Logarithmic geometry and stacks in resolution of singularities and moduli: gluing and degeneration

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LMS lecture series

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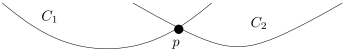
Gluing on curves

- Gluing and degeneration are intertwined.
- Gluing is at the core of WDVV/Quantum Cohomology.
- Start with one nodal curve:



$$C = C_1 \overset{p}{\cup} C_2.$$

implying

$$\begin{aligned} Hom(C, X) &= Hom(C_1, X) \underset{Hom(p,X)}{\times} Hom(C_2, X) \\ &= Hom(C_1, X) \underset{X}{\times} Hom(C_2, X) \end{aligned}$$

Gluing on moduli

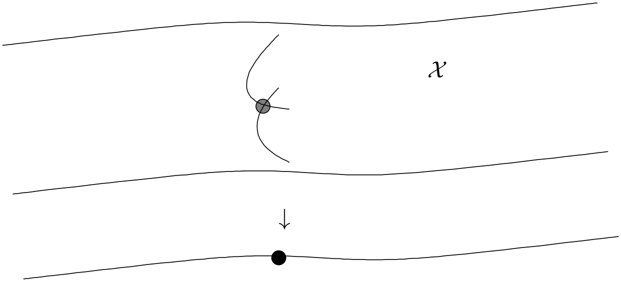
- Spreading out on moduli, consider $\overline{\mathcal{M}}(X/B,\tau)$, and cutting one edge $\overline{\mathcal{M}}(X/B,\tau')$.
- The following diagram is cartesian

$$\begin{array}{ccc} \overline{\mathcal{M}}(X/B,\tau) & \longrightarrow & \overline{\mathcal{M}}(X,\tau') \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

- Evaluation maps are virtually smooth,
- so diagram is compatible with virtual fundamental classes.
- Our goal: logarithmic version.
- Digression: orbifold version.

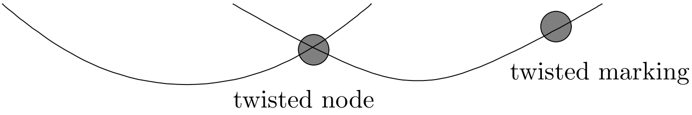
Maps to stacks

- Mirror symmetry requires maps to orbifolds – stacks.
- If the target is a stack, the source is obliged to become a stack:



twisted curves

$$\Sigma_i \subset \mathcal{C} \to C$$



- $\mathcal{C}$ is a Deligne–Mumford stack with C as its coarse moduli space.
- Over a node $xy = 0$ of C, $\mathcal{C}$ has chart

$$[\{uv = 0\}/\boldsymbol{\mu}_r] \qquad (u, v) \mapsto (\zeta u, \zeta^{-1} v).$$

- At a marking, $\mathcal{C}$ has a chart $[\mathbb{A}^1/\mu_s]$, with standard action $u \mapsto \zeta u$
- The substack Σ_i at the i-th marking is locally defined by $u = 0$.

Twisted stable maps

We write $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ for the resulting moduli space.

Theorem (ℵ–Vistoli, Chen–Ruan)

The category $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ is a proper Deligne–Mumford stack with projective coarse moduli space.

Gluing twisted curves

$$\mathcal{C} = \mathcal{C}_1 \overset{\Sigma}{\sqcup} \mathcal{C}_2.$$

Therefore

$$\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{X}) \quad = \quad \underline{\mathrm{Hom}}(\mathcal{C}_1, \mathcal{X}) \underset{\underline{\mathrm{Hom}}(\Sigma, \mathcal{X})}{\times} \underline{\mathrm{Hom}}(\mathcal{C}_2, \mathcal{X}).$$

- but Σ is no longer a point but a gerbe!
- $\underline{\mathrm{Hom}}(\Sigma, \mathcal{X}) = \mathcal{I}\mathcal{X}$ (with a grain of salt).

Gluing of twisted stable maps

Proposition

The evaluation maps $\overline{\mathcal{M}}(\mathcal{X},\tau) \to \mathcal{I}\mathcal{X}^m$ are virtually smooth.
Given an edge of τ with splitting τ' we have a cartesian splitting diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}(\mathcal{X},\tau) & \longrightarrow & \overline{\mathcal{M}}(\mathcal{X},\tau') \\ \downarrow & & \downarrow \\ \mathcal{I}\mathcal{X} & \longrightarrow & \mathcal{I}\mathcal{X} \times \mathcal{I}\mathcal{X} \end{array}$$

of stacks with compatible virtual fundamental classes.

This leads to a wonderful WDVV for orbifolds, leading to Chen–Ruan cohomology, quantum cohomology, Tseng's orbifold upgrade of the Givental formalism, Coates–Corti–Itritani–Tseng, the crepant resolution conjecture, etc.

Lessons learned

In any generalization of GW theory,

1. The structure of curves should reflect the structure of targets, and vice versa.
2. The structure of points should say something about where one evaluates.
3. Gluing should be a fibered diagram, with compatible virtual structure, as above.

Degeneration formula - first case

- $\mathcal{X} \to B$ 1-parameter degeneration, smooth total space, of smooth X to $Y_1 \cup^D Y_2$, with Y_i, D smooth.
- A.M. Li–Ruan, J. Li: $GW(\mathcal{X}/B), GW(Y_i, D)$.
- Theorem: $GW(\mathcal{X}/B) = GW(Y_1, D) * GW(Y_2, D)$
- Original constructions and proofs: expanded degenerations, a bit of logarithmic deformation theory.
- ℵ-Fantechi: expanded degenerations, orbifolds.
- Kim, Chen: expanded degenerations, logarithmic.
- Kim–Lho–Ruddat: purely logarithmic.
- ℵ–Marcus–Wise: all give the same result.

The general case

- $\mathcal{X} \to B$ a SNC degeneration!
- This was the original purpose of logarithmic GW theory.
- In an ideal world, Siebert's dream would have been realized directly, with no caveats.
- Brett Parker in a series of papers has an approach for a full degeneration formula using his theory of exploded manifolds.
- Ranganathan, Maulik–Ranganathan: a full degeneration formula using expanded degenerations controlled by combinatorics.
- There is also an orbifold approach by Fan, Wu and You. Unlike the simplest case, there is a difference between this approach and the logarithmic approach.
- logarithmic geometry (Gross–Siebert, ACGS, Gross) provides a gluing mechanism, which works in many important cases but not all.
- I feel that we are still missing a simplifying component of the theory, and would welcome ideas.

End of segment

Next: Log GW