Logarithmic geometry and stacks in resolution of singularities and moduli: introduction to stacks

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The inevitable rise of stacks

The story truly starts with Deligne–Mumford (1969):

- Algebraic geometry is about varieties
- The best varieties are smooth and projective.
- For instance: if $X \to S$ is smooth and projective, S connected,
- and assume X_η is geometrically irreducible
- then X_s is geometrically irreducible, $\forall s \in S$.

The case of M_g

- Let M_g be the moduli space of smooth curves of genus $g > 1$, see later lectures.
- Know from topology: $(M_g)_{\mathbb{C}}$ irreducible, so $(M_g)_{\mathbb{Q}}$ geometrically irreducible.
- That's the generic fiber of $M_g \to \operatorname{Spec} \mathbb{Z}$.
- Want: $(M_g)_{\mathbb{F}_p}$ irreducible.

To apply the principle above, there are issues:

- M_g is not proper
- M_g is not smooth: locally at $[C]$ it is smooth / $\operatorname{Aut} C$.

The case of M_g: issues resolved

issues:

- M_g is not proper
- M_g is not smooth: locally at $[C]$ it is smooth / Aut C.

Resolution:

- $M_g \subset \overline{M}_g$ a compactification.
- A stack $\overline{\mathcal{M}}_g \to \overline{M}_g$ *smooth and proper* over $\operatorname{Spec}\mathbb{Z}$

Theorem (Deligne–Mumford 1969)

$(M_g)_{\mathbb{F}_p}$ *irreducible.*

Proof.

- $\overline{\mathcal{M}}_g \to \operatorname{Spec}\mathbb{Z}$ smooth and proper.
- $(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ geometrically irreducibe.
- $\Rightarrow (\overline{\mathcal{M}}_g)_{\mathbb{F}_p}$ irreducible $\Rightarrow (\overline{M}_g)_{\mathbb{F}_p}$ irreducible $\Rightarrow (M_g)_{\mathbb{F}_p}$ irreducible

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How I got infected

- In 1990 Harris gave his Moduli of Curves course for the first time.
- He introduced representable functors, coarse moduli spaces of non-representable ones, going deeper into tangent spaces and such.
- Ian Morrison was in the audience — hence the book.
- Angelo Vistoli was in the audience, and was — I think deliberately — provoked to ask "but how do you account for automorphisms?",
- and was recruited to give lectures which were immediately inspiring and eventually transformative for my career.

...and recruited

- I do not have his notes. I was called upon to lecture on occasion, and relied on Barbara Fantechi's notes, resulting in what follows.
- An important point: as geometers we are inclined to think about "variety / group action".
- Deligne and Mumford, in a way following Giraud, take a categorical approach, where a stack truly encodes a moduli question.

End of introduction to stacks

Next: stacks I