|\; \begin{array}{l} L \simeq \mathbb{P}^1 \\ p_1, p_2, q, r \text{ distinct} \end{array} \right\}}$$

- $\overline{\mathcal{M}}_{0,4} \simeq \mathbb{P}^1$ via cross ratio. $\quad \overline{\mathcal{M}}_{0,4} \smallsetminus \mathcal{M}_{0,4} = \{0, 1, \infty\}$

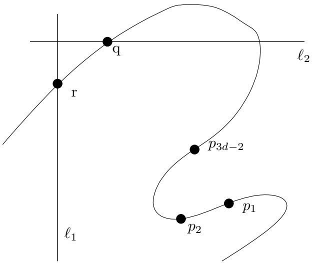
$$\begin{aligned} 0 &= (p_1, p_2 \mid q, r), \\ 1 &= (p_1, q \mid p_2, r), \text{ and} \\ \infty &= (p_1, r \mid p_2, q), \end{aligned}$$



A one-parameter family

$C \to B$ parametrized by a curve B:

- Each curve C_b contains $p_1, \ldots p_{3d-2}$ (but not necessarily $p_{3d-1} = \ell_1 \cap \ell_2$).
- One point $q \in C_b \cap \ell_2$ is marked.
- One point $r \in C_b \cap \ell_1$ is marked.



Geometry of the equation

Forgetting $p_3, \ldots, p_{3d-2}$ and stabilizing we get

$$B \xrightarrow{\lambda} \overline{\mathcal{M}}_{0,4}$$
$$C \mapsto CR(p_1, p_2, q, r)$$

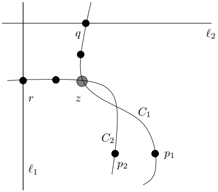
Since points on $\mathbb{P}^1$ are homologically equivalent we get

$$\deg_B \lambda^{-1}(p_1, p_2 | q, r) = \deg_B \lambda^{-1}(p_1, q | p_2, r).$$

The right hand side

$\deg_B \lambda^{-1}(p_1, q|p_2, r)$:

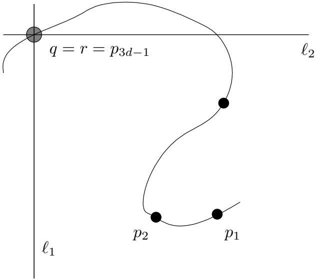
- components C_1, C_2 of degrees d_1, d_2 satisfying $d_1 + d_2 = d$.
- $p_1 \in C_1$ as well as $3d_1 - 2$ points among the $3d - 4$ points $p_3, \dots p_{3d-2}$.
- We have $p_2 \in C_2$ as well as the remaining $3d_2 - 2$ of the points.
- We select one point $z \in C_1 \cap C_2$ where the two curves attach.
- We mark one point $q \in C_1 \cap \ell_2$ and $r \in C_2 \cap \ell_1$.



$$= \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 > 0}} \binom{3d-4}{3d_1-2} \cdot N_{d_1} N_{d_2} \cdot d_1 d_2 \cdot d_1 \cdot d_2.$$

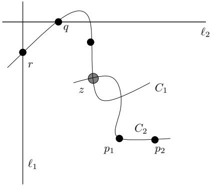
The left hand side: N_d part

Curves counted in $\deg_B \lambda^{-1}(p_1, p_2 | q, r)$ come in two flavors:
there are *irreducible* curves passing through $q = r = \ell_1 \cap \ell_2$ This is precisely N_d.



The left hand side: reducible part

- C_1, C_2 of respective degrees d_1, d_2 satisfying $d_1 + d_2 = d$.
- $3d_1 - 1$ among $p_3, \dots p_{3d-2}$ are on C_1.
- $p_1, p_2 \in C_2$ as well as the remaining $3d_2 - 2$ points.
- We select one point $z \in C_1 \cap C_2$. to attached.
- We mark one point $q \in C_1 \cap \ell_1$ and $r \in C_1 \cap \ell_2$.



$$\deg_B \lambda^{-1}(p_1, p_2 | q, r)$$

$$= \quad N_d \quad + \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 > 0}} \binom{3d-4}{3d_1-1} \cdot N_{d_1} N_{d_2} \cdot d_1 d_2 \cdot d_1^2$$

as needed

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GW theory

Gromov–Witten theory gives us a nice, organized, rigorous way to make these calculations.

End of segment

Next: GW 1