Logarithmic geometry and stacks in resolution of singularities and moduli: stacks part 2

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Algebraic stacks

A. Categorical setup: Show $\mathcal{X}$ is a category fibered in groupoids.
B. Topological requirement: Show that $\mathcal{X}$ is a stack.
C. Geometric nature: Show $\mathcal{X}$ is an algebraic stack.

- Think A, B (and representability of Isom) come with good housekeeping, C might be tricky.
- Need a bag of tricks.

Approaches

- easy approach: show $\mathcal{X} \simeq [V/G]$ when it works
- hard approach: use Artin's axioms to construct V using deformation theory

$\overline{\mathcal{M}}_g$ is algebraic

- Recall that $\overline{\mathcal{M}}_g$ is the stack of stable curves of genus g: projective curves with at most nodes with ample dualizing sheaf.
- Theorem: $\overline{\mathcal{M}}_g$ is algebraic.
- Here A, B come from good housekeeping.
- It is algebraic because $\overline{\mathcal{M}}_g = [H_0/PGL_{N+1}]$.
- Why?

$\overline{\mathcal{M}}_g = [H_0/PGL_{N+1}]$

- A 3-canonical map $\phi_{|3K|} : C \to \mathbb{P}^N$ of a stable curve is an embedding.
- So $(C, \phi) \in H$ is a point in the Hilbert scheme.
- One carefully shows (tedious) that there is a locally closed subscheme $H \supset H_0$ of 3-canonically embedded stable curves.
- One carefully shows (lovely) that $\overline{\mathcal{M}}_g = [H_0/PGL_{N+1}]$.

$\overline{\mathcal{M}}_{g,n}$ is algebraic

- That's the stack of stable n-pointed curves.
- Theorem: $\overline{\mathcal{M}}_{g,n}$ is algebraic.
- The previous approach works, but is a bit tedious.
- Common trick: pick your favorite , nonisomorphic, 1-pointed curves (D_i, q_i) of huge genus G_i.
- Given an arbitrary stable n-pointed curve $(C, p_1, \ldots, p_n)$ of genus g,
- if you clutch p_i with q_i you get a stable curve $C' = C \cup_{p_1 = q_i} \bigcup D_i$ of genus $g' = g + \sum G_i$.
- One checks that this gives a closed embedding $\overline{\mathcal{M}}_{g,n} \subset M_{g'}$,
- giving the claim!

$\mathfrak{M}_{g,n}$ is algebraic

- That's the stack of prestable *n*-pointed curves, not DM, not even of finite type!
- Knee-jerk reaction: must use Artin's criteria.
- Olsson's trick: adding 3 distinct points on each component makes a curve stable.
- caution: can do this on the open set where C has at most K components.
- get an open $M^{[K]} \subset \overline{\mathcal{M}}_{g,n+3K}$ and smooth $M^{[K]} \to \mathfrak{M}_{g,n}$.
- A smooth covering of $M^{[K]}$ gives a smooth covering of $\mathfrak{M}_{g,n}$.

$\overline{\mathcal{M}}_{g,n}(X,d)$ is algebraic

- That's the stack of stable maps. It maps to $\mathfrak{M}_{g,n}$.
- The fiber $Z_S = S \times_{\mathfrak{M}_{g,n}} \overline{\mathcal{M}}_{g,n}(X,d)$ is open in the Hom scheme $Hom_{S,d}(C,X)$.
- That exists because of Hilbert schemes.
- If S is a smooth covering of $\mathfrak{M}_{g,n}$ then Z_S is a smooth covering of $\overline{\mathcal{M}}_{g,n}(X,d)$.

Hard cases

- $\mathcal{K}_{g,n}(\mathcal{M}, \beta)$. In A-V we used Artin. Olsson then devised a way out using log curves!
- Punctured curves: we build a covering by hand.

Coarse moduli spaces

We started with $\mathcal{M}_g \to M_g$, following Mumford coarse moduli space of a functor.

Theorem (Keel-Mori)

Let $\mathcal{X}$ be a separated DM stack. There is an algebraic space X and a morphism $\mathcal{X} \to X$

- *Universal for morphisms to algebraic spaces,*
- $\mathcal{X}(\bar{k}) \to X(\bar{k})$ *is bijective on isomorphsim classes.*

Example

if $\mathcal{X} = [V/G]$ then $X = V/G$.

Root construction: global section case

- Say $D \subset X$ divisor defined by a global section f of $\mathcal{O}_X(D)$.
- You can view it as a morphism $f : X \to \mathbb{A}^1$.
- The scheme $X(\sqrt[r]{f}) := Spec_X \mathcal{O}_X[t]/(t^r - f)$ depends on the choice of f.
- It has μ_r action $t \mapsto \zeta \cdot t$.
- The quotient stack $X(\sqrt[r]{D}) := [X(\sqrt[r]{f})/\mu_r]$ is independent of choices.
- How to globalize?

Root construction and the first Artin fan

- The stack $\mathcal{A}^1 := [\mathbb{A}^1/\mathbb{G}_m]$ is the moduli stack "line bundle with section" (Olsson).
- (X, D) corresponds to $X \to \mathcal{A}^1$ via 1_D.
- Cadman–Vistoli: $X(\sqrt[r]{D})$ is the cartesian product

$$\begin{array}{ccc} X(\sqrt[r]{D}) & \longrightarrow & \mathcal{A}^1 \\ \downarrow & & \downarrow \otimes r \\ X & \longrightarrow & \mathcal{A}^1. \end{array}$$

- $T \to X(\sqrt[r]{D})$ corresponds to
  - $f : T \to X$
  - $\mathcal{L} \in Pic(T), s \in \Gamma(\mathcal{L})$
  - $\phi : \mathcal{L}^r \to f^*O(D)$

  such that $\phi(s^r) = f^*1_D$.

$\mathcal{P}roj$

- Let S be a scheme and $R = \mathcal{O}_S \oplus R_1 \oplus \cdots$ a graded algebra.
- We have an action of $\mathbb{G}_m$ on $\mathrm{Spec}_S(R) \to S$.
- Let $Z = V(R_+)$.

Definition

$\mathcal{P}roj_S R := [\ \ \mathrm{Spec}_S(R) \smallsetminus Z \ \ / \ \ \mathbb{G}_m].$

- If R is generated by R_1 this is the usual *Proj*.

Weighted projective stacks

- $S = \operatorname{Spec} k, R = k[x_1, \ldots, x_n]$.
- Give x_i weight w_i.
- when $w_i = 1$ then $\mathcal{P}roj(R) = Proj(R) = \mathbb{P}^{n-1}$.
- In general $\mathcal{P}(w_1, \ldots, w_n) := \mathcal{P}roj(R)$ is a weighted projective $n-1$ stack.
- Remark: Globalizing this has some interesting twists!

Weighted blowups — Cox construction

- $Y = \operatorname{Spec} k[x_1, \ldots, x_n], X = V(x_1, \ldots, x_k),$

$$R = O_Y[s, x_1', \ldots, x_k']/(x_i - s_i^{w_i} x_i').$$

- $B := Spec_Y(R)$
- $Z := V(x_1', \ldots, x_k')$
- $B_+ = B \smallsetminus Z.$
- s has $\mathbb{G}_m$-weight -1, and x_i' has weight w_i.
- The weighted blowup is $[B_+/\mathbb{G}_m]$.
- Big exercise: with weights 1 this is the usual blowup.

End of stacks II

Next: The unreasonable effectiveness of toric varieties