# Logarithmic geometry and stacks in resolution of singularities and moduli: stacks part 1

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# Stacks as moduli

Given a moduli question $X$, there are three stages to encoding it in an algebraic stack:

A. Categorical setup: Show $X$ is a category fibered in groupoids.
B. Topological requirement: Show that $X$ is a stack.
C. Geometric nature: Show $X$ is an algebraic stack.

- While A and B are abstract, they usually come from setting up things well.
- This is the case for the key examples we consider:
  - $\mathcal{M}_g$, and
  - $[V/G]$ where $V$ is a variety and $G \subset \operatorname{Aut} V$ an algebraic group.
- Condition C is potentially the more tricky one.
- As we'll see, it is automatic when $X = [V/G]$.
- Two subcases of interest: $V =$point, or $G = \{1\}$.

# $M_g$ and $\mathcal{M}_g$

- "a curve of genus $g$": smooth projective geometrically integral over a field.
- $M_g = \{$ curves of genus $g\}/$isomorphisms.
- A family $X \to S$: projective and flat morphism, $X_s$ a curve of genus $g$.

What is this?

Deligne–Mumford: a category.

# The category $\mathcal{M}_g$

- a morphism between two families is a cartesian diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array},$$

- so $X_1 \to S_1 \times_{S_2} X_2$ is an isomorphism.
- Also note that by taking $S_1 \times_{S_2} X_2$, there is always an $X_1$ filling the diagram!
- Note: a structure functor: $\mathcal{M}_g \to \mathfrak{Sch}$ sending $X \to S$ to $S$.

# Categories fibered in groupoids

The fiber $\mathcal{M}_g(S)$ of $\mathcal{M}_g \to \mathfrak{Sch}$ over $S$ (really $id_S$) is $\{X \to S\}$ with diagrams

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S & = & S \end{array},$$

**Definition**

$F : \mathcal{C} \to \mathfrak{Sch}$ is a category fibered in groupoids if pullbacks exist, and are unique up to unique isomorphisms.

So the fiber over $id_S$ is indeed a groupoid.

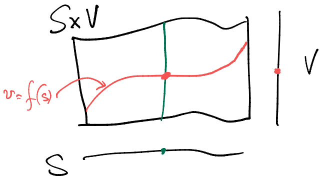
# basic examples

$M_g$: Example: $M_g$ = the moduli of curves of genus $g$

$V$: Example: Let $V$ be a variety. Let $Sch_V \to \mathfrak{Sch}$ be the functor sending $S \to V$ to the source $S$.

- An exercise asks you to verify this is a CFG.
- Wait - moduli of what?
- A morphism $S \to V$ is encoded by its graph
- The graph of a morphism is a family of points!!

$V$: Example: Let $V$ be a variety. Let $Sch_V \to \mathfrak{Sch}$ be the moduli of points in $V$.



# The category of CFGs

- We define an arrow between $\mathcal{C}_i \to \mathfrak{Sch}$ as a strictly commuting

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{f} & \mathcal{C}_2 \\ {\scriptstyle \mathcal{F}_1}\downarrow & & \downarrow{\scriptstyle \mathcal{F}_2} \\ \mathfrak{Sch} & = & \mathfrak{Sch}, \end{array}$$

  so $\mathcal{F}_1 = \mathcal{F}_2 \circ f$.
- One checks that an arrow $\mathfrak{Sch}_{V_1} \to \mathfrak{Sch}_{V_2}$ is the same as $f : V_1 \to V_2$,
- and also the same as an object $\mathfrak{Sch}_{V_2}(V_1)$.

# Quotients

- Say a (flat) group (scheme) $G$ acts on $V$.
- Want: $[V/G]$
- First case: free action with geometric quotient $Y$
- Then $X \to Y$ is a principal $G$-bundle.
- and for $S \to Y$ we get a pullback

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & Y, \end{array}$$

- with $P \to X$ equivariant, and $P \to S$ a principal bundle.
- This of course can be pulled back if you have $S_1 \to S_2$.

# The category $[V/S]$

- We define $[V/G]$ to have objects

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array},$$

- with $P \to X$ equivariant, and $P \to S$ a principal bundle,
- and arrows cartesian diagrams:

$$\begin{array}{ccccc} P_1 & \longrightarrow & P_2 & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2, & & \end{array}$$

- and note that when $Y$ exists $[V/G] \simeq \mathfrak{Sch}_Y$.
- !!!
- Exercise: a Category Fibered in Groupoids.

# Fibered products

- Suppose given a diagram of categories fibered in groupoids:

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{\pi_X} & S. \end{array}$$

- Define $X \times_S Y$ where objects over a scheme $T$ are $(\xi, \eta, \phi)$: $\xi \in X(T), \eta \in Y(T)$, and isomorphism $\phi : \pi_X(\xi) \to \pi_Y(\eta)$.
- One shows this gives rise naturally to a CFG.
- An exercise for you is to show that

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & [X/G] \end{array}$$

is cartesian.

# From CFG to stack

- In algebraic geometry schemes are glued together from affine schemes.
- In particular a family of curves $X \to S$ is glued together from families $X_i \to S_i$, where $S_i$ is a covering of $S$.
- This is true in the Zariski topology (sort of by definition), but also in the étale, smooth or various flat topologies (Grothendieck).
- What you need is decsent data: isomorphism $\phi_{ij} : (X_i)|_{S_{ij}} \to (X_j)|_{S_{ij}}$,
- which must satisfy the cocycle condition on triple intersections.
- We say that every descent datum is effective.

# From CFG to stack - continued

- There is also an easier condition for constructing isomorphisms from $X_1 \to S$ to $X_2 \to S$.
- You need isomorhisms

$$f_i : (X_1)|_{S_i} \to (X_2)|_{S_i},$$

- which exactly agree on the intersections:

$$(f_i)|_{S_{ij}} = (f_j)|_{S_{ij}}.$$

- In other words, $Isom_S(X_1, X_2)$ forms a sheaf.

# Stacks

## Definition

A category fibered in groupoids $\mathcal{C} \to \mathfrak{Sch}$ is a stack if isoms form a sheaf, and every descent datum is effective.

This is abstract, but many reasonable moduli problems satisfy it.
For instance, $M_g$ or $[V/G]$.
Often it is automatic from scheme theory.

# Representability

- We say that $\mathcal{C} \to \mathfrak{Sch}$ is represented by $V$ if it is isomorphic to $\mathfrak{Sch}_V$.
- We say $\mathcal{C}_1 \to \mathcal{C}_2$ is representable if for every map from scheme $S \to C_2$ we have that $S \times_{\mathcal{C}_2} \mathcal{C}_1$ is representable.
- Example/exercise: for every scheme $T$ any morphism $T \to \mathcal{M}_g$ is representable.
- Indeed think of it as a family $X \to T$, and for $S$ a family $X' \to T$,
- and $S \times_{\mathcal{M}_g} T = Isom_{S \times T}(X, X')$, representable by a Hilbert scheme.

# Smoothness for representable morphisms

- A representable $\mathcal{C}_1 \to \mathcal{C}_2$ is smooth (or respectively étale, or covering)
- if for every $S \to \mathcal{C}_2$ the morphism of schemes $S \times_{\mathcal{C}_2} \mathcal{C}_2 \to S$ is smooth (or respectively étale, or covering).

**Definition**

A stack $\mathcal{C} \to \mathfrak{Sch}$ is algebraic if *Isom* functors are representable, and there is a scheme $V$ and a smooth covering $V \to \mathcal{C}$. It is a Deligne–Mumford stack if the covering can be taken étale.

- This is abstract, but many reasonable moduli problems satisfy it.
- For instance, $M_g$ or $[V/G]$.
- Existence of $V \to \mathcal{C}$ is "only" automatic for $[V/G]$, with $G$ smooth:
- In this case $V \to [V/G]$ is smooth, since $P = S \times_{[V/G]} V \to S$ is a principal $G$-bundle.

# End of stacks I

Next: Geometry and Combinatorics I