Proceedings of the International Congress of Mathematicians
Hyderabad, India, 2010

An exercise in mirror symmetry

Richard P. Thomas

Abstract. This expository article is an attempt to illustrate the power of Kontsevich's homological mirror symmetry conjecture through one example, the heuristics of which lead to an algebro-geometric construction of knot invariants.

Mathematics Subject Classification (2010). Primary 14J33; Secondary 53D37, 57M27, 53C26.

Keywords. Mirror symmetry, Khovanov cohomology.

1. Introduction

This paper can be thought of as a companion to the paper [32], giving the background, mirror symmetric motivation, and helpful pictures that are missing there. Along the way we give a geometric description of Manolescu's isomorphism [18] between an open subset of a Hilbert scheme of points on an ALE space and the Slodowy slice to a nilpotent matrix with two equal Jordan blocks considered by Seidel and Smith, along the lines of the construction in [11]. We use a description of these ALE spaces as blow ups which is probably well known to experts but was new to me, giving maps between them that are crucial to our construction.

Heuristics. We treat mirror symmetry as a heuristic device to motivate constructions on one side of the mirror that reflect better known constructions on the other. We make no rigorous claims for our putative mirrors; for instance we are not claiming that a hyperkähler resolution of a singularity is mirror to a hyperkähler smoothing. Though we will use examples where this ansatz works well, in the key example it *fails* (see Section 5.1) and has to be augmented with a deformation.

Acknowledgements. I would like to thank Gordon Brown, Chris Murphy, Wilson Sutherland, Simon Donaldson, Paul Seidel, Mikhail Khovanov, Shing-Tung Yau, Daniel Huybrechts, Tom Bridgeland and Rahul Pandharipande for educating me, and my collaborator Ivan Smith with whom much of this work was done. Thanks also to Arthur Greenspoon and Ivan Smith for carefully reading the manuscript.

2. Symplectic geometry

We begin by surveying some standard constructions in symplectic geometry. We skate over many technical issues, in particular Floer cohomology, gradings, the construction of the Fukaya category, and the difficulties in doing symplectic parallel transport in noncompact spaces. Most of these are dealt with manfully in the wonderful papers of Paul Seidel [23, 24, 25].

2.1. Parallel transport. A family of projective manifolds

$$p\colon \mathcal{X} \to B$$

will not in general be locally trivial over its smooth locus $B^* \subset B$; the complex structure will vary. As Paul Seidel once taught me, symplectic geometry is what is left when you look for what *is* locally constant. (I liked this because it sounded like it might subordinate symplectic geometry to algebraic geometry.) Here the symplectic form ω is given by pulling back the Fubini-Study form via a projective embedding. Over B^* there is a connection on the family: take the annihilator of the fibrewise tangent bundle $T_{\mathcal{X}/B}$ under ω to define the horizontal subbundle of $T\mathcal{X}$. Parallel transport along this connection preserves the symplectic form, and so identifies smooth fibres $\mathcal{X}_{b_0}$, $\mathcal{X}_{b_1}$ by symplectomorphisms, once we pick a path between their images b_0, b_1 in the base B^*. In particular the monodromy around a loop in B^* can be taken to be a *symplectomorphism* of any such fibre $(X,\omega) \cong (\mathcal{X}_b, \omega|_{\mathcal{X}_b})$.

This connection is not flat. Any two tangent vectors $v_1, v_2 \in T_bB^*$ have unique horizontal lifts $\tilde{v}_i \in \Gamma(T_{\mathcal{X}}|_{\mathcal{X}_b})$. Thinking of

$$h := \omega(\tilde{v}_1, \tilde{v}_2)$$

as a Hamiltonian function on $\mathcal{X}_b$ it defines an infinitesimal symplectomorphism of $\mathcal{X}_b$ by the Hamiltonian vector field X_h whose contraction with $\omega|_{\mathcal{X}_b}$ is dh. This $\mathrm{Ham}(\mathcal{X}_b, \omega|_{\mathcal{X}_b})$-valued 2-form on B^* is the curvature of the connection. Therefore isotopic loops in B^* give rise to different but Hamiltonian isotopic monodromies. We get a homomorphism

$$\pi_1(B^*) \to \mathrm{Aut}(X, \omega)$$

to the group of symplectomorphisms modulo Hamiltonian isotopies.

Pick a singular point x_0 lying above a point $b_0 \in B$ in the discriminant locus, and a path in B^* to $b \in B^*$. The locus L of points of $\mathcal{X}_b$ that flow to x_0 by parallel transport along the path is called the *vanishing cycle* of the singularity x_0. Because the flow preserves the symplectic structure, L is isotropic (where it is smooth): $\omega|_{TL} \equiv 0$. If x_0 is an isolated critical point then L is in fact Lagrangian.

The curvature of the symplectic connection blows up as we approach such singular points. Taking smaller and smaller loops in B^* around b_0 the monodromy symplectomorphism approaches the identity away from the vanishing cycle.

2.2. The ordinary double point. We start with a basic affine local model. Consider the family

$$f\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}, \qquad f(\mathbf{x}) = \sum_{i=1}^{n+1} x_i^2. \tag{2.1}$$

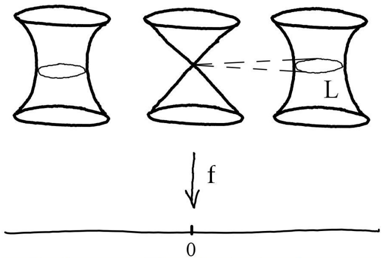

Figure 1. Vanishing cycle L of the family (2.1).

Over 0 we get the n-dimensional ordinary double point $\sum x_i^2 = 0$, while over $\epsilon \neq 0$ we find its smoothing $X_\epsilon = \{\sum x_i^2 = \epsilon\}$. We use the symplectic structure inherited from the standard Kähler form on $\mathbb{C}^{n+1}$.

Using the $O(n+1)$ symmetry it is easy to see that the vanishing cycle L over $\epsilon \neq 0$ along the straight line path to $0 \in \mathbb{C}$ is the real slice

$$x_i \in \sqrt{\epsilon}.\mathbb{R} \subset \mathbb{C}$$

of X. Scaling coordinates by $\epsilon^{-1/2}$ this is just the sphere

$$L = S^n = \left\{\sum x_i^2 = 1\right\} \subset \mathbb{R}^{n+1}.$$

In fact take $\epsilon \in (0, \infty)$, without loss of generality, and take real and imaginary parts: $x_j = a_j + ib_j$. Consider $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$ as lying in $\mathbb{R}^{n+1}$ and $(\mathbb{R}^{n+1})^*$ respectively, and give $\mathbb{C}^{n+1} = \mathbb{R}^{n+1} \oplus (\mathbb{R}^{n+1})^* = T^*\mathbb{R}^{n+1}$ its canonical symplectic structure. Then the equation $f = \epsilon$ becomes

$$\sum a_i^2 - b_i^2 = \epsilon, \quad \sum a_i b_i = 0.$$

In particular $|\mathbf{a}|^2 = \epsilon + |\mathbf{b}|^2 > 0$ so we may divide $\mathbf{a}$ and multiply $\mathbf{b}$ by $|\mathbf{a}|$ to give a symplectomorphism of $f^{-1}(\epsilon)$ to

$$T^*S^n = \{(\mathbf{a}, \mathbf{b}) \in T^*\mathbb{R}^{n+1} \colon |\mathbf{a}| = 1,\ \mathbf{b}(\mathbf{a}) = 0\}.$$

The monodromy on going once anticlockwise around $\epsilon = 0$ is Seidel's generalised Dehn twist T_L [22] about L (first suggested by Arnol'd). This is (Hamiltonian isotopic to) the time π flow by the Hamiltonian $\phi(|\mathbf{b}|)$, where ϕ is a smooth monotonic function with $\phi(x) = x$ for small $x \geq 0$ and $\phi \equiv$const for large x. This flow is discontinuous across the vanishing cycle $\mathbf{b} = 0$, but after time π comes back to the antipodal map there and so becomes continuous again. (Alternatively use the standard metric to identify T^*S^n with TS^n. The latter has a canonical vector field which at a point $v \in T_pS^n$ is the horizontal lift $\tilde{v}$ of v to $T_{(p,v)}(TS^n)$. Flowing down $\tilde{v}/|\tilde{v}|$ is again discontinuous, cutting T^*S^n along its zero section then regluing after time π. Then use a bump function to glue this symplectomorphism to the identity away from the zero section.)

When $n = 1$ this reduces to the classical Dehn twist along an embedded S^1 in a Riemann surface: cut along S^1, rotate everything to one side of it through 2π, then reglue. Figure 2 shows its action on one of the cotangent fibres $\mathbb{R} \subset T^*S^1$. More generally given any middle dimensional cycle, the action of the Dehn

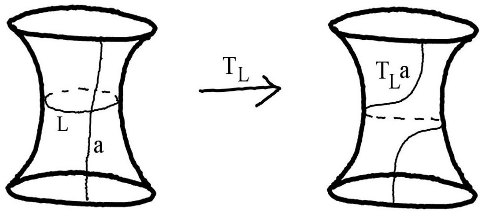

Figure 2. Action of the Dehn twist on a cotangent fibre a of T^*S^n.

twist, i.e. the monodromy around $\epsilon = 0$, can be described similarly: for every transverse intersection point with the vanishing cycle L, the cycle picks up a copy of L (connect summed to it at the intersection point). In particular in any projective family acquiring an ordinary double point we have the above local model near the vanishing cycle L (by Weinstein's theorem) and the action on middle degree homology H_n is given by the Picard-Lefschetz reflection

$$a \mapsto (a.[L])[L] + a. \tag{2.2}$$

One can keep more of the symplectic information by instead using the Fukaya A_∞-category [8, 25]. This has as objects certain Lagrangian submanifolds (with some extra decorations) and morphisms the Floer cochain complex $CF^*(L_1, L_2)$ whose generators are intersection points of generic Hamiltonian perturbations of L_1, L_2 with differential given by counting holomorphic discs running between the intersection points with boundary in the L_i. (The result is independent of the choices of (almost) complex structure and Hamiltonian isotopy up to quasi-isomorphism.) The tautological evaluation map in this Fukaya category

$$CF^*(L, L') \otimes L \to L' \tag{2.3}$$

has a cone in the derived category $\mathcal{F}(X_\epsilon, \omega)$ of twisted complexes in the Fukaya category. Under certain conditions on the Maslov degree of the intersection points, this cone is equivalent to the (graded) Lagrangian connect sum of L' and L at its intersection points [7, 22, 23, 33]. The induced action of the Dehn twist on the derived Fukaya category indeed takes L' to the above cone [23], clearly categorifying the Picard-Lefschetz reflection (2.2) to which it reduces at the level of cohomology. Another way of saying this is that there is an exact triangle

$$HF^*(L, L') \otimes L \to L' \to T_L(L') \tag{2.4}$$

in $\mathcal{F}(X_\epsilon, \omega)$.

2.3. Families of quadrics. Another way of seeing the smoothing of the ordinary double point – i.e. a smooth fibre of (2.1) – is by fibring *it* over $\mathbb{C}$ using the last coordinate $x_{n+1} = t$:

$$\left\{\sum_{i=1}^{n} x_i^2 = \epsilon - t^2\right\} \subset \mathbb{C}^n_{x_i} \times \mathbb{C}_t \longrightarrow \mathbb{C}_t. \tag{2.5}$$

This expresses the n-dimensional affine quadric as a family of $(n-1)$-dimensional affine quadrics – the fibres $\sum_{i=1}^n x_i^2 = \text{const}$ where t is fixed. Each contains a canonical Lagrangian S^{n-1} real slice, except the two singular fibres where $\epsilon - t^2$ vanishes and the vanishing cycle collapses to a point. Picking a path between $t = \pm\epsilon^{1/2}$, the S^{n-1}-bundle over it (collapsing at the endpoints) gives a Lagrangian S^n as in Figure 3. This is the vanishing cycle of the degeneration of the total

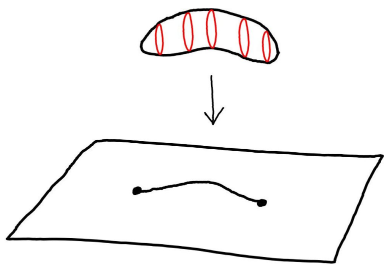

Figure 3. Lagrangian S^n fibred by S^{n-1}s over a matching path between the critical points of the fibration (2.5).

space given by tending $\epsilon \to 0$ (so that the path and the vanishing cycle both collapse). Monodromy around this simply rotates the path anticlockwise through 180^o, exchanging the endpoints and giving another way to view the Dehn twist.

This picture generalises by considering a degree k polynomial p on the right hand side of (2.5):

$$X = X_{\underline{\lambda}} := \left\{\sum_{i=1}^{n} x_i^2 = p(t)\right\} \subset \mathbb{C}^n_{x_i} \times \mathbb{C}_t. \tag{2.6}$$

We fix p monic, with set $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ of *distinct, unordered* roots with centre of mass $0 \in \mathbb{C}$. Then $X_{\underline{\lambda}}$ is smooth (but acquires ordinary double points when p has double roots). By the same reasoning, paths between zeros λ_i of p give $O(n)$-invariant Lagrangian spheres in $X_{\underline{\lambda}}$. Such a sphere is the vanishing cycle of the degeneration given by bringing the two roots of p at its endpoints together along the path to produce an ordinary double point. We will be particularly interested in the $n = 2$ case of this construction, in which case the fibres are the *type-A_{k-1} ALE surfaces $S_{\underline{\lambda}}$*.

We get a smooth family of $X_{\underline{\lambda}}$s over C_k^0, the configuration space of k distinct unordered points $\underline{\lambda}$ in the plane $\mathbb{C}$ with centre of mass the origin. Now $\pi_1(C_k^0) =$

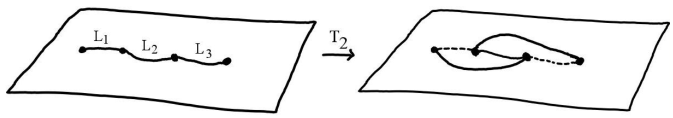

Figure 4. Action of the Dehn twist T_i on the A_{k-1}-chain.

B_k, the braid group on k strands: a loop in C_k can be considered as a motion, as time runs from 0 to 1, of the k points through $\mathbb{C}$ (never touching, and starting and ending at the same set of points, possibly permuted); plotting the graph of this motion in $\mathbb{C} \times [0,1]$ gives a braid. So the monodromy is a representation

$$B_k \to \mathrm{Aut}(X, \omega),$$

which is *faithful* [15]. Take as basepoint of C_k^0 a configuration of k points along the real line $\mathbb{R} \subset \mathbb{C}$, with the obvious A_{k-1}-chain of paths given by the intervals between them. Then the braid given by rotating the ith and $(i+1)$st points about each other in $\mathbb{C}$ while fixing the others gives the generator T_i of B_k. The corresponding automorphism $T_i \in \mathrm{Aut}(X, \omega)$ is the monodromy about the ordinary double point that $X_{\underline{\lambda}}$ acquires when the two points are brought together along the interval between them. Thus it is the Dehn twist in the Lagrangian sphere L_i fibring over that interval. It takes our A_{k-1}-chain of Lagrangian spheres to a different A_{k-1}-chain, as shown in Figure 4. The T_i satisfy the braid relations

$$\begin{array}{rcll} T_iT_jT_i & \cong & T_jT_iT_j, & |i-j| = 1, \\ T_iT_j & \cong & T_jT_i, & |i-j| > 1, \end{array} \tag{2.7}$$

in $\mathrm{Aut}(X, \omega)$ and so also in $\mathrm{Aut}(\mathcal{F}(X, \omega))$. (To be more careful one has to show that the T_i can be lifted to act on the decorations in the derived Fukaya category, in particular the grading.)

2.4. Spaces of matrices.

The family (2.6) is a baby version of another natural family over C_k^0; the space M_k^0 of complex $k \times k$ trace-free matrices with distinct eigenvalues. This has a natural Kähler, and so symplectic, form ω inherited from $\mathbb{C}^{k^2}$. Consider the map

$$M_k^0 \to C_k^0 \tag{2.8}$$

taking a matrix to its set of eigenvalues $\underline{\lambda} \in C_k^0$. It has smooth fibre $M_{\underline{\lambda}}$, the $\mathrm{ad}_{SL(k)}$-orbit of similar matrices with the same eigenvalues $\underline{\lambda}$. We get the monodromy representation

$$B_k \to \mathrm{Aut}(M_{\underline{\lambda}}, \omega). \tag{2.9}$$

In fact the family (2.1) for $n = 2$ is the above family (2.8) when $k = 2$, and (2.6) is also a Slodowy slice (at a nilpotent matrix with Jordan blocks of size $(1, k-1)$) of the fibration (2.8). The monodromies can also be described as coisotropic family Dehn twists modelled on relative versions of the 2-dimensional Dehn twist of Section 2.2 with $n = 2$; see [16, Section 3.4].

A different slice of the family (2.8) when $k = 2m$ is considered by Seidel and Smith [27]. Let SS_{2m} denote the space of trace-free matrices A with distinct eigenvalues and the following block form

$$A := \begin{pmatrix} A_1 & I_2 & 0 & 0 & 0 \\ A_2 & 0 & I_2 & 0 & 0 \\ \dots & & & \dots & \\ A_{m-1} & 0 & 0 & 0 & I_2 \\ A_m & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.10}$$

where A_i is any 2×2 matrix, A_1 is trace-free, and I_2 is the 2×2 identity matrix. Again the eigenvalue map makes this a smooth symplectic bundle

$$SS_{2m} \to C^0_{2m}, \tag{2.11}$$

with monodromy representation $B_{2m} \to \mathrm{Aut}(SS_{\underline{\lambda}}, \omega)$ on a fibre $SS_{\underline{\lambda}}$.

2.5. The Manolescu isomorphism. Manolescu [18] found another beautiful relationship between the Seidel-Smith family and the basic family (2.6) over C^0_{2m}. Namely, he showed that SS_{2m} can be identified with an explicit open subset of the relative Hilbert scheme of m points on the smooth fibres of the family of ALE surfaces given by (2.6) with $n = 2$ and $\deg p = 2m$.

Manolescu described his isomorphism by ingenious algebraic manipulation, but it is possible to describe it geometrically as follows. We fix m and work on one fibre $SS_{\underline{\lambda}}$, *fixing* the degree $2m$ monic polynomial $p_{\underline{\lambda}}(x)$ with roots $\underline{\lambda}$ that is the characteristic polynomial of matrices in $SS_{\underline{\lambda}}$.

Since the A_i commute with the other 2×2 blocks in A (2.10), we can evaluate the determinant of $xI_{2m} - A$ blockwise to give the 2×2 matrix polynomial

$$A(x) := I_2 x^m - A_1 x^{m-1} - A_2 x^{m-2} - \dots - A_m \tag{2.12}$$

with determinant $\det(A(x)) = p_{\underline{\lambda}}(x)$.

In fact it is convenient to work with the matrices

$$B(x) := A(x)J = Jx^m - (A_1 J)x^{m-1} - (A_2 J)x^{m-2} - \dots - (A_m J), \tag{2.13}$$

where multiplication by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J^2 = -1,$$

is invertible, preserves determinants, and takes trace-free matrices to symmetric matrices. Therefore writing the polynomial-valued 2×2 matrices $B(x)$ in the form

$$B(x) = \begin{pmatrix} V(x) & U(x) \\ W(x) & X(x) \end{pmatrix}, \tag{2.14}$$

we have that U and $-W$ are monic of degree m, U and W have equal coefficients of x^{m-1} (the $\mathrm{tr}\, A_1 = 0$ condition), and V, X have degree $m - 1$ and satisfy

$$\det(B(x)) = V(x)X(x) - U(x)W(x) = p_{\underline{\lambda}}(x). \tag{2.15}$$

Matrices $B(x)$ (2.14) satisfying these conditions are entirely equivalent to matrices $A \in SS_{\underline{\lambda}}$ (2.10).

Considering $B(x)$ to be an endomorphism of the trivial rank 2 bundle over $\mathbb{C}_x$, we study it via its spectral curve. Plotting the two eigenvalues $y_1(x), y_2(x)$ of $B(x)$ gives a curve

$$C_B := \{(x,y)\colon \det(yI_2 - B(x)) = 0\} \subset \mathbb{C}_x \times \mathbb{C}_y$$

double covering $\mathbb{C}_x$. Expanding out gives the equation of $C_B \subset \mathbb{C}_x \times \mathbb{C}_y$ as

$$y^2 - \operatorname{tr}(B(x))y + p_{\underline{\lambda}}(x) = 0. \tag{2.16}$$

Over this curve is the natural line subbundle $\mathrm{Eig} \to C_B$ of the trivial rank two bundle given by the corresponding eigenspace of $B(x)$. At $(x,y) \in C_B$, $yI_2 - B(x)$ has rank ≤ 1 and top row $(y - V(x) \quad -U(x))$, so an obvious element of the kernel is its perpendicular

$$\begin{pmatrix} U(x) \\ y - V(x) \end{pmatrix}.$$

This defines a generator of the eigenspace except when it vanishes, i.e. except at the points $(\alpha_i, V(\alpha_i))$, where α_i are the m roots of $U(x)$. (And from (2.14) or (2.16) one sees that indeed $y = V(x)$ is on one branch of C_B at the roots of $U(x)$; the other branch being $y = X(x)$.)

So we have exhibited a section of Eig vanishing on the length-m divisor $D = \{(\alpha_i, V(\alpha_i))\}$, or, more precisely,

$$D = \{U(x) = 0 = y - V(x)\} \in \operatorname{Hilb}^m C_B. \tag{2.17}$$

In particular, at smooth points of C_B, we find that

$$\mathrm{Eig} \cong \mathcal{O}_{C_B}(D). \tag{2.18}$$

Write the equation (2.16) of the curve $C_B \subset \mathbb{C}_x \times \mathbb{C}_y$ as

$$y(\operatorname{tr}(B(x) - y) = p_{\underline{\lambda}}(x).$$

Plotting the graph of the *other* eigenvalue

$$Y = \operatorname{tr}(B(x)) - y$$

of $B(x)$ embeds C_B in

$$S_{\underline{\lambda}} := \{yY = p_{\underline{\lambda}}(x)\} \subseteq \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_Y. \tag{2.19}$$

This is the *affine blow up* of $\mathbb{C}^2_{xy}$ in the points $(\lambda_i, 0)$ defined by $y = 0 = p_{\underline{\lambda}}(x)$, and is isomorphic to the ALE surface (2.6) (with $n = 2$ and $k = 2m$). (The usual blow up is given by the same equation in $\mathbb{C}^2 \times \mathbb{P}^1_Y$ but we are removing the locus $Y = \infty$ – the proper transform of $y = 0$ – to get $S_{\underline{\lambda}}$. Since by (2.15) the curve

$C_B \subset \mathbb{C}^2$ never hits $y = 0$ except at the roots of $p_{\underline{\lambda}}(x)$ it more naturally lies in the blow up (2.19) of $\mathbb{C}^2$ than in $\mathbb{C}^2$ itself. This will help us to invert the construction below. What is going on here is that a point $(x, y) \in C_B$ determines the other eigenvalue $Y = p_{\underline{\lambda}}(x)/y$ by (2.15) except when $y = 0$. At such points, i.e. when x is one of the roots λ_i, the fact that $y = 0$ tells us nothing as we already knew that C_B goes through $(\lambda_i, 0)$ by (2.15). To invert the construction we will need to know the gradient of C_B at this point instead, and this determines the other eigenvalue. The blow up (2.19) achieves this.)

Manolescu's map then maps A (2.10) (or equivalently $B(x)$ (2.13)) to the image of the divisor D (2.17) under the inclusion

$$\operatorname{Hilb}^m C_B \subset \operatorname{Hilb}^m S_{\underline{\lambda}}.$$

By its definition (2.17) we see that D projects to the length m subscheme $\{U(x) = 0\} \in \operatorname{Hilb}^m \mathbb{C}_x$ under the obvious projection $S_{\underline{\lambda}} \to \mathbb{C}_x$. In other words no part of D is tangent to the fibres of this projection and the restriction of the projection to D is an isomorphism. This proves one half of the following.

Theorem 2.20. [18, Prop 2.7] *The above construction gives an isomorphism between the space $SS_{\underline{\lambda}}$ and the open subset of $\operatorname{Hilb}^m S_{\underline{\lambda}}$ consisting of subschemes whose projection to $\mathbb{C}_x$ also have length m.*

The proof of the converse is now easy. Fix $D \in \operatorname{Hilb}^m S_{\underline{\lambda}}$ whose projection to $\mathbb{C}_x$ has length m. This defines a unique degree m monic polynomial $U(x)$ with those roots. The function $y|_D$ defines a function on the projection of D in $\mathbb{C}_x$, and there is a unique degree $m-1$ polynomial $V(x)$ on $\mathbb{C}_x$ whose restriction takes the same values. Similarly $Y|_D$ defines $X(x)$. Finally a degree m polynomial $W(x)$, with leading two coefficients -1 and the x^{m-1} coefficient of $U(x)$ respectively, is uniquely determined by comparing coefficients in the equation (2.15), using the fact that the coefficient of x^{2m-1} in $p_{\underline{\lambda}}$ is $\sum \lambda_i = \operatorname{tr} A = 0$. This determines $B(x)$ (2.14), as required.

More geometrically, we are saying that D determines the curve C_B through it, and (at least at smooth points of C_B) the eigensheaf $\operatorname{Eig} = \mathcal{O}_{C_B}(D)$ (2.18). Pushing this down gives the trivial rank two bundle, on $\mathbb{C}_x$, while the scalar endomorphism y descends to an endomorphism $B(x)$ of this trivial rank two bundle. This is the classical spectral curve construction for Higgs bundles [10]. I only recently discovered that the link to Hilbert schemes was discovered 15 years ago by Hurtubise [11].

2.6. Digression – fixed point locus.

In [28] Seidel and Smith also consider the involution on $SS_{\underline{\lambda}}$ given by replacing each A_i by its transpose. The fixed point locus consists of those matrices $A(x)$ (2.12) which are symmetric; after multiplying by J we get those matrices $B(x)$ (2.13) which are trace-free.

This fixes the eigenvalues of $B(x)$ (since its determinant is also fixed (2.15)) and so the (smooth) spectral curve,

$$C_B := \{y^2 = p_{\underline{\lambda}}\}. \tag{2.21}$$

Restricted to this locus, the above gives a geometric description of the algebraic construction in [28] (a precursor [29] of Manolescu's construction). The result is an embedding of the fixed point locus of $SS_{\underline{\lambda}}$ in

$$\mathrm{Sym}^m C_B.$$

The image is the complement of the "hyperelliptic locus" of $\mathrm{Sym}^m C_B$ – i.e. it is the length-m subschemes of the hyperelliptic curve $C_B \to \mathbb{C}_x$ whose projection to $\mathbb{C}_x$ also have length m. In [28] Seidel and Smith use this to make a beautiful link between their construction of Khovanov cohomology (of Section 2.8) to Ozsváth-Szabó theory. So in this setting the passage from Ozsváth-Szabó theory to Khovanov cohomology is a form of complexification, replacing the Riemann surface (2.6 with $n = 1$) by the hyperkähler ALE surface (2.6 with $n = 2$) – i.e. replacing (2.21) by (2.19) – and taking Hilb^m of either.

2.7. ALE spaces as affine blow ups. Buried in the description of the Manolescu embedding we saw how to describe the ALE surfaces $S_{\underline{\lambda}}$ (2.6) as affine blow ups. Here we emphasise the construction and a consequence.

Fixing monic p with roots $\underline{\lambda}$, we consider the ALE surface

$$S_{\underline{\lambda}} = \{xy = p(t)\} \subset \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_t \tag{2.22}$$

with its obvious projection to $\mathbb{C}_x \times \mathbb{C}_t$. This is an isomorphism except over the points $x = 0 = p(t)$ of $\mathbb{C}^2$, where the fibre is an exceptional copy of $\mathbb{C}$. This is the affine blow up of $\mathbb{C}^2$ in $x = 0 = p(t)$: the usual blow up given by the same formula in $\mathbb{C}_x \times \mathbb{P}^1_y \times \mathbb{C}_t$ but with $y = \infty$ (the proper transform of the t-axis $x = 0$) removed.

The usual A_{k-1}-chain of Lagrangian S^2s in (2.22) can be seen as follows. Pick an A_{k-1}-chain of paths in $\mathbb{C}_t$ between the roots of $p(t)$. Multiplying by the radius ϵ circle about the origin in $\mathbb{C}_x$ gives k Lagrangian $S^1 \times [0, 1]$ tubes in $\mathbb{C}^2$. Blow up $\mathbb{C}^2$ symplectically by removing balls of radius ϵ about each point of $x = 0 = p(t)$ and collapsing the Hopf fibration on the boundary S^3s. This collapses the tubes to Lagrangian S^2s forming our A_{k-1}-chain; see for instance [31].

As Ivan Smith explained to me, this can also be seen as a "spinning" ([26] is a good recent reference) of $\mathbb{C}_t$ over the roots $\underline{\lambda}$ of $p(t)$. The fibres of the projection to $\mathbb{C}_t$ are conics $\mathbb{C}^*$ (the fibres of $\mathbb{C}_x \times \mathbb{C}_t \to \mathbb{C}_t$ with the t-axis ($x = 0$) removed) except over the roots of $p(t)$ where we get the singular conics $\mathbb{C} \cup_0 \mathbb{C}$ (the exceptional fibre union the original fibre $\mathbb{C}_x$).

What is nice about the description as an affine blow up is that it demonstrates natural maps between the ALE spaces that are compatible with the A_k-chains. Ignoring the centre of mass condition for simplicity, let

$$S_{k-1} \subset \overline{S}_{k-1}$$

denote the ALE surface (2.22) with $\underline{\lambda} = (1, 2, \ldots, k)$ inside the full blow up of $\mathbb{C}^2$ in the points $(0, 1), (0, 2), \ldots, (0, k)$.

Then $\overline{S}_k$ is the blow up of $\overline{S}_{k-1}$ in the point $(0, \infty, k+1)$. On removing $y = \infty$ we get a projection $S_k \to S_{k-1}$. And since we have removed the blow up point

$(0, \infty, k+1)$, we also get an inclusion $S_{k-1} \hookrightarrow S_k$ which is a right inverse. These maps are holomorphic; there are also maps preserving the real symplectic structure once we remove a ball about $(0, \infty, k+1)$ from $\overline{S}_{k-1}$, which will be sufficient for our needs in the next Section.

2.8. The Seidel-Smith construction. Seidel and Smith managed to produce an invariant of links using the space SS_{2m} (2.10). Via the Manolescu isomorphism, and using plait closure in place of braid closure, the construction should become the following. (Since the technical details have only been carried out carefully [27] in the open subset $SS_{\underline{\lambda}} \subset \mathrm{Hilb}^m S_{\underline{\lambda}}$, the following is partly conjectural, and should be thought of only as motivation for the mirror construction. In particular $\mathrm{Hilb}^m S_{\underline{\lambda}}$ is not an exact symplectic manifold, so the definition of Floer cohomology needs some care.)

We fix one of the ALE surfaces (2.22), writing it as

$$S_{2m-1} := \left\{ xy = \prod_{i=1}^{2m} (t - \lambda_i) \right\} \subset \mathbb{C}_{x,y} \times \mathbb{C}_t,$$

where $\underline{\lambda}$ is a collection of $2m$ distinct numbers $\lambda_i \in \mathbb{C}$ (with average zero). We also choose an A_{2m-1}-configuration of paths γ_i running between them, as in Figure 4, and so an A_{2m-1}-chain of Lagrangian spheres $L_i \subset S_{2m-1}$.

In turn this defines the Lagrangian $(S^2)^m$

$$\mathcal{L} = \mathcal{L}_m := L_1 \times L_3 \times \ldots \times L_{2m-3} \times L_{2m-1} \tag{2.23}$$

in the Hilbert scheme

$$H_m := \mathrm{Hilb}^m S_{2m-1},$$

via the map $L_1 \times \ldots \times L_{2m-1} \subset (S_{2m-1})^m \to \mathrm{Sym}^m S_{2m-1} \dashrightarrow \mathrm{Hilb}^m S_{2m-1}$. (Since the L_{2i-1} are disjoint, the map's image lies in the complement of the large diagonal, over which $\mathrm{Hilb}^m S_{2m-1} \to \mathrm{Sym}^m S_{2m-1}$ is an isomorphism.)

The relative Hilbert schemes of the family of $S_{\underline{\lambda}}$s (2.6) gives a quasi-projective family over C^0_{2m}. Taking monodromy, we see that the braid group lifts to the symplectomorphism group of $\mathrm{Hilb}^m S_{2m-1}$. The Kähler form is the one pulled back via the resolution $\mathrm{Hilb}^m \to \mathrm{Sym}^m$, minus $\epsilon[E]$, where E is the exceptional divisor. By making $\epsilon \to 0$ we can ensure that the action of $\beta \in B_{2m}$ is arbitrarily close, away from the exceptional locus, to the action of $\beta \times \ldots \times \beta$ on $\mathrm{Sym}^m S_{2m-1}$.

Then for any $\beta \in B_m$ define the braid invariant

$$SS^*(\beta) := HF^{*+m+w}(\mathcal{L}, \beta\mathcal{L}) \tag{2.24}$$

to be the Floer cohomology of $\mathcal{L}$ and its image under β (assuming the technical details can be overcome to define this, and as a graded $\mathbb{C}$-vector space rather than a module over a Novikov ring). Here the writhe w is the number of positive minus the number of negative crossings in the braid β.

In fact $SS^*(\beta)$ should be an invariant of the isotopy class of the link given by the plait closure of β. By a result of Birman [3], modified slightly in [2], and the fact that Floer cohomology is functorial under (graded) symplectomorphisms (so that $HF^*(\mathcal{L}, \alpha\beta\mathcal{L}) = HF^*(\alpha^{-1}\mathcal{L}, \beta\mathcal{L})$, for instance), to deduce this it is sufficient to prove the following; see Figure 5 and the further explanation below.

1. $T_1\mathcal{L} \cong \mathcal{L}[-1]$,
2. $T_{2i-1}T_{2i}\mathcal{L} \cong T_{2i-1}^{-1}T_{2i}^{-1}\mathcal{L}$,
3. $T_{2i}T_{2i-1}T_{2i+1}T_{2i}\mathcal{L} \cong \mathcal{L}$, and
4. $HF^*(\mathcal{L}_m, \beta\mathcal{L}_m) \cong HF^{*+1}(T_{2m}^{\pm 1}\mathcal{L}_{m+1}, \beta\mathcal{L}_{m+1})$.

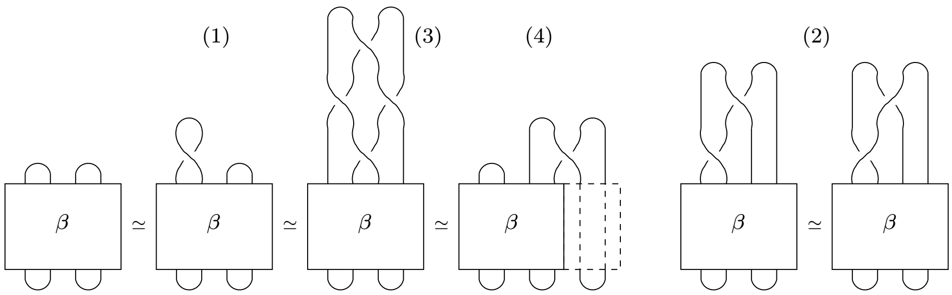

Figure 5. Equivalent plait closures of a braid $\beta \in B_4$.

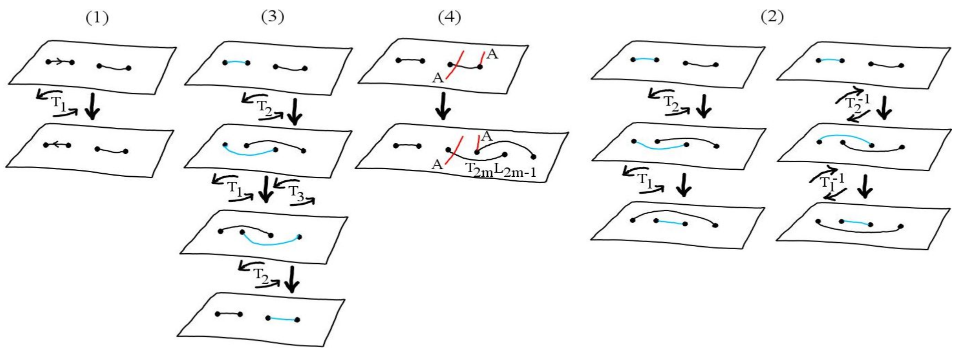

Figure 6. Action of the moves (1) – (4) on the Lagrangians L_i fibring over the paths shown. This gives the action on $\mathcal{L} \subset \mathrm{Hilb}^m S_{2m-1}$, which is a product (2.23) of L_i s.

We now explain these relations and Figures, starting with (1). As we have already seen, T_1 simply flips the path running between the first two roots λ_1, λ_2. This preserves L_1 but shifts its grading by the $[-1]$ on the right hand side of (1). Since we are skating over the issue of grading we content ourselves with noting only that it reverses the orientation of L_1 (this is equivalent to the action on grading mod 2). Since the other L_{2i-1}, $i \geq 2$ are untouched by T_1 the relation (1) follows.

Secondly we consider (3). As shown in Figure 6, $T_{2i}T_{2i-1}T_{2i+1}T_{2i}$ simply swaps $L_{2i\pm1}$ (and leaves the other L_{2j-1} alone). But in $\mathrm{Hilb}^m S_{2m-1}$ the order of the factors of $\mathcal{L}$ is unimportant, so (3) follows.

Relation (4) (stabilisation as we increase the number of strands in our braid, or Markov II as it is called in [27]) is slightly more involved. The left hand side is computed in H_m, with β an element of B_{2m}. The right hand side takes place in H_{m+1}, with β considered as an element of B_{2m+2} via the standard inclusion $B_{2m} \hookrightarrow B_{2m+2}$. Here we are using the inclusion of ALE spaces $S_{2m-1} \subset S_{2m+1}$ of Section 2.7.

In Figure 6 is drawn part of an arbitrary $O(2)$-invariant Lagrangian A which is generated in $\mathcal{F}(S_{2m-1})$ by L_i, $i \leq 2m-1$ ($\beta\mathcal{L}_m$ in (4) being a product of such things). We have drawn intersections of A with L_{2m-1} in either the root λ_{2m} or elsewhere. This corresponds to a splitting

$$HF^*(L_{2m-1}, A) \;\cong\; HF^{*+1}(L_{2m}, A) \oplus HF^*(T_{2m}L_{2m-1}, A) \tag{2.25}$$

coming from the exact triangle (cf. (2.4))

$$L_{2m-1} \to T_{2m}L_{2m-1} \to L_{2m} \tag{2.26}$$

in $\mathcal{F}(S_{2m-1})$. (One can show that $HF^*(\,\cdot\,, A)$ applied to the second arrow vanishes for $A = L_i$, $i \leq 2m-1$, and so for any A, to give the splitting (2.25).) The first summand in (2.25) corresponds to the intersections at the root λ_{2m}; these come from intersections with the next Lagrangian L_{2m} along via cup product with the $HF^1(L_{2m}, L_{2m-1})$ class of the intersection of L_{2m-1} and L_{2m} (the extension class of the triangle (2.26)). The other intersection points are those which survive when L_{2m-1} is Dehn twisted about L_{2m}, as shown in Figure 6, and form the second summand of (2.25).

Since A has no intersections with L_{2m+1} the first summand is isomorphic to $HF^{*+1}(T_{2m}L_{2m+1}, A)$, as can also be seen from Figure 6. The upshot is that if $\mathcal{A}$ is a product of Lagrangians of the form A, the intersection points used to calculate $HF^*(\mathcal{L}_m, \mathcal{A})$ can be matched with intersection points used to calculate $HF^{*+1}(T_{2m}\mathcal{L}_{m+1}, \mathcal{A})$. More precisely their Floer cohomologies can be matched using (2.25). Applied to $\mathcal{A} = \beta\mathcal{L}_m$ this gives (4).

Finally we come to relation (2). We calculate on S_{2n-1} that both $T_{2i-1}T_{2i}$ and $T_{2i-1}^{-1}T_{2i}^{-1}$ leave L_{2j+1} alone for $j \neq i, i-1$, and take L_{2i-1} to L_{2i}. This is clear from Figure 6. Their actions on L_{2i+1} differ, however. They both take it to connect sums of L_{2i-1}, L_{2i} and L_{2i+1}, but in the opposite direction:

$$T_{2i-1}T_{2i}L_{2i+1} \;\cong\; L_{2i+1}\#L_{2i}\#L_{2i-1}, \tag{2.27}$$

$$T_{2i-1}^{-1}T_{2i}^{-1}L_{2i+1} \;\cong\; L_{2i-1}\#L_{2i}\#L_{2i+1}. \tag{2.28}$$

Here # is the *graded Lagrangian connect sum* [22, 33], and is *not* symmetric. It can be described in an $O(2)$-symmetric manner by the connect-summed paths in Figure 6 – with the connect sums in opposite directions corresponding to paths above and below their intersection point.

The two Lagrangians (2.27, 2.28) are certainly *not* Hamiltonian isotopic in S_{2m-1}, so that $T_{2i-1}T_{2i}\mathcal{L}$ and $T_{2i-1}^{-1}T_{2i}^{-1}\mathcal{L}$ are *not* Hamiltonian isotopic in either the product $(S_{2m-1})^m$ or symmetric product $\operatorname{Sym}^m S_{2m-1}$. However Seidel and Smith prove they *are* Hamiltonian isotopic in SS_{2m}, and therefore also in $\operatorname{Hilb}^m S_{2m-1}$. We want to think about this categorically as follows.

In the derived Fukaya category, we see $T_{2i-1}T_{2i}\mathcal{L}$ and $T_{2i-1}^{-1}T_{2i}^{-1}\mathcal{L}$ as extensions of the same objects in the opposite direction. On deforming the symplectic space $\operatorname{Sym}^m S_{2m-1}$ to $\operatorname{Hilb}^m S_{2m-1}$ (by "inflating" the exceptional divisor – subtracting a small amount of the class of the exceptional divisor from the degenerate symplectic form pulled back from $\operatorname{Sym}^m S_{2m-1}$) the Lagrangians $T_{2i-1}^{\pm 1}T_{2i}^{\pm 1}\mathcal{L}$ deform because both $\mathcal{L}$ and the symplectomorphisms T_i do. However the pieces $L_{2i\pm 1}\times L_{2i}$ of the extensions do *not* deform as Lagrangians – the class $[E]$ restricts to a nonzero class thereon (because $L_{2i\pm 1}$ intersects L_{2i} inside S_{2m-1}). And then for general reasons, if two extensions of the same pieces deform while the pieces do not then the deformations of the extensions become isomorphic. The algebro-geometric analogue of this will be clearer to see in Section 5.

Using slightly different techniques in a fibre of SS_{2m}, Seidel and Smith prove carefully that they get an invariant of links up to isotopy. Conjecturally their invariant can be derived from the famous Khovanov cohomology $KH^{*,*}\otimes\mathbb{C}$ [14] by a certain collapse of the latter's bigrading. In the algebro-geometric mirror described later, it will in fact be possible to get the full bigrading and prove the isomorphism to $KH^{*,*}\otimes\mathbb{C}$.

3. Simultaneous resolution

In each of the examples (2.1), (2.6), (2.8) and (2.11) – in the first two cases only in dimension $n=2$ – the families have a remarkable property. The complete family $\mathcal{X}\to B$ (*including* the singular fibres now) can be pulled back to a new family $\mathcal{X}'\to B'$ via a finite basechange $B'\to B$, such that $\mathcal{X}'$ admits a *simultaneous resolution*

$$\pi\colon \overline{\mathcal{X}}\to\mathcal{X}'.$$

This is a map which is birational, and a resolution of singularities on each fibre. In particular on each smooth fibre it restricts to an isomorphism. So the smooth fibres fit together with the *resolutions* of the singular fibres in a smooth family $\overline{\mathcal{X}}\to B'$. Thus the smoothings and resolutions of the singular fibres of $\mathcal{X}\to B$ are diffeomorphic (something which is obviously not true for the $n=1$ dimensional node (2.1), for instance) and is related to the fact that they are hyperkähler [12].

3.1. Surface ordinary double point.

The simplest case is the smoothing of the surface ordinary double point,

$$\mathcal{X}=\{x^2+y^2+w^2=t\}\subset\mathbb{C}^3_{xyw}\times\mathbb{C}_t\to\mathbb{C}_t.$$

If we pull this back by the double cover $t \mapsto t^2$ of the base then the total space becomes singular itself, with the threefold ordinary double point singularity

$$\mathcal{X}' = \{x^2 + y^2 + w^2 = t^2\} \subset \mathbb{C}^3_{xyw} \times \mathbb{C}_t \to \mathbb{C}_t. \tag{3.1}$$

Setting $X = x + iy, Y = x - iy, T = t + w, W = t - w$ this becomes

$$\mathcal{X}' = \{XY = TW\} \subset \mathbb{C}^4$$

fibring over $\mathbb{C}$ by the function $(T+W)/2$. Blowing up the Weil divisor $(X = 0 = T)$ gives a resolution $\overline{\mathcal{X}} \to \mathcal{X}'$ which is an isomorphism away from the origin. More explicitly, $\overline{\mathcal{X}}$ is the graph of the rational function $X/T = W/Y \colon \mathcal{X}' \to \mathbb{P}^1$ in $\mathcal{X}' \times \mathbb{P}^1$:

$$\overline{\mathcal{X}} := \{(X, Y, T, W, [\lambda : \mu]) \in \mathbb{C}^4 \times \mathbb{P}^1 \colon XY = TW,\ \mu X = \lambda T,\ \mu W = \lambda Y\}.$$

Then $\overline{\mathcal{X}} \to \mathcal{X}'$ is an isomorphism on all of the smooth fibres of (3.1), and replaces the central fibre's surface ordinary double point by its minimal resolution – i.e. its blow up with a $\mathbb{P}^1$ exceptional set C. (So the exceptional set of the whole family is this $C \cong \mathbb{P}^1$, which is *not* a divisor: $\overline{\mathcal{X}} \to \mathcal{X}'$ is a *small* resolution).

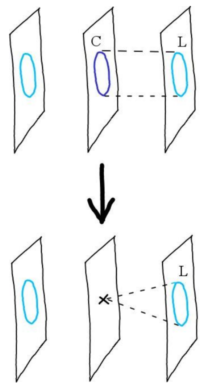

Figure 7. Simultaneous resolution $\overline{\mathcal{X}}$ of the family (3.1), with the Lagrangian vanishing cycles $L \cong S^2$ limiting to the holomorphic exceptional curve $C \cong \mathbb{P}^1$.

We picture this in Figure 7. By its definition as a vanishing cycle, under symplectic parallel transport the *Lagrangian* L limits to the *holomorphic* exceptional $\mathbb{P}^1 = C$. This is remarkable but no contradiction; the pull back of the standard Kähler form from $\mathcal{X}'$ is symplectic on the general fibre (and zero on restriction to L) but degenerate on the central fibre (it is precisely zero along C). One could perturb to get a nondegenerate Kähler form on $\overline{\mathcal{X}}$, giving nonzero area to C, but this would then also have nonzero area on the (homologous) L which would therefore cease to be Lagrangian.

One can also ask what the limit of the Dehn twists is on the central fibre. Consider the graph in $\mathcal{X}_\epsilon \times \mathcal{X}_\epsilon$ of the monodromy about the circle of radius ϵ. As

$\epsilon \to 0$, this approaches the identity away from the vanishing cycle L. Arbitrarily close to L we can always find $\epsilon > 0$ and a point that the Dehn twist takes to any other given point. So in the limit we get all of $C \times C$ (since C is the limit of L_ϵ).

The upshot is that as $\epsilon \to 0$ the limit of the Dehn twists about the L_ϵ is the *holomorphic* correspondence

$$\Delta \cup (C \times C) \tag{3.2}$$

in $\overline{\mathcal{X}}_0 \times \overline{\mathcal{X}}_0$, where Δ is the diagonal.

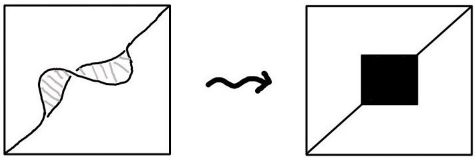

Figure 8. The graph of the Dehn twist limits to the correspondence $\Delta \cup (C \times C)$. (Despite the crude picture, the two irreducible components Δ and $C \times C$ have the same dimension 2.)

The family of $S_{\underline{\lambda}}$s over C_k^0 (2.22) also admits a simultaneous resolution after basechange, with the A_{k-1}-chain of Lagrangian S^2s limiting to the A_{k-1}-chain of holomorphic $\mathbb{P}^1$s in the minimal resolution. When $k = 2m$, taking the relative Hilbert scheme of this new family gives (a birational model of) a similar simultaneous resolution for the space SS_{2m} (2.11) via Manolescu's embedding. Instead of describing these examples in detail we pass straight to the final, and universal example. The previous examples can be obtained from this by taking slices.

3.2. Adjoint quotient and the Flag variety. We partially compactify the adjoint quotient (2.8) with the space of *all* trace-free $k \times k$ matrices, mapping via the roots of its characteristic polynomial to $\operatorname{Sym}^k \mathbb{C}$:

$$M_k \to \operatorname{Sym}^k \mathbb{C}. \tag{3.3}$$

We basechange by the projection $\mathbb{C}^k \to \operatorname{Sym}^k \mathbb{C}$ that forgets the order of k-tuples. In other words we consider the space of matrices with a chosen ordering of the roots (with multiplicities) of its characteristic polynomial:

$$M_k' \to \mathbb{C}^k.$$

At a point $(A, \lambda_1, \dots, \lambda_{2k}) \in M_k'$ with distinct roots, so that the matrix has distinct eigenvalues λ_i with eigenspaces L_i, there is a canonical associated flag $0 < V_1 < \dots < V_{k-1} < V$ given by $V_i = \oplus_{j \le i} L_j$. This is preserved by A, and characterised by the property that A acts on V_i/V_{i-1} with weight λ_i. Therefore the space $\overline{M}_k$ defined as

$$\Big\{\big(A, \underline{\lambda}, (0 < V_1 < \dots < V_{k-1} < V)\big) \colon AV_i \subseteq V_i\ \forall i,\ A \text{ acts on } V_i/V_{i-1} \text{ as } \lambda_i\Big\} \tag{3.4}$$

has a forgetful map to M'_k which is an isomorphism over the good locus of matrices with distinct eigenvalues. In fact $\overline{M}_k \to M'_k$ is a simultaneous resolution, restricting over each fibre of $M'_k \to \mathbb{C}^k$ to a resolution of singularities. The central fibre is the cotangent bundle T^*Fl of the Flag variety, because its fibre over a point $(0 < V_1 < \cdots < V_{k-1} < V) \in Fl$ is

$$\{A : V \to V : AV_i \subseteq V_{i-1}\}.$$

It provides a resolution of the central fibre of $M_k \to \operatorname{Sym}^k \mathbb{C}$, i.e. of the nilpotent cone of matrices with no nonzero eigenvalues. The general fibre is diffeomorphic to it; in fact it is *symplectomorphic* to T^*Fl with its canonical real symplectic structure as the cotangent bundle of a real manifold.

A similar picture to Figure 7 holds. While Fl is a *holomorphic* subvariety of the central fibre, it is the limit of *Lagrangian* vanishing cycles $Fl \subset T^*Fl$ in the general fibre.

In the central fibre T^*Fl live the divisors

$$N_i := \pi_i^* T^* Fl_i \subset T^*Fl, \tag{3.5}$$

where $\pi_i \colon Fl \to Fl_i$ is the map to the partial flag variety that forgets the ith term V_i in the flag. In the general fibre (seen as symplectomorphic to T^*Fl) they are coisotropic with characteristic foliation $\pi_i|_{N_i}$ a fibration by isotropic S^2s. As λ_i and λ_{i+1} come together in the base $\operatorname{Sym}^k \mathbb{C} = \{\text{eigenvalues}\}$ (3.3), N_i is the relative vanishing cycle that collapses along this characteristic foliation to a family of surface ordinary double points. Doing the family generalised Dehn twist about N_i ([21, Section 1.4], [19, Section 2.3]) should give the braid group of symplectic monodromies of (2.9). The limit of the graphs of these symplectomorphisms is the subvariety

$$\Delta \cup (N_i \times_{Fl_i} N_i) \;\subset\; T^*Fl \times T^*Fl. \tag{3.6}$$

4. Homological mirror symmetry

Kontsevich's homological mirror symmetry conjecture [17] is an amazing categorical expression of Witten's formulation of mirror symmetry in terms of A- and B-models. It has become a vast subject that we will only touch on through our example.

Roughly speaking, Kontsevich says that two closed Calabi-Yau manifolds should be considered as mirror pairs when the derived Fukaya category of one is isomorphic to the derived category of coherent sheaves on the other. Symplectic geometry (the "A-model") on one side is equated with complex geometry (the "B-model") on the other side. In particular the plentiful automorphisms of a symplectic manifold should be mirrored not by holomorphic automorphisms of the mirror (of which there are few) but by autoequivalences of its derived category.

4.1. Surfaces. For the examples of the last section, passing from the general fibre to the resolution of the central fibre (using symplectic parallel transport and simultaneous resolution) gives a cheap way to swap complex and symplectic structures. As we have seen, Lagrangian submanifolds can become, in the limit, holomorphic (in fact complex Lagrangian, in the canonical holomorphic symplectic structure). Taking the structure sheaves of these limits means we have turned objects of the derived Fukaya category into objects of the derived category of coherent sheaves.

So it seems a reasonable guess that the mirror of the (symplectic) general fibre might be related to the (holomorphic) resolution of the central fibre. (That mirror symmetry is so simple here, not even changing the topology, is a feature of hyperkähler manifolds, with the mirror map being related to hyperkähler rotation. To make this more precise would involve complexifying our symplectic forms with B-fields, putting connections with curvature $B|_L$ on our Lagrangians L, introducing coisotropic branes, worrying about noncompactness, and working much harder. But we use mirror symmetry here only as a motivational guide.)

So in the simplest case we would like to think of the mirror of the symplectic manifold T^*S^2 (the smoothing of the surface ordinary double point) as something like the complex surface $S = T^*\mathbb{P}^1$ (the resolution of the surface ordinary double point). As usual we denote the Lagrangian S^2 by L and the holomorphic $\mathbb{P}^1$ by C, so we would like mirror symmetry to relate

$$L \in \mathcal{F}(T^*S^2) \quad \text{to} \quad \mathcal{O}_C(-1) \in D(S),$$

where D denotes the bounded derived category of coherent sheaves with compact support. (Work of Auroux and Seidel suggests one should remove certain loci from T^*S^2 and $T^*\mathbb{P}^1$ before they can sensibly be considered as mirror, but for our heuristic purposes we can ignore this.) The twist by the line bundle $\mathcal{O}(-1)$ is unimportant (since it defines an autoequivalence of $D(S)$) and is just for convenience.

Since the graph of the Dehn twist T_L about L limits (3.2) to the holomorphic subvariety

$$\Delta \cup (C \times C) \overset{\iota}{\hookrightarrow} S \times S, \tag{4.1}$$

it is natural to use this as a holomorphic correspondence on S. In fact we would like to lift this to an action on $D(S)$, mirror to the induced action of T_L on $\mathcal{F}(T^*S^2)$. So we might use the structure sheaf of (4.1) as a Fourier-Mukai kernel. For convenience we twist by the line bundle $\mathcal{L}$ which is $\mathcal{O}_S(C)$ on Δ glued to $\mathcal{O}(-1,-1)$ on $C \times C$ (both are isomorphic to $\mathcal{O}_C(-2)$ on Δ_C):

$$T_C := \pi_{2*}\big(\iota_*\mathcal{L} \otimes \pi_1^*(\,\cdot\,)\big) \colon D(S) \to D(S).$$

(Here $\pi_1, \pi_2 \colon S \times S \to S$ are the obvious projections, and the functors $\otimes$ and π_{2*} are derived. It turns out that using the untwisted structure sheaf gives the inverse of the functor T_C; I don't know if this is significant or a coincidence.) Equivalently, the action of T_C on $E \in D(S)$ is

$$E \mapsto T_C E = \operatorname{Cone}\big(R\operatorname{Hom}(\mathcal{O}_C(-1), E) \otimes \mathcal{O}_C(-1) \to E\big), \tag{4.2}$$

where the arrow is the obvious evaluation map. (Taking E to be a complex of injectives, this map is canonical rather than defined up to homotopy, so the cone turns out to be functorial here [30].) Compare its mirror (2.3, 2.4).

More generally, the simultaneous resolution of the family of ALE surfaces (2.22) has central fibre the minimal resolution of

$$\{xy = t^k\} \subseteq \mathbb{C}^3. \tag{4.3}$$

Call this S, with its A_{k-1}-chain of exceptional -2-curves $C_i \subset S$ (the limit of an A_{k-1}-chain of Lagrangian vanishing cycles L_i on a general fibre). In fact the sheaves $A_i := \mathcal{O}_{C_i}(-1)$ satisfy the following homological definition of an A_{k-1}-chain in any derived category of coherent sheaves.

Definition 4.4. [30] *Objects $A_i \in D(S)$, $i = 1, \dots, k-1$ form an A_{k-1}-chain of n-spherical objects if for all i, j,*

- $\mathrm{Ext}^*(A_i, A_i) \cong H^*(S^n, \mathbb{C})$,
- $A_i \otimes \omega_S \cong A_i$,
- $\bigoplus_p \mathrm{Ext}^p(A_i, A_j) = \begin{cases} \mathbb{C} & |i-j| = 1, \\ 0 & |i-j| > 1. \end{cases}$

For us $n = 2$, and the second, Calabi-Yau condition always holds since the canonical bundle ω_S of S is trivial. One can then define the Dehn twists about the A_i as in (4.2) by

$$T_{A_i} E := \mathrm{Cone}\big(R\,\mathrm{Hom}(A_i, E) \otimes A_i \to E\big), \tag{4.5}$$

or by Fourier-Mukai transform with the kernel

$$\mathrm{Cone}\big(A_i^\vee \boxtimes A_i \to \mathcal{O}_\Delta\big).$$

Here $^\vee$ denotes derived dual, and the arrow is restriction to the diagonal followed by evaluation (trace).

Theorem 4.6. [15, 30] *If the A_i form an A_{k-1}-chain then the $T_i = T_{A_i}$ define a (weak) faithful action of the braid group $B_k \hookrightarrow \mathrm{Aut}(D(S))$.*

In particular, the T_i are invertible and satisfy the braid relations

$$\begin{array}{rcll} T_i T_j T_i & \cong & T_j T_i T_j, & |i-j| = 1, \\ T_i T_j & \cong & T_j T_i, & |i-j| > 1. \end{array}$$

So our putative mirrors of Dehn twists really satisfy the same relations as the original twists (2.7). And we have put things in a more categorical framework, allowing twists around arbitrary spherical objects, as mirror symmetry suggests should be possible – according to Kontsevich's conjecture, all mirror symmetry needs to see is categorical properties, rather than specific geometry. For more on mirror symmetry for ALE surfaces see [13].

4.2. Higher dimensions. Our other examples of families over C_k^0 fit into a similar hyperkähler mirror symmetry picture. In fact they all follow from the case of the space of matrices of Section 3.2 by taking slices. In much the same way as described above, the family Dehn twists around the divisors N_i limit to the Fourier-Mukai transforms with kernels the structure sheaves of the limits $\Delta \cup (N_i \times_{Fl_i} N_i)$ (3.6) of the graphs of these symplectomorphisms. Up to twisting by a line bundle, these are the relative versions of the derived category Dehn twist (4.5), with action

$$E \mapsto \text{Cone}\left(\iota_{i*}p_i^* p_{i*}\iota_i^! E \to E\right).$$

Here the arrow is evaluation, and p_i and ι_i are the obvious maps

$$\begin{array}{l} N_i \overset{\iota_i}{\hookrightarrow} T^*Fl \\ \downarrow^{p_i} \\ T^*Fl_i. \end{array} \tag{4.7}$$

Again these define autoequivalences $T_i\colon D(T^*Fl) \to D(T^*Fl)$ which satisfy the braid relations [1, 16]. In fact the T_i (both here and on the slices S of the last section) even admit natural transformations between them which satisfy the relations of the braid cobordism category, and these give rise to maps between the Khovanov cohomology groups of links of the next Section, when we fix a link cobordism. But we refer to [16] for this further extension of mirror symmetry.

The braid relations in this case are *much* harder than those in 2 dimensions. But Manolescu's isomorphism means that they follow from the simple two dimensional case for the spaces relevant to Khovanov cohomology.

5. Hilbert schemes of ALE spaces and Khovanov cohomology

By now it should be clear how one would go about trying to mirror the Seidel-Smith construction to define Khovanov cohomology in a derived category of coherent sheaves. There is a slice of (3.4) that provides a simultaneous resolution of (the basechange of) SS_{2m}. The derived category of its central fibre carries a braid group action and a complex Lagrangian submanifold that $\mathcal{L}$ (2.23) limits to. Taking its structure sheaf (and possibly twisting by a line bundle) as an object of the derived category, one would like to show that the Exts from this object to its image under a braid give an invariant of the link closure of the braid.

Such a programme has been carried out in beautiful work of Cautis and Kamnitzer [5]. In fact they use a compactification of the above space related, via the geometric Satake correspondence, to the $sl(2)$ representations of the Reshetikhin-Turaev tangle calculus. This has the huge advantage of being generalisable to other Lie algebras [6]. However, as mentioned above, it is also hard work, involving calculations in high dimensions.

Manolescu's isomorphism suggests we might work with something like the Hilbert scheme of points on $S = S_{2m-1}$, the minimal resolution of the A_{2m-1}-

singularity (4.3). This reduces most of the work to much simpler calculations with sheaves on the surfaces S_{2m-1}. In fact, by [4, 9] the category

$$D_m := D(\operatorname{Hilb}^m S_{2m-1})$$

has a canonical identification with the Σ_m-equivariant derived category of $(S_{2m-1})^m$, where the symmetric group Σ_m permutes the factors:

$$D(\operatorname{Hilb}^m S_{2m-1}) \;\cong\; D(S_{2m-1}^m)^{\Sigma_m}. \tag{5.1}$$

5.1. However.

One would expect the right hand side of (5.1) to be mirror to the Σ_m-equivariant Fukaya category of $S_{\underline{\lambda}}$ (2.19), which is *not* the Fukaya category of its Hilbert scheme, but can be thought of as playing the role of the Fukaya category of the singular symplectic space $\operatorname{Sym}^m S_{2m-1}$. Considering the Hilbert scheme as a symplectic deformation of this (subtracting a small multiple of the exceptional divisor of Hilb $\to$ Sym from the degenerate symplectic form one gets by pulling back from Sym) suggests the mirror might be a *deformation* of $\operatorname{Hilb}^m S_{2m-1}$. We will indeed use such a deformation related to the exceptional divisor.

This is an example where our naive description of mirror symmetry fails. The mirror of the smoothing $\operatorname{Hilb}^m S_{\underline{\lambda}}$ of a hyperkähler singularity $\operatorname{Hilb}^m S_{\underline{0}}$ appears not to be the obvious choice $\operatorname{Hilb}^m S_{2m-1}$ (which is birational to a resolution of $\operatorname{Hilb}^m S_{\underline{0}}$) but a *deformation* thereof.

5.2. The construction.

Any $E \in D(S_{2m-1}^m)$ defines an element

$$\Sigma_m.E := \bigoplus_{\sigma\in\Sigma_m} \sigma^* E \;\in\; D(S_{2m-1}^m)^{\Sigma_m}, \tag{5.2}$$

with its obvious Σ_m-linearisation. Thus from the spherical objects $L_i := \mathcal{O}_{C_i}(-1) \in D(S_{2m-1})$ we define

$$\mathcal{L} = \mathcal{L}_m := \Sigma_m.(L_1 \boxtimes L_3 \boxtimes \ldots \boxtimes L_{2m-1}) \in D(S_{2m-1}^m)^{\Sigma_m}. \tag{5.3}$$

Equivalently, the object $\mathcal{L} \in D(\operatorname{Hilb}^m S_{2m-1})$ can be described via the Haiman-BKR equivalence (5.1) as follows. As in (2.23), the composition

$$C_1 \times C_3 \times \ldots \times C_{2m-1} \hookrightarrow S_{2m-1}^m \to \operatorname{Sym}^m S_{2m-1} \dashrightarrow \operatorname{Hilb}^m S_{2m-1}$$

is an embedding since the C_{2i-1} do not intersect each other so their product avoids the diagonal locus over which the last map is not regular. Then

$$\mathcal{L} = \mathcal{O}_{C_1\times C_3\times\ldots\times C_{2m-1}}(-1,-1,\ldots,-1) \in D(\operatorname{Hilb}^m S_{2m-1}). \tag{5.4}$$

Any autoequivalence $T \in \operatorname{Aut}(D(S_{2m-1}))$ induces a canonical autoequivalence $\Phi(T) \in \operatorname{Aut}(D(S_{2m-1}^m)^{\Sigma_m})$ [20]. Its action on objects of the form (5.3) is the obvious one:

$$\Phi(T)\Big(\Sigma_m.(E_1 \boxtimes \ldots \boxtimes E_m)\Big) \;=\; \Sigma_m.\big(T(E_1)\boxtimes \ldots \boxtimes T(E_m)\big). \tag{5.5}$$

We apply this to the spherical twists $T_i := T_{\mathcal{O}_{C_i}(-1)}$:

$$\mathsf{T}_i := \Phi(T_i)[1] \in \operatorname{Aut}\left(D(S^m_{2m-1})^{\Sigma_m}\right). \tag{5.6}$$

Since Φ is a homomorphism, these define generators of a braid group action $B_{2m} \to \operatorname{Aut}(D_m)$. (The braid relations are homogeneous, so the extra shift [1] makes no difference.) Thus any $\beta \in B_{2m}$ gives an autoequivalence $\mathsf{T}_\beta \in \operatorname{Aut}(D_m)$. We define the braid invariant

$$kh^*(\beta) := \operatorname{Ext}^*_{D_m}(\mathcal{L}, \mathsf{T}_\beta\, \mathcal{L}[m]). \tag{5.7}$$

The shifts in the definitions (5.6, 5.7) match with the shift $w + m$ in the mirror Seidel-Smith construction (2.24).

5.3. Maps between ALE spaces.

To study the dependence of (5.7) on m we will need the holomorphic analogue (or hyperkähler rotation) of the symplectic maps between ALE spaces of Section 2.7. So let S_{k-1} be the minimal resolution of $A_{k-1} := \{x^k = yz\} \subset \mathbb{C}^3$. We will exhibit a natural inclusion $S_{k-1} \subset S_k$ taking the A_{k-1}-chain of -2-curves $C_i \cong \mathbb{P}^1,\ i = 1, \dots, k-1$ in the former to the first $k-1$ curves of the A_k-chain $C_1, \dots, C_{k-1}, C_k$ in the latter.

Consider the blow up of $\mathbb{C}^2$ in the ideal (x^k, y). Call this $\overline{A}_{k-1}$. It can be constructed inductively via blow ups and a blow down in smooth centres:

1. Blow up the origin in $\mathbb{C}^2$, giving an exceptional divisor $E_1 \cong \mathbb{P}^1$.

2. Blow up the point $\infty \in E_1$ (its intersection with the proper transform of the x-axis). We get a new exceptional divisor E_2, and the proper transform of E_1 which is a -2-curve C_1.

(r) At the rth stage, blow up $\infty \in E_{r-1}$ to produce a new exceptional divisor E_r, and the proper transform of E_{r-1} is a -2-curve C_r.

After the kth step we get a surface $\overline{S}_{k-1}$ with an A_{k-1}-chain of -2-curves C_i and a -1-curve E_k; see Figure 9. Now blow down the $C_i,\ i = 1, \dots, k-1$ to get $\overline{A}_{k-1}$.

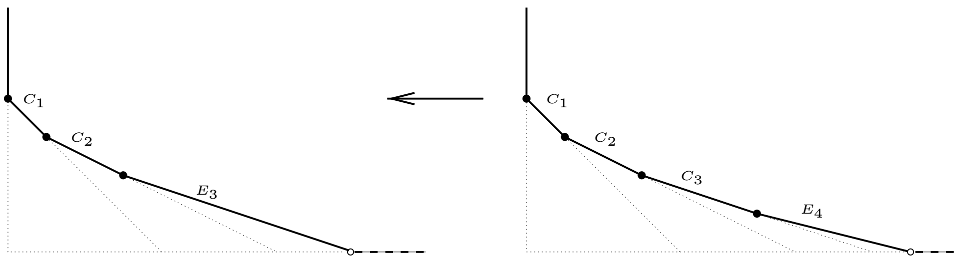

Figure 9. Newton polygon diagram of the blow up map $\overline{S}_2 \leftarrow \overline{S}_3$. On removing the divisors corresponding to the dashed lines (the proper transforms of the x-axis) we get an inclusion $S_2 \subset S_3$ in the opposite direction.

Now $\overline{A}_{k-1} = \mathrm{Bl}_{(x^k,y)}\,\mathbb{C}^2 = \{\mu x^k = \lambda y\} \subset \mathbb{C}^2_{x,y} \times \mathbb{P}^1_{[\lambda:\mu]}$. Therefore if we remove the proper transform $\overline{\{y=0\}} = \{\mu = 0\}$ of the x-axis we can set $[\lambda : \mu] = [z : 1]$ to get the affine variety

$$\{x^k = yz\} \subset \mathbb{C}^2_{x,y} \times \mathbb{C}_z,$$

which is precisely A_{k-1}. Thus $\overline{A}_{k-1}$ and $\overline{S}_{k-1}$ are partial compactifications of A_{k-1} and S_{k-1} respectively (since $\overline{S}_{k-1}$ is the minimal resolution of S_{k-1}).

We obtained $\overline{S}_k$ from $\overline{S}_{k-1}$ by blowing up the latter in the point $\infty \in E_k$. But $\infty = \overline{\{y=0\}} \cap E_k$ lies in the divisor $\overline{\{y=0\}}$ that we remove from $\overline{S}_{k-1}$ to get S_{k-1}, so the inclusion $S_{k-1} \subset \overline{S}_{k-1}$ lifts to the blow up: $S_{k-1} \subset \overline{S}_k$. Its image is clearly contained in the open subset S_k, and maps the curves $C_i \subset S_{k-1}$ to the corresponding curves $C_i \subset S_k$, as claimed.

As in Section 2.8, to prove that kh^* is a link invariant under plait closure it is sufficient to prove the following; again see Figure 5.

1. $\mathsf{T}_1\mathcal{L} \cong \mathcal{L}$,
2. $\mathsf{T}_{2i-1}\mathsf{T}_{2i}\,\mathcal{L} \cong \mathsf{T}_{2i-1}^{-1}\mathsf{T}_{2i}^{-1}\mathcal{L}$,
3. $\mathsf{T}_{2i}\mathsf{T}_{2i-1}\mathsf{T}_{2i+1}\mathsf{T}_{2i}\,\mathcal{L} \cong \mathcal{L}$, and
4. $\mathrm{Ext}^*_{D_m}(\mathcal{L}_m, \mathsf{T}_\beta\,\mathcal{L}_m[m]) \cong \mathrm{Ext}^*_{D_{m+1}}(\mathsf{T}_{2m}^{\pm1}\mathcal{L}_{m+1}, \mathsf{T}_\beta\,\mathcal{L}_{m+1}[m+1])$.

In the last relation we use the inclusion $S_{2m-1} \hookrightarrow S_{2m+1}$ exhibited above.

Theorem 5.8. [32] *The relations (1), (3) and (4) hold in the categories D_m, but (2) does not.*

The proof is reduced by (5.5) to simple computations in $D(S_{2m-1})$ mirroring those of Section 2.8.

Firstly, $T_1L_{2i+1} \cong L_{2i+1}$ for $i \geq 1$ by (4.2), because $\mathrm{Ext}^*(L_1, L_{2i+1}) = 0$. Since $\mathrm{Ext}^*(L_1, L_1) \cong H^*(S^2, \mathbb{C})$ we get the exact triangle in $D(S_{2m-1})$

$$L_1 \oplus L_1[-2] \longrightarrow L_1 \longrightarrow T_1L_1.$$

The first map is the identity on the first factor, so $T_1L_1 \cong L_1[-1]$. Therefore by (5.5), $\mathsf{T}_1\mathcal{L} \cong \mathcal{L}[-1][1] = \mathcal{L}$, which proves relation (1).

For (2) we note the following calculation on S_{2m-1}. If $A, B \cong \mathbb{P}^1$ are (possibly reducible) rational curves in S_{2m-1} intersecting in a single transverse point, then $\mathrm{Ext}^*(\mathcal{O}_A, \mathcal{O}_B) = \mathbb{C}[-1]$ and the resulting exact triangle

$$\mathcal{O}_B \to T_{\mathcal{O}_A}\mathcal{O}_B \to \mathcal{O}_A \tag{5.9}$$

expresses $T_{\mathcal{O}_A}\mathcal{O}_B$ as the nontrivial extension

$$T_{\mathcal{O}_A}\mathcal{O}_B \cong \mathcal{O}_{A\cup B}(1,0). \tag{5.10}$$

By $(1,0)$ we mean to twist by the line bundle which is the gluing of the trivial bundle on A and the degree 1 bundle $\mathcal{O}_B(A \cap B)$ on B. (A similar result to (5.10) holds when $\mathcal{O}_A, \mathcal{O}_B$ and $T_{\mathcal{O}_A}\mathcal{O}_B$ are all twisted by the same line bundle.) If we denote this extension by $\mathcal{O}_B \# \mathcal{O}_A$ it is the mirror of the Lagrangian connect sum of (2.27, 2.28). So, for instance, we picture

$$T_i L_{i-1} = L_{i-1} \# L_i = \mathcal{O}_{C_{i-1} \cup C_i}(0, -1)$$

as the path in $\mathbb{C}$, running from λ_{i-1} *over* λ_i to λ_{i+1}, over which its mirror is S^1-fibred. Similarly the connect sum in the opposite direction, which is $T_{i-1}L_i = \mathcal{O}_{C_{i-1} \cup C_i}(-1, 0)$, corresponds to the path *under* λ_i. (See [34] for more on these pictures for objects of $D(S_{2m-1})$.)

Applying this twice we find that

$$T_{2i-1}T_{2i}L_{2i+1} = T_{2i-1}\mathcal{O}_{C_{2i} \cup C_{2i+1}}(-1, 0) = \mathcal{O}_{C_{2i-1} \cup C_{2i} \cup C_{2i+1}}(-1, 0, 0), \qquad (5.11)$$

the second equality following from (5.10) applied to $A = C_{2i-1}$ and $B = C_{2i} \cup C_{2i+1}$. Similarly,

$$T_{2i-1}^{-1}T_{2i}^{-1}L_{2i+1} \cong \mathcal{O}_{C_{2i-1} \cup C_{2i} \cup C_{2i+1}}(0, 0, -1). \qquad (5.12)$$

Finally $T_{2i-1}T_{2i}$ and $T_{2i-1}^{-1}T_{2i}^{-1}$ both take L_{2i-1} to L_{2i}, by similar calculations mirroring Figure 6, and they leave L_{2j+1} alone for $j \neq i, i-1$.

Since (5.11) and (5.12) are *not* isomorphic it follows from (5.5) that $\mathsf{T}_{2i-1}\mathsf{T}_{2i}\,\mathcal{L} \ncong \mathsf{T}_{2i-1}^{-1}\mathsf{T}_{2i}^{-1}\mathcal{L}$, i.e. (2) does not hold.

Repeated calculations with (5.10) on S_{2m-1} show that $T_{2i}T_{2i-1}T_{2i+1}T_{2i}$ also leaves L_{2j+1} alone for $j \neq i, i-1$, but swaps $L_{2i \pm 1}$ (see Figure 6):

$$T_{2i}T_{2i-1}T_{2i+1}T_{2i}L_{2i\pm 1} = L_{2i \mp 1}.$$

Relation (3) then follows again from (5.5).

Finally (4) follows just as in the mirror situation of Section 2.8. The exact triangle (2.26) holds just as well in $D(S_{2m-1})$ – see (5.9) – giving the splitting

$$\begin{aligned} \operatorname{Ext}^*(L_{2m-1}, A) &\cong \operatorname{Ext}^{*+1}(L_{2m}, A) \oplus \operatorname{Ext}^*(T_{2m}L_{2m-1}, A) \\ &\cong \operatorname{Ext}^{*+1}(T_{2m}L_{2m+1}, A) \oplus \operatorname{Ext}^*(T_{2m}L_{2m-1}, A) \end{aligned}$$

that replaces (2.25) for any $A \in D(S_{2m-1})$ generated by L_i, $i \leq 2m-1$. Relation (4) follows easily; see [32] for full details.

5.4. Deformation. As suggested in Section 5.1, to get something which acts as a better mirror of $\operatorname{Hilb}^m S_{\underline{\lambda}}$ in which relation (2) holds, we should deform by something concentrated on the diagonal.

The exceptional divisor E of $H_m := \operatorname{Hilb}^m(S_{2m-1}) \to \operatorname{Sym}^m(S_{2m-1})$ has a class $[E] \in H^1(\Omega_{H_m})$ despite the noncompactness. (For instance the exact sequence $0 \to \Omega_{H_m} \to \Omega_{H_m}(\log D) \to \mathcal{O}_D \to 0$ has extension class in $\operatorname{Ext}^1(\mathcal{O}_D, \Omega_{H_m})$; its image in $\operatorname{Ext}^1(\mathcal{O}_{H_m}, \Omega_{H_m}) = H^1(\Omega_{H_m})$ is $[E]$.) Via the holomorphic symplectic form $\Omega_{H_m} \cong T_{H_m}$, and we get a canonical class $e \in H^1(T_{H_m})$, the space of first order deformations of H_m.

Using some twistor theory we get a canonical family $\mathcal{H} \to \mathbb{P}^1$ of holomorphic symplectic deformations $(\mathcal{H}_t, \sigma_t)$ of $H_m = \mathcal{H}_0$ in the direction of e; see [32]. The

Lagrangian $\mathcal{L}$ (5.4) deforms along this deformation because it is disjoint from $[E]$. We show in [32] that the functors T_β also deform. Both sides of the relations (1), (3) and (4) therefore also deform along $\mathcal{H}$, and by rigidity of the complexes involved the equalities continue to hold.

Finally then we come to (2). As in (5.11, 5.12) we have (cf. (2.27, 2.28)),

$$\begin{aligned} T_{2i-1}T_{2i}L_{2i+1} &\cong L_{2i+1}\#L_{2i}\#L_{2i-1}, \\ T_{2i-1}^{-1}T_{2i}^{-1}L_{2i+1} &\cong L_{2i-1}\#L_{2i}\#L_{2i+1}, \end{aligned}$$

are extensions of the same objects but in opposite directions. On deforming $\mathrm{Hilb}^m S_{2m-1}$ along $\mathcal{H}$, the Lagrangians $T_{2i-1}^{\pm1}T_{2i}^{\pm1}\mathcal{L}$ deform because both $\mathcal{L}$ and the symplectomorphisms T_i do. However the pieces $L_{2i\pm1}\times L_{2i}\times\ldots$ of the extensions do *not* deform (essentially because $[E]$ restricts to a nonzero class on their support since $L_{2i\pm1}$ intersects L_{2i} inside S_{2m-1}). For general reasons, if two extensions of the same pieces deform while the pieces do not then the deformations of the extensions become isomorphic.

The baby model to keep in mind is to deform S itself so that $[C_1]$ and $[C_2]$ do not remain of type $(1,1)$, but their sum $[C_1]+[C_2]$ does. Then neither of $\mathcal{O}_{C_1}(-1)$ or $\mathcal{O}_{C_2}(-1)$ deform, but their extensions in different directions,

$$\mathcal{O}_{C_1\cup C_2}(0,-1) \quad \text{and} \quad \mathcal{O}_{C_1\cup C_2}(-1,0)$$

both deform and become isomorphic to $\mathcal{O}_C(-1)$, where C is the unique (smooth) rational curve that degenerates back to $C_1\cup C_2$ on the central fibre.

5.5. Bigrading and Khovanov cohomology. There are also $\mathbb{C}^*$-actions on the spaces S_i with respect to which the inclusion maps $S_{k-1}\subset S_k$ are equivariant [32]. Since the constructions of this paper are equivariant with respect to this $\mathbb{C}^*$-action, we get extra $\mathbb{C}^*$-action, and so a bigrading, on the link invariant kh^*.

Finally, using the method of [5], one can show that the resulting $kh^{*,*}$ is in fact Khovanov cohomology $KH^{*,*}\otimes\mathbb{C}$ (up to a shift in bigrading). Building up a link from standard cobordisms one presents both $kh^{*,*}$ and $KH^{*,*}\otimes\mathbb{C}$ as iterated cones on the same standard pieces. (For $KH^{*,*}$ this is Khovanov's famous "cube of resolutions".) Because of some vanishing of Ext groups, this iterated cone is unique.

References

[1] R. Bezrukavnikov, I. Mirković, and D. Rumynin, *Singular localization and intertwining functors for reductive Lie algebras in prime characteristic.* Nagoya Math. Jour. **184**, 1–55, 2006. math.RT/0602075.

[2] S. Bigelow, *A homological definition of the Jones polynomial* Geom. & Top. Monogr. **4**, 29–41, 2002. math.GT/0201221.

[3] J. Birman, *On the stable equivalence of plat representations of knots and links*, Canad. Jour. Math. **28**, 264–290, 1976.

[4] T. Bridgeland, A. King and M. Reid, *The McKay correspondence as an equivalence of derived categories*, Jour. A.M.S. **14**, 535–554, 2001. math.AG/9908027.

[5] S. Cautis and J. Kamnitzer, *Knot homology via derived categories of coherent sheaves I, $sl(2)$-case*, Duke Math. Jour. **142**, 511–588, 2008. math.AG/0701194.

[6] S. Cautis and J. Kamnitzer, *Knot homology via derived categories of coherent sheaves II, $sl(m)$ case*, Invent. Math. **174**, 165–232, 2008. arXiv:0710.3216.

[7] K. Fukaya, *Mirror symmetry of Abelian variety and multi theta functions*, Jour. Alg. Geom. **11**, 393–512, 2002.

[8] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction*, AMS/IP Stud. in Adv. Math., **46**, 2009.

[9] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, Jour. A.M.S. **14**, 941–1006, 2001. math.AG/0010246.

[10] N. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **55**, 59–126, 1987.

[11] J. Hurtubise, *Integrable systems and algebraic surfaces*, Duke Math. Jour. **83**, 19-50, 1996.

[12] D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135**, 63–113, 1999.

[13] A. Ishii, K. Ueda and H. Uehara, *Stability conditions on A_n-singularities*, math.AG/0609551.

[14] M. Khovanov, *A categorification of the Jones polynomial.* Duke Math. Jour., **101**, 359-426, 2000. math.QA/9908171.

[15] M. Khovanov and P. Seidel, *Quivers, Floer cohomology, and braid group actions*, J. Amer. Math. Soc. **15**, 203–271, 2002. math.QA/0006056.

[16] M. Khovanov and R. P. Thomas, *Braid cobordisms, triangulated categories, and flag varieties*, Homology, Homotopy and Applications **9**, 19–94, 2007. math.AG/0609335.

[17] M. Kontsevich, *Homological Algebra of Mirror Symmetry*, International Congress of Mathematicians, Zürich 1994. Birkhäuser, 1995. alg-geom/9411018.

[18] C. Manolescu, *Nilpotent slices, Hilbert schemes, and the Jones polynomial*, Duke Math. Jour. **132**, 311–369, 2006. math.SG/0411015.

[19] T. Perutz, *Lagrangian matching invariants for fibred four-maniolds I*, Geom. & Top. **11**, 759–828, 2007. math.SG/0606061.

[20] D. Ploog, *Equivariant autoequivalences for finite group actions*, Adv. in Math. **216**, 62–74, 2007. math.AG/0508625.

[21] P. Seidel, unpublished notes, 1997.

[22] P. Seidel, *Graded Lagrangian submanifolds*, Bull. Soc. Math. France. **128**, 103–146, 2000. math.SG/9903049.

[23] P. Seidel, *A long exact sequence for symplectic Floer cohomology.* Topology **42**, 1003–1063, 2003. math.SG/0105186.

[24] P. Seidel, *Lectures on four-dimensional Dehn twists*, In "Symplectic 4-manifolds and algebraic surfaces", 231–267, LNM **1938**, Springer, 2008. math.SG/0309012.

[25] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zürich Lectures in Advanced Mathematics, EMS, 2008.

[26] P. Seidel, *Suspending Lefschetz fibrations, with an application to local mirror symmetry*, to appear in Comm. Math. Phys., 2010. arXiv:0907.2063.

[27] P. Seidel and I. Smith, *A link invariant from the symplectic geometry of nilpotent slices*, Duke Math. Jour. **134**, 453–514, 2006. math.SG/0405089.

[28] P. Seidel and I. Smith, *Localization for involutions in Floer cohomology*, arXiv:1002.2648.

[29] P. Seidel and I. Smith, *Symplectic geometry of the adjoint quotient, I & II*, Lectures, MSRI, April 2004.

[30] P. Seidel and R. P. Thomas, *Braid group actions on derived categories of sheaves*, Duke Math. Jour. **108**, 37–108, 2001. math.AG/0001043.

[31] I. Smith, *Quadrics, quilts and representation varieties*, in preparation.

[32] I. Smith and R. P. Thomas, *Khovanov homology from ALE spaces*, preprint.

[33] R. P. Thomas, *Moment maps, monodromy and mirror manifolds*, Symplectic geometry and mirror symmetry (Seoul, 2000), 467–498, World Sci. Publishing, 2001.

[34] R. P. Thomas, *Stability conditions and the braid group*, Comm. Anal. Geom. **14**, 135–161, 2006. math.AG/0212214.

Department of Mathematics
Imperial College
London, UK
E-mail: richard.thomas@imperial.ac.uk