Logarithmic geometry and stacks in resolution of singularities and moduli: A story of fans

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The complex of a toroidal object

- A variety with divisor (X, D) is toroidal if étale locally it is like a toric $(X(\Sigma), D(\Sigma))$.
- example: X smooth and D a NCD.
- KKMSD: $X \mapsto \{(X(\Sigma), D(\Sigma))\} \mapsto \Sigma$ combines to a polyhedral cone complex.
- not an equivalence, but
- enough to encode tor. modifications by subdivisions, tor. alterations by lattice alterations, etc.
- a drawback: these are skeletons, and skeletons are dead — lack *algebraic* geometry

Log

- Olsson: X (saturated) log scheme is encoded by $\phi_X : X \to \mathrm{Log}$.
- For instance X log smooth if and only if ϕ_X is smooth.
- Is there an incarnation of Σ that is more like Log?

Artin (Olsson) fans

- If X, D are smooth the answer is clearly $\mathcal{A}^1 \subset \text{Log}$.
- With Cadman, Fantechi, Wise we got plenty of mileage just with $\mathcal{A}^1$.
- Kato had "Kato fans" F_X. Gross, Chen had some inspiring ideas.
- which showed that things are tricky.
- With wise we proposed the following for log smooth schemes, and generalized with Chen, Marcus and Wise:

Theorem

Let X be a (reasonably nice) log scheme. The morphism $\phi_X : X \to \text{Log}$ has a universal factorization $X \to \mathcal{A}_X \to \text{Log}$, with $\mathcal{A}_X \to \text{Log}$ étale and representable. In particular $\mathcal{A}_X$ is toric: locally of the form $[V/T]$.

Artin fans take off

- With these we showed equivalence of certain log and relative invariants for a smooth divisor,
- birational invariance of log GW invariants,
- a general boundedness result.
- Dhruv shows that tropical maps are realizable on $\mathcal{A}_X$.
- With Chen, Gross, Siebert we use these in log and punctured GW theory.
- Johnston generalizes birational invariance and boundedness to punctured invariants ...

Breathing life into a skeleton

- On the one hand, an Artin fan is combinatorial: $\Sigma(\mathcal{A}_X) = \Sigma(X)$, with correspondence of strata.
- On the other, it is algebraic. Here is a bridge:

Theorem (Ulirsch)

The Berkovich analytification $\mathcal{A}_X^\beth$ is homeomorphic, with corresponding strata, to $\Sigma(X)$.

End of segment

Next: Log Geometry I