# Notes on General Relativity

Ryan Barnett

**Abstract**

Notes for: Tensor Calculus and General Relativity (MA7), Spring 2017

# 1 Special Relativity

We start by stating the **Postulates of Special Relativity**:

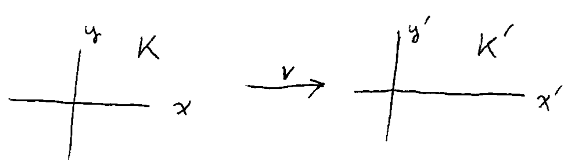
1. The speed of light in vacuum $c \approx 3.0 \times 10^8$ m/s is the same in all inertial reference frames.
2. The laws of nature are the same in all inertial reference frames.

An **inertial reference frame** is a frame of reference in which Newton's first law holds (i.e. it is not accelerating). While Postulate 2 is consistent with Newtonian physics, Postulate 1 is not. Postulate 1 has been confirmed by numerous experimental tests (e.g. the Michelson-Morley experiment).

## 1.1 The Lorentz Transformation

Let's start by considering the relationship between spatial and temporal coordinates in different reference frames. We take two reference frames $K$ and $K'$ as shown in the figure. For convenience we take the origins to coincide at time $t = t' = 0$.

$$K \text{ coordinates}: \ \mathbf{r} = (x, y, z) \text{ and } t$$
$$K' \text{ coordinates}: \ \mathbf{r}' = (x', y', z') \text{ and } t'$$



In Newtonian physics, the coordinates transform according to a **Galilean transformation**:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \text{ (absolute time).} \end{aligned}$$

These transformation rules are at odds with Postulate 1. For instance, consider a particle moving at the speed of light in $K'$: $x' = ct'$. Then its speed in $K$ will be $c + v$.

This motivates us to seek an alternative transformation rule that is consistent with the Postulates of special relativity. Let's assume the transformation to be linear and write

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{1}$$

$$\begin{aligned} y' &= y \\ z' &= z. \end{aligned}$$

We will attempt to deduce $A, B, C, D$ by insisting that Postulate 1 is satisfied and considering four types of motion. Note that since $x$ and $ct$ have the same dimensions (length) the elements in the above matrix will be dimensionless.

**Case 1**: We first consider a particle moving at the speed of light along the $x$-axis in the $K$ frame: $x = ct, y = 0, z = 0$. Then by the first Postulate, in the $K'$ frame we must have $x' = ct', y' = 0, z' = 0$. Inserting this into Eq. 1 gives

$$A + B = C + D.$$

**Case 2**: We next consider a particle sitting at the origin of $K'$: $x' = y' = z' = 0$. Then in frame $K$ we will have $x = vt, y = 0, z = 0$. Insertion into Eq. 1 gives

$$Cc = -Dv$$

**Case 3**: We now consider a particle sitting at the origin of $K$. Going through the same procedure for this case gives

$$Av = -Cc$$

**Case 4**: Finally, we consider a particle moving along the $y$-axis in $K$ at the speed of light: $x = 0, y = ct, z = 0$. Postulate 1 requires $\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 = c^2$, $z' = 0$. With Eq. 1 we find

$$A^2 = 1 + C^2.$$

The equations for these four Cases can be solved to determine $A, B, C, D$. The solution gives

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{2}$$

where

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}.$$

We have arrived at a **Lorentz transformation**. We will refer to $\gamma$ as the **relativistic factor**. A rotation-free Lorentz transformation (as in the above) is a **Lorentz boost**.

One can verify that the inverse transformation (from $K'$ to $K$) can be obtained by replacing $v \to -v$. That is,

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \frac{v}{c}\gamma \\ \frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{3}$$

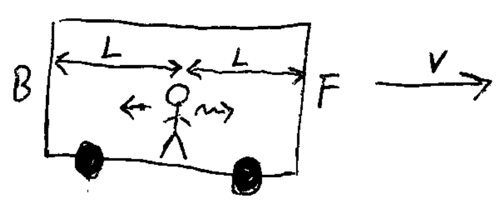
with $y = y'$, $z = z'$.

Special relativity forces us to abandon the notion of absolute time, and to consider four-dimensional **spacetime**. A point $(ct, x, y, z)$ in spacetime is called an **event**.

## 1.2 Some Consequences of the Lorentz Transformation

### 1.2.1 Simultaneity

In Newtonian physics, events simultaneous in one frame are simultaneous in another. We will now illustrate how things are different in special relativity.




Consider a passenger standing in the middle of a train car which is moving at velocity $v$ as shown in the picture. The person emits a pulse of light at time $t' = 0$. We will consider the time it takes for the pulse to reach the front and back of the train in different reference frames. We take $K'$ to be the rest frame of the train. Then

$$x'_F = L, \;\; x'_B = -L, \;\; t'_F = t'_B = L/c$$
$$\Delta t' = t'_F - t'_B = 0 \;\; \text{(Simultaneous in } K'\text{)}$$

In this subscripts $F, B$ denote the front and back of the train. For instance, $t'_F$ is the time in $K'$ at which the light reaches the front of the train. On the other hand, in $K$ (the platform frame) through the Lorentz transformation we have

$$t_F = \gamma(t'_F + x'_F v/c^2) = \frac{\gamma L}{c}\left(1 + \frac{v}{c}\right), \quad t_B = \gamma(t'_B + x'_B v/c^2) = \frac{\gamma L}{c}\left(1 - \frac{v}{c}\right)$$
$$\Delta t = t_F - t_B = 2\gamma \frac{Lv}{c^2} \quad \text{(Not simultaneous in } K'\text{)}$$

### 1.2.2 Time Dilation

Consider a clock sitting at the origin in $K'$. The time intervals in $K$ and $K'$ are related by the Lorentz transformation:

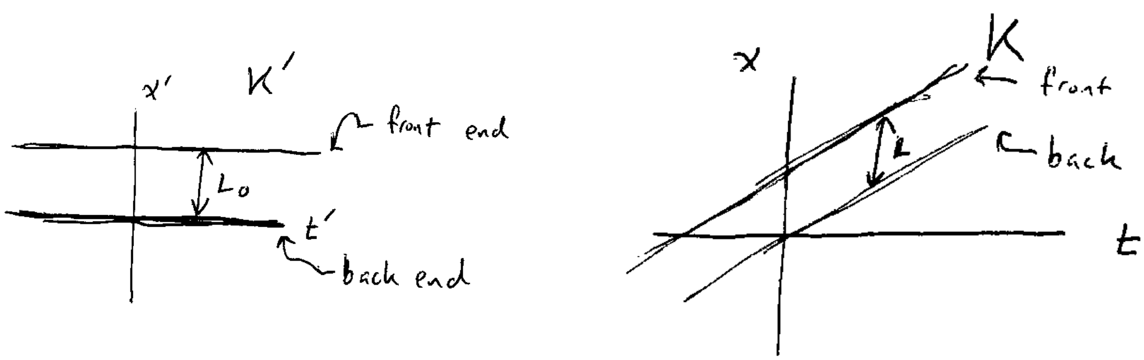
$$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x') = \gamma \Delta t'$$

($\Delta x' = 0$ since the clock is motionless in $K'$). The time recorded by a clock in its rest frame is referred to as the **proper time** $\tau$ and so

$$\Delta\tau = \Delta t/\gamma$$

Since $\gamma > 1$ (for $v \neq 0$) the time interval in $K$ is longer: “Moving clocks run slowly”. This phenomenon is known as **time dilation**.

### 1.2.3 Length Contraction




Consider a rod of length $L_0$ at rest in $K'$ and oriented along the $x'$-axis:

$$x'_F = L_0, \quad x'_B = 0, \quad \Delta x' = x'_F - x'_B = L_0.$$

Now transform to $K$:

$$x_F = \frac{1}{\gamma}L_0 + vt, \quad x_B = vt, \quad \Delta x = x_F - x_B = L_0/\gamma.$$

We see that $\Delta x = \Delta x'/\gamma < \Delta x'$. This is known as **length contraction**.

## 1.3 Relativistic Addition of Velocities

Consider a particle moving at velocity $w$ in $K'$: $dx'/dt' = w$. What is the particle's velocity $u = dx/dt$ as measured in $K$? In Newtonian physics, the result is

$$u = v + w.$$

On the other hand, by taking the differential of the Lorentz transformation, we have

$$\begin{aligned} dx &= \gamma(dx' + vdt') = \gamma dt'(w + v) \\ dt &= \gamma(dt' + vdx'/c^2) = \gamma dt'(1 + vw/c^2). \end{aligned}$$

Dividing these equations gives

$$u = \frac{dx}{dt} = \frac{v + w}{1 + \frac{vw}{c^2}}. \tag{4}$$

This gives the correct way to add velocities in special relativity. In comparing this with the Newtonian result, we see that the denominator serves to enforce the speed limit of light. Note that this formula will be modified if the particle is not moving along the $x'$-axis. As our derivation did not rely on $w$ being constant, Eq. 4 is true even if the particle in $K'$ is accelerating.

Eq. 4 can also be arrived at by considering a combination of Lorentz boosts. Take reference frame $K$ and $K'$ to be as before. Now consider a third frame $K''$ in which the particle discussed previously is at rest. That is, the origin of $K''$ is moving away from the origin of $K'$ at velocity $w$ along the $x'$-axis. Now define the **rapidity** $\psi_v$ through

$$\frac{v}{c} = \tanh(\psi_v)$$

with analogous relations for $\psi_u$ and $\psi_v$.

Then the matrix which transforms or "boosts" $(ct, x)$ to $(ct', x')$ is (compare to Eq. 2)

$$\lambda(v) \equiv \begin{pmatrix} \gamma & -\gamma\frac{v}{c} \\ -\gamma\frac{v}{c} & \gamma \end{pmatrix} = \begin{pmatrix} \cosh(\psi_v) & -\sinh(\psi_v) \\ -\sinh(\psi_v) & \cosh(\psi_v) \end{pmatrix} = e^{-\sigma\psi_v} \tag{5}$$

where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (check). In this we take the exponential of a square matrix $A$ to be defined through $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Writing the Lorentz boost in such a way enables us to find a simple way to evaluate combinations of boosts. The boost from $K$ to $K''$ is given by

$$\lambda(u) = e^{-\sigma\psi_u} = \lambda(w)\lambda(v) = e^{-\sigma\psi_w}e^{-\sigma\psi_v} = e^{-\sigma(\psi_w+\psi_v)}.$$

Thus the rapidities combine in a simple way:

$$\psi_u = \psi_v + \psi_w. \tag{6}$$

Eq. 4 can be deduced from Eq. 6 (check).

## 1.4 Relativistic Acceleration

Now we move on to find how to transform a particle's acceleration between inertial reference frames. We start by taking the differential of Eq. 4 (recall that $u = dx/dt$ and $w = dx'/dt'$):

$$du = \frac{1}{\gamma^2}\frac{1}{(1+\frac{vw}{c^2})^2}dw.$$

Next, take the differential of an equation from our Lorentz transformation:

$$dt = \gamma(dt' + \frac{v}{c^2}dx') = \gamma(1+\frac{vw}{c^2})dt'.$$

Dividing these equations gives

$$\frac{du}{dt} = \frac{d^2x}{dt^2} = \frac{1}{\gamma^3}\frac{1}{\left(1+\frac{vw}{c^2}\right)^3}\frac{d^2x'}{dt'^2}$$

which relates $\frac{d^2x}{dt^2}$ and $\frac{d^2x'}{dt'^2}$.

**Example**: The relativistic rocket. We consider a passenger aboard a rocket who always "feels" a constant acceleration $a = \frac{d^2x'}{dt'^2}$. What is the rocket's velocity $u(t)$ measured in the $K$ frame?

We take the inertial frame $K'$ to be moving away from the origin of $K$ with the same speed as the rocket at a particular time (a momentarily comoving frame – more on this later). Then at this particular time, $u = v$ and $w = 0$. This gives a differential equation for $u$:

$$\frac{du}{dt} = \left(1-\left(\frac{u}{c}\right)^2\right)^{3/2} a.$$

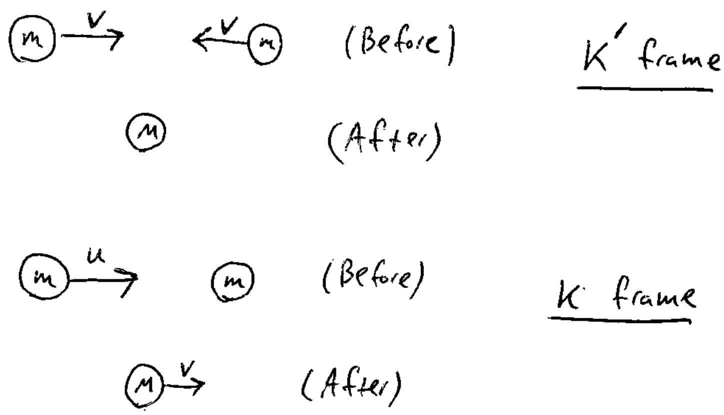
For simplicity, let's assume the rocket starts from rest: $u(0) = 0$. Solving this differential equation gives

$$u(t) = \frac{at}{\sqrt{1+\left(\frac{at}{c}\right)^2}}.$$

This expression gives very sensible results in limiting cases. For short times, $u \approx at$ as in Newtonian physics, while for long times, $u \approx c$.

## 1.5 Relativistic Energy and Momentum

We now move on to generalise the familiar expressions of energy and momentum from Newtonian physics to special relativity. We will motivate this by considering a collision in different inertial reference frames. We use the frames $K$ and $K'$ as described in Sec. 1.1. In $K'$, two particles, each of mass $m$, moving with opposite velocities collide to form a particle of mass $M$. In $K$, one of these particles is motionless (see figure).




Let's first look at this collision using Newtonian mechanics. Here, we will have $M = 2m$ and $u = 2v$. In both frames mass and momentum are conserved:

$$\begin{aligned}
\text{frame } K'\text{:} \qquad & 2m = M \qquad && \text{(Mass)} \\
& mv - mv = M \cdot 0 && \text{(Momentum)} \\
\text{frame } K\text{:} \qquad & 2m = M && \text{(Mass)} \\
& mu = Mv && \text{(Momentum)}
\end{aligned}$$

However, in special relativity, through the velocity addition result, we have

$$u = \frac{w + v}{1 + \frac{wv}{c^2}} = \frac{2v}{1 + (v/c)^2}. \tag{7}$$

With this, the Newtonian momentum conservation equation in $K$ does not hold: $mu \neq Mv$.

This motivates us to consider the case where mass depends on velocity by the replacement $m \to f(v)m$ where $f(v)$ is some function to be determined. With this replacement and taking $f(0) = 1$ we have

$$\begin{aligned}
\text{frame } K'\text{:} \qquad & 2mf(v) = M \qquad && \text{(Mass)} \\
& f(v)mv - f(v)mv = M \cdot 0 && \text{(Momentum)} \\
\text{frame } K\text{:} \qquad & mf(u) + m = Mf(v) && \text{(Mass)} \\
& mf(u)u = Mf(v)v && \text{(Momentum)}
\end{aligned}$$

Solving the equations in $K$ for $f(u)$ gives $f(u) = v/(u - v)$. Combining this with Eq. 7 gives $f(u) = 1/\sqrt{1 - (u/c)^2}$. So we see that $f(u)$ is nothing other than the relativistic factor: $f(u) = \gamma(u)$.

We should check that all these equations are satisfied with this choice for $f$. This is easiest to do by utilising the rapidities introduced earlier. Here we will have $\psi_u = \psi_v + \psi_v = 2\psi_v$. The mass equation in frame $K'$ gives $M = 2m\cosh(\psi_u/2)$.[1] Inserting this value into the

[1]Note that the relativistic factor of a particle moving at velocity $v$ is $\gamma(v) = \cosh(\psi_v)$

equations in $K$ we find the following identities:

$$\begin{aligned} \cosh(\psi_u) + 1 &= 2\cosh^2(\psi_u/2) \\ \sinh(\psi_u) &= 2\sinh(\psi_u/2)\cosh(\psi_u/2) \end{aligned}$$

so everything checks.

We say a particle with **rest mass** $m$ moving at velocity $v$ has a **relativistic mass** $\gamma(v)m$. We further define the relativistic momentum to be $p = \gamma(v)mv$. Extending this to several components we have

$$\mathbf{p} = \gamma(v)m\mathbf{v} \tag{8}$$

where $\mathbf{v} = d\mathbf{r}/dt$. With these revised expressions, we have that momentum and mass are conserved in the above collision. We further interpret the conservation of mass as conservation of energy. Multiplying the relativistic mass by $c^2$ (to obtain correct units) we define the relativistic energy to be

$$E = \gamma(v)mc^2. \tag{9}$$

This interpretation becomes quite plausible for $v/c \ll 1$. In this regime we have

$$E \approx mc^2 + \frac{1}{2}mv^2.$$

The second term in the above is the Newtonian kinetic energy.

## 1.6 A More Systematic Notation

We will now introduce some terminology and a more systematic notation which will be used in general relativity. We label spacetime coordinates as follows:

$$x^0 = ct \quad x^1 = x \;\; x^2 = y \;\; x^3 = z$$

(the superscripts here are not to be confused with exponents). We thus label an event by[2]

$$x^\mu = (ct, x, y, z) = (ct, \mathbf{r}).$$

It also proves convenient to use the **Einstein summation convention**. With this convention, summation is implied in an expression when an index appears twice. For instance,

$$X^\mu Y_\mu \equiv \sum_{\mu=0}^{3} X^\mu Y_\mu = X^0 Y_0 + \ldots + X^3 Y_3.$$

We will later discuss the relevance of superscript and subscript indices. Suppression of the summation symbol often proves to be economical, but we must be careful not to write ambiguous expressions.[3]

[2]This equation may appear a little sloppy. Strictly speaking, $x^\mu$ are the components of the spacetime point. Such expressions are common in GR.

[3]For instance, we might be tempted to write $(X^\mu Y_\mu)^2 = X^\mu Y_\mu X^\mu Y_\mu$ but this in not correct. Instead, we need to introduce another index: $(X^\mu Y_\mu)^2 = X^\mu Y_\mu X^\nu Y_\nu$.

We next introduce the Minkowski metric:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

This gives us a way of taking "dot products" of spacetime vectors. The spacetime line element $ds$ is defined through

$$(ds)^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{11}$$

which is analogous to $d\mathbf{r} \cdot d\mathbf{r}$ in Euclidean space. Let's consider how $(ds)^2$ changes under our Lorentz boost Eq. 2:

$$\begin{aligned} (cdt)^2 - (dx)^2 &= \gamma^2(cdt' + vdx'/c)^2 - \gamma^2(dx' + vdt')^2 \\ &= \gamma^2(c^2 - v^2)(dt')^2 - \gamma^2(1 - (v/c)^2)(dx')^2 \\ &= c^2(dt')^2 - (dx')^2. \end{aligned}$$

Since $y = y'$ and $z = z'$, we therefore have

$$(ds)^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'}. \tag{12}$$

The line element is **invariant** under our Lorentz transformation. In fact $(ds)^2$ is invariant under all Lorentz transformations (think about, say, boosts in the $y$ and $z$ directions). In Eq. 12 we have placed primes on *indices* to denote that these quantities are in the $K'$ reference frame. This is known as the kernel-index convention.

Now let's introduce a more compact notation for Lorentz transformations:

$$dx^{\mu'} = \Lambda^{\mu'}{}_{\nu} dx^\nu.$$

For instance, for the Lorentz transformation we have been regularly using, $\Lambda^{\mu'}{}_{\nu}$ is the matrix appearing in Eq. 2. Inserting this into the spacetime line element and demanding that this quantity is invariant gives

$$\eta_{\mu\nu} = \Lambda^{\sigma'}{}_{\mu} \eta_{\sigma'\rho'} \Lambda^{\rho'}{}_{\nu} \tag{13}$$

or in matrix form

$$\eta = \Lambda^T \eta \Lambda. \tag{14}$$

We take this condition as the defining relation for Lorentz transformations. More precisely, the a transformation $dx^{\mu'} = \Lambda^{\mu'}{}_{\nu} dx^\nu$ is a Lorentz transformation if and only if $\Lambda$ satisfies Eq. 13. It can be verified that the collection of matrices satisfying Eq. 14 form a group under matrix multiplication. This group is known as the Lorentz group.

Suppose that two vectors transform under a Lorentz transformation as $X^{\mu'} = \Lambda^{\mu'}{}_{\nu} X^\nu$ and $Y^{\mu'} = \Lambda^{\mu'}{}_{\nu} Y^\nu$. By using Eq. 14 it follows that

$$X^\mu Y^\nu \eta_{\mu\nu} = X^{\mu'} Y^{\nu'} \eta_{\mu'\nu'}.$$

So we see that this "dot product" is invariant. This is analogous to the way the dot product of two Cartesian vectors is invariant when they are rotated about the origin, since rotation matrices are orthogonal. However, unlike Cartesian vectors, the dot product of $X^\mu$ with itself can be negative. $X^\mu$ is a **timelike**, **spacelike**, or **null** vector if $X^\mu X^\nu \eta_{\mu\nu}$ is positive, negative, or zero respectively.

## 1.7 Momentarily Comoving Reference Frame

For a particle moving at constant velocity, we can transform to an inertial frame in which the particle is always at rest. What about an accelerating particle? For this, we introduce the **Momentarily Comoving Frame** (MCF) which is an inertial frame in which the particle is momentarily at rest. Let's revisit the line element $(ds)^2$. Let $K'$ be a MCF of the particle. Then at the moment the particle is at rest we have

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = c^2(dt')^2 - (dx')^2 - (dz')^2 = c^2(dt')^2 = c^2(d\tau)^2$$

where $d\tau$ is the differential of the particle's proper time (the **proper time** is the time a clock travelling with the particle would keep). So we have the important relation

$$(ds)^2 = c^2(d\tau)^2. \tag{15}$$

This holds even if the particle is accelerating.

Let's revisit time dilation. We can rewrite Eq. 15 as

$$(c^2 - v^2)(dt)^2 = c^2(d\tau)^2$$

where $v^2 = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}$. Then we can compute a proper time interval as

$$\Delta\tau = \int_{t_i}^{t_f} dt\sqrt{1 - (v/c)^2}$$

where $t_i$ and $t_f$ denote the initial and final times in frame $K$. Note that this reduces to our previous result $\Delta\tau = \Delta t/\gamma$ when $v$ is constant.

## 1.8 Four-Velocity and Four-Momentum

Dividing Eq. 15 by $(d\tau)^2$ we obtain

$$u^\mu u^\nu \eta_{\mu\nu} = c^2$$

where

$$u^\mu = \frac{dx^\mu}{d\tau}.$$

is the **four-velocity**. The **four-momentum** of a massive particle is defined as

$$p^\mu = m\frac{dx^\mu}{d\tau}.$$

Note that $p^\mu p^\nu \eta_{\mu\nu} = m^2c^2$. We see that these are both timelike vectors.

Since $dx^{\mu'} = \Lambda^{\mu'}{}_\nu dx^\nu$ and $d\tau$ is invariant, we have that the four-velocity and four-momentum transform simply under a Lorentz transformation: $u^{\mu'} = \Lambda^{\mu'}{}_\nu u^\nu$, $p^{\mu'} = \Lambda^{\mu'}{}_\nu p^\nu$. This is a primary reason for why it is often best to work with these four vectors. Note, for instance, that the **coordinate velocity** $d\mathbf{r}/dt$ does not transform in such a simple way.

Consider a particle moving with velocity $v$ along the $x$-axis of $K$. Now let's consider boosting to a MCF. In this frame, all components except for the 0-component of $p^{\mu'}$ will be zero. Using Eq. 15 we find

$$p^{\mu'} = (mc, 0, 0, 0).$$

Next, let's transform back to $K$:

$$p^\mu = (\gamma mc, \gamma mv, 0, 0) = (E/c, p, 0, 0)$$

where $p = \gamma m dx/dt$ is the relativistic momentum introduced earlier.

The above argument can be repeated for a particle moving in an arbitrary direction. One finds

$$p^\mu = (E/c, \mathbf{p}). \tag{16}$$

For the four-velocity, one similarly finds

$$u^\mu = (\gamma c, \gamma \mathbf{v}). \tag{17}$$

The four-momentum is very useful since it combines our conserved quantities into a single vector. Conservation of energy and momentum can be expressed simply as

$$\frac{dp^\mu}{d\tau} = 0.$$

Inserting Eq. 16 into our condition $p^\mu p^\nu \eta_{\mu\nu} = m^2c^2$ gives an expression for the relativistic energy

$$E = \sqrt{(mc^2)^2 + (pc)^2} \tag{18}$$

(compare to Eq. 9). For $\frac{pc}{mc^2} \ll 1$ we have

$$E \approx mc^2 + \frac{p^2}{2m}$$

where the second term, again, is the Newtonian kinetic energy.

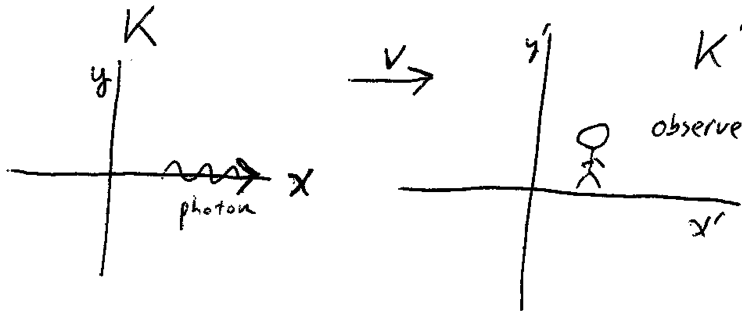
## 1.9 Photons and the Doppler Effect

**Photons** are particles (or quanta) of light. They have zero mass. From Eq. 18, we see that we can express the energy of a photon as $E = pc$. Using Eq. 16, we see that

$$p^\mu p^\nu \eta_{\mu\nu} = 0.$$

So the four-momentum of a photon is a null vector. Also, by considering Eq. 17, we see that the four-velocity of a photon is ill-defined (instead we take a $m \to 0$, $v \to c$ limiting case of the four-momentum).

Borrowing a basic result from modern physics, the energy of a photon is given by $E = h\nu$ where $h$ is Planck's constant and $\nu$ is the photon's frequency (units of inverse time). The photon's wavelength $\lambda$ satisfies $\lambda\nu = c$. How does the photon's wavelength change when viewed from a different reference frame (as shown in the drawing)?




In frame $K$, the photon's four-momentum is

$$p^\mu = (E/c, \mathbf{p}) = \left(\frac{h}{\lambda}, \frac{h}{\lambda}\hat{\mathbf{x}}\right).$$

Now transform to frame $K'$:

$$p^{\mu'} = \left(\frac{h}{\lambda}\gamma(1 - v/c), \frac{h}{\lambda}\gamma(1 - v/c)\hat{\mathbf{x}}\right) \equiv \left(\frac{h}{\lambda'}, \frac{h}{\lambda'}\hat{\mathbf{x}}\right).$$

The observer in frame $K'$ will therefore detect a photon of wavelength

$$\lambda' = \sqrt{\frac{1 + v/c}{1 - v/c}}\lambda.$$

When $\lambda' > \lambda$ we say that the photon is **redshifted** while if $\lambda' < \lambda$ we say the photon is **blueshifted**. Similar expressions can be derived for the photon's frequency and energy.

We can arrive at this result in a more elegant way by considering the following. The four-velocity of the observer in the observer's rest frame (or a MCF if the observer is accelerating) is $(u^{\mu'})_{\text{obs}} = (c, 0, 0, 0)$. We note with the above that

$$p^{\mu'}(u^{\nu'})_{\text{obs}}\eta_{\mu'\nu'} = \frac{hc}{\lambda'}.$$

But noting the transformation properties of these four-vectors, a very similar result will hold in frame $K$ (or any inertial frame)

$$p^\mu(u^\nu)_{\text{obs}}\eta_{\mu\nu} = \frac{hc}{\lambda'} = \frac{hc}{\lambda_{\text{obs}}} \tag{19}$$

where $\lambda_{\text{obs}} = \lambda'$ is the observed value for the photon's wavelength. So instead of transforming the photon's four-momentum to the rest frame of the observer, we could just directly evaluate the above. In practice, this is often the simplest way to proceed. Note Eq. 19 will hold even if the observer is accelerating.

## 1.10 Final Examples from Special Relativity

### 1.10.1 Uniformly Accelerating Particle Parametrised by Proper Time

We take the position of the particle (confined to the $x$-axis) to be given by

$$x(t) = \frac{c^2}{a}\sqrt{1+(at/c)^2}.$$

Note that $\frac{dx}{dt} = \frac{at}{\sqrt{1+(at/c)^2}}$ gives the result we found previously for the relativistic rocket. Using $(ds)^2 = c^2(d\tau)^2$ we find

$$c^2(d\tau)^2 = c^2(dt)^2 - (dx)^2 = \frac{c^2}{1+(at/c)^2}(dt)^2.$$

So

$$\tau = \int \frac{dt}{\sqrt{1+(at/c)^2}} = \frac{c}{a}\sinh^{-1}(at/c) + \text{const}.$$

Setting the integration constant to zero so that $\tau = 0$ when $t = 0$, we find that the particle's trajectory or **world line** parametrised by $\tau$ is given by

$$x^\mu = \left(\frac{c^2}{a}\sinh(a\tau/c), \frac{c^2}{a}\cosh(a\tau/c), 0, 0\right).$$

As a check, we can evaluate the particle's four-velocity and verify $u^\mu u^\nu \eta_{\mu\nu} = c^2$.

### 1.10.2 Rindler Coordinates

Up until now, we have been treating the rocket essentially as a point particle. Our aim here is to find a coordinate system which is glued to the moving rocket (do not worry if this example is confusing at first or second pass). We continue to take all motion to be confined to the $x$-axis.

First, let's consider the following: suppose that we take all components of the rocket to be accelerating with the same uniform acceleration. For instance, we take the trajectory of the front and back of the rocket to be given by

$$\begin{aligned} x_F(t) &= \frac{c^2}{a}\sqrt{1+(at/c)^2} + \left(\bar{x}_F - \frac{c^2}{a}\right) \\ x_B(t) &= \frac{c^2}{a}\sqrt{1+(at/c)^2} + \left(\bar{x}_B - \frac{c^2}{a}\right) \end{aligned}$$

so that at $t = 0$, $x_F = \bar{x}_F$ and $x_B = \bar{x}_B$. By differentiating these relations, we find the expected results for uniform acceleration. Now we note that at *all times* the rocket will have the same length $x_F(t) - x_B(t) = \bar{x}_F - \bar{x}_B$ in the $K$ frame. But from our considerations of length contraction, we expect that this rocket will be stretched out in its rest frame (if we can define such a thing!).

We instead seek a way to accelerate the rocket as a rigid object.[4] First, let's consider the trajectory from the previous example: $x(t) = \frac{c^2}{a}\sqrt{1+(at/c)^2}$. This satisfies

$$x^2 - c^2t^2 = X_a^2$$

where $X_a \equiv c^2/a$. Through our Lorentz transformation we also find

$$x'^2 - c^2t'^2 = X_a^2. \tag{20}$$

Now consider the following. Take the front and back of the rocket to have different uniform accelerations with $a_F < a_B$ (so that it maintains a constant rest length we want to accelerate the front less than the back). Let's further take the trajectories of the front and back to be given by

$$x_F^2 - c^2t^2 = \left(\frac{c^2}{a_F}\right)^2 = X_{a_F}^2$$
$$x_B^2 - c^2t^2 = \left(\frac{c^2}{a_B}\right)^2 = X_{a_B}^2.$$

Then at $t = 0$ both front and back ends of the rocket will be at rest with $x_F = X_{a_F}$, $x_B = X_{a_B}$, and $x_F - x_B = X_{a_F} - X_{a_B}$. Now consider transforming to frame $K'$ through Eq. 20. At time $t' = 0$, we see that the rocket is at rest in this frame and will have the *same* rest length: $x'_F = X_{a_F}$, $x'_B = X_{a_B}$, and $x'_F - x'_B = X_{a_F} - X_{a_B}$. Thus, we have found a way to accelerate the front and back of the rocket so that the the rest length of the rocket is fixed. In fact, we can determine the acceleration of any intermediate point within the rocket in such a way. For instance the centre of the rocket will accelerate with uniform acceleration $a_C$ given by $1/a_C = (1/a_F + 1/a_B)/2$.

With such considerations, we see that the family of trajectories

$$x^2 - c^2t^2 = X^2 \tag{21}$$

(for different $X$ values) will describe how to accelerate the rocket as a rigid object. Additionally, $X$ will give the position in a coordinate system glued to the accelerating rocket. From $X = c^2/a$ we can read off how fast we need to accelerate this point so that the whole rocket accelerates as a rigid object.

We therefore take $X$ to label position in the rocket's frame. What about time? Eq. 21 is suggestive. Suppose we choose a time coordinate $T$ such that

$$t = \frac{1}{c}X\sinh(a_BT/c).$$

Then $x = X\cosh(a_BT/c)$ (this satisfies Eq. 21). From the previous example, we identify $T$ as the time kept by a clock sitting at the back of the rocket. This is a sensible time coordinate but

[4] But can rigid objects actually exist in relativity? For instance, if you push on one end of a rigid rod, and the other end moves instantly, then information would be propagated faster than the speed of light. The concept of rigidity was debated in the early days of relativity by some eminent figures. To be more precise, in this section, we are taking the rod to be rigid in the sense of "Born Rigididy". For a discussion see arXiv:1105.3899.

we could have put the clock somewhere else in the rocket. Note that clocks sitting at different positions in the rocket will tick at different rates. $X$ and $T$ are called Rindler coordinates.

For a massive particle under no external forces we have $dp^\mu/d\tau = 0$ in an inertial frame. In the Rindler Coordinate system this reads (with some work)

$$\begin{aligned} \frac{d^2T}{d\tau^2} + \frac{2}{X}\frac{dX}{d\tau}\frac{dT}{d\tau} &= 0 \\ \frac{d^2X}{d\tau^2} + \left(\frac{a_B}{c}\right)^2 X \left(\frac{dT}{d\tau}\right)^2 &= 0. \end{aligned}$$

Solving these equations would give, say, the trajectory of a ball dropped in the rocket as seen from inside the rocket. The ball will fall as if it is under a gravitational field. In our investigation of general relativity, we will look at similar systems of equations in non-inertial reference frames with one important difference: unlike the above, the gravitational fields in GR cannot be removed everywhere with a coordinate transformation.

This completes our overview of special relativity.[5]

[5] There's a neat game which allows you to "experience" special relativity including Lorentz contractions and red/blue shifts at `http://gamelab.mit.edu/games/a-slower-speed-of-light/`.

# 2 Mathematics of General Relativity

## 2.1 The Equivalence Principle and What's Ahead

Let's consider the following thought experiment. Consider two lifts. One is deep in outer space, while the other is freely falling under the earth's gravitational field. A passenger is aboard one of the two lifts. By doing experiments over small regions of spacetime the passenger cannot discern between the two scenarios. That is, the passenger would feel weightless in both lifts. The passenger could do local kinematic experiments and observe that Newton's first law is obeyed for either case.[6] This is due to the equivalence of gravitational and inertial mass. This is summarised by the **equivalence principle**:

- In a freely falling laboratory occupying a small region of spacetime, special relativity holds.

When gravitational fields are present, it is generally not possible to find a coordinate system which is inertial everywhere. In general relativity, gravity is accounted for by the curvature of spacetime. Free particles follow the straightest possible paths in this curved spacetime. From our discussion of special relativity, we have the line element and the equations of motion of a massive particle under no forces are given by

$$(ds)^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad \frac{d^2 x^\mu}{d\tau^2} = 0.$$

We will find that these equations generalise to

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

($g_{\mu\nu}$ and $\Gamma^\mu_{\nu\sigma}$ to be defined soon).

## 2.2 General Coordinate Systems for Euclidean space

Before jumping into our discussion of differentiable manifolds, we will start with something more familiar: three-dimensional Euclidean space. We will find that many of the results from Euclidean space will carry over naturally to the general case. We will start labeling indices with letters near the beginning of the alphabet $a, b, c, d, \ldots$ (hopefully we will not need to go too far into the alphabet!). We will reserve Greek indices for spacetime.

### 2.2.1 General Basis Vectors

Let

$$(x, y, z) = (x^1, x^2, x^3)$$

[6] On the other hand, over larger regions of spacetime the passenger *would* be able to discern between the two lifts. For instance, if the lift is falling under the earth's gravitational field, since the gravitational field is radial, the horizontal distance between two balls released from rest from the left and right hands of the passenger will eventually decrease in apparent violation of Newton's first law. The balls would keep a constant horizontal separation for the lift deep in space.

be the usual Cartesian coordinates and

$$(u, v, w) = (u^1, u^2, u^3)$$

be an arbitrary coordinate system. We will restrict our attention to cases where the Cartesian coordinates can be written as functions of the $(u, v, w)$ coordinates and vice versa. Write

$$\mathbf{r} = x(u, v, w)\hat{\mathbf{x}} + y(u, v, w)\hat{\mathbf{y}} + z(u, v, w)\hat{\mathbf{z}}$$

and consider the vectors

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial u},\ \mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial v},\ \mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial w}$$

or, more succinctly,

$$\mathbf{e}_a = \frac{\partial \mathbf{r}}{\partial u^a}.$$

These vectors are not necessarily mutually orthogonal or normalised. Now consider another collection of vectors defined by

$$\mathbf{e}^1 = \nabla u,\ \mathbf{e}^2 = \nabla v,\ \mathbf{e}^3 = \nabla w$$

or $\mathbf{e}^a = \nabla u^a$. In this, $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$ is the usual gradient operator. Note we need to write the $u^a$ coordinates as functions of the Cartesian coordinates to evaluate the above.

We will now establish a useful relationship between these vectors:

$$\mathbf{e}^a \cdot \mathbf{e}_b = \nabla u^a \cdot \frac{\partial \mathbf{r}}{\partial u^b} = \frac{\partial u^a}{\partial x}\frac{\partial x}{\partial u^b} + \frac{\partial u^a}{\partial y}\frac{\partial y}{\partial u^b} + \frac{\partial u^a}{\partial z}\frac{\partial z}{\partial u^b} = \frac{\partial u^a}{\partial x^c}\frac{\partial x^c}{\partial u^b} = \frac{\partial u^a}{\partial u^b}$$

In these manipulations, we have made use of the chain rule. We have found the orthogonality relation

$$\mathbf{e}^a \cdot \mathbf{e}_b = \delta^a_b$$

where $\delta^a_b$ is the **Kronecker delta function**: $\delta^a_b = 1$ if $a = b$, $\delta^a_b = 0$ if $a \neq b$. This orthogonality relation can be used to establish the linear independence of $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$. That is, consider $c_1, c_2, c_3$ which satisfy

$$c_1\mathbf{e}^1 + c_2\mathbf{e}^2 + c_3\mathbf{e}^3 = 0.$$

Taking the dot product of this equation with $\mathbf{e}_a$, and using the orthogonality relation will give $c_1 = c_2 = c_3 = 0$. A similar argument can be used to establish that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are linearly independent. We will call $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the natural basis and $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ the dual basis.

A general vector $\mathbf{X}$ in Euclidean space can therefore be written in either basis:

$$\mathbf{X} = X^a\mathbf{e}_a = X_a\mathbf{e}^a.$$

(summation implicit). We call $X^a$ the **contravariant** components of $\mathbf{X}$ and $X_a$ the **covariant** components of $\mathbf{X}$. Using this orthogonality relation, the components of $\mathbf{X}$ can be extracted as:

$$\mathbf{e}_a \cdot \mathbf{X} = \mathbf{e}_a \cdot X_b\mathbf{e}^b = X_b\delta^b_a = X_a.$$

Similarly, $X^a = \mathbf{e}^a \cdot \mathbf{X}$.

### 2.2.2 The Metric

We define the **metric** $g_{ab}$ as

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b.$$

Similarly, we define $g^{ab} = \mathbf{e}^a \cdot \mathbf{e}^b$. From these definitions, we see that these quantities are symmetric: $g_{ab} = g_{ba}$, $g^{ab} = g^{ba}$. Now consider two arbitrary vectors, $\mathbf{X}$ and $\mathbf{Y}$ which can be written as

$$\mathbf{X} = X^a \mathbf{e}_a = X_a \mathbf{e}^a$$
$$\mathbf{Y} = Y^a \mathbf{e}_a = Y_a \mathbf{e}^a.$$

There are four ways we can write the dot product of these vectors:

$$\mathbf{X} \cdot \mathbf{Y} = g_{ab} X^a Y^b = g^{ab} X_a Y_b = X^a Y_a = X_a Y^a.$$

Let's consider $X_a Y^a = g^{ab} X_a Y_b$. Since this is true for arbitrary $\mathbf{X}$, it must be that

$$Y^a = g^{ab} Y_b.$$

Similarly, $X^a Y_a = g_{ab} X^a Y^b$ tells us that

$$Y_a = g_{ab} Y^b.$$

We see that the metric can be used to convert contravariant to covariant components and vice versa. We also have for arbitrary $Y^a$,

$$Y^a = g^{ab} Y_b = g^{ab} g_{bc} Y^c.$$

Therefore,

$$g^{ab} g_{bc} = \delta^a_c.$$

That is, $g^{ab}$ is the inverse of $g_{ab}$.

**Example**: Cylindrical coordinates. Let's apply what we have established so far to cylindrical coordinates. We take $u = \rho$, $w = \phi$, and $v = z$ where $x = \rho\cos(\phi)$, $y = \rho\sin(\phi)$. We find (check)

$$\begin{aligned}
\mathbf{e}_1 &= \cos(\phi)\hat{\mathbf{x}} + \sin(\phi)\hat{\mathbf{y}} \\
\mathbf{e}_2 &= -\rho\sin(\phi)\hat{\mathbf{x}} + \rho\cos(\phi)\hat{\mathbf{y}} \\
\mathbf{e}_3 &= \hat{\mathbf{z}} \\
\mathbf{e}^1 &= \cos(\phi)\hat{\mathbf{x}} + \sin(\phi)\hat{\mathbf{y}} \\
\mathbf{e}^2 &= -\frac{1}{\rho}\sin(\phi)\hat{\mathbf{x}} + \frac{1}{\rho}\cos(\phi)\hat{\mathbf{y}} \\
\mathbf{e}^3 &= \hat{\mathbf{z}}
\end{aligned}$$

and

$$g_{ab} = \mathrm{diag}(1, \rho^2, 1), \quad g^{ab} = \mathrm{diag}(1, \frac{1}{\rho^2}, 1).$$

### 2.2.3 Why Do We Need Two Types of Basis Vectors?

Why do we need two types of basis vectors? First we note that if $(u, v, w)$ are just the Cartesian coordinates, then the natural basis and the dual basis are the same: $\hat{\mathbf{x}} = \mathbf{e}_1 = \mathbf{e}^1$ with similar relations for the $y$ and $z$ basis vectors. For the Cartesian basis, we will use the notation $\hat{\mathbf{x}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$ with similar relations for the other vectors.

Now let's go back to arbitrary $(u, v, w)$. Some quantities are most naturally described by the dual basis. For instance, consider a scalar function $f(u, v, w)$. The gradient of this is

$$\nabla f = \frac{\partial f}{\partial x^a}\hat{\mathbf{x}}^a = \frac{\partial f}{\partial u^b}\frac{\partial u^b}{\partial x^a}\hat{\mathbf{x}}^a = \frac{\partial f}{\partial u^a}\mathbf{e}^a.$$

So we see that gradients are naturally expressed in the dual basis.

On the other hand, some quantities are better described by the natural basis. Suppose we have a curve in Euclidean space parameterised by $t$: $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$. A vector tangent to this curve is given by

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial u^a}\frac{du^a}{dt} = \dot{u}^a\mathbf{e}_a$$

where $\dot{u}^a \equiv \frac{du^a}{dt}$. Thus tangent vectors to curves are expressed in terms of the natural basis.

### 2.2.4 Arc Length

Let's take the curve parametrised by $t$ from the previous section and restrict $t$ to $t_1 \leq t \leq t_2$. Using our machinery, we see that the arc length $\ell$ of this curve is

$$\ell = \int_{t_1}^{t_2} dt\sqrt{\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}} = \int_{t_1}^{t_2} dt\sqrt{\dot{u}^a\dot{u}^b g_{ab}}.$$

> **Example**: Arc length of spiral. Let's consider a curve which is naturally described in cylindrical coordinates. Take a spiral with $\rho = R = \text{const}$, $\phi = t$, $z = t$, and $0 \leq t \leq T$. Then, using the metric from the previous example we find (check)
>
> $$\ell = \int_0^T dt\sqrt{g_{11}\dot{\rho}^2 + g_{22}\dot{\phi}^2 + g_{33}\dot{z}^2} = T\sqrt{1 + R^2}.$$

### 2.2.5 Coordinate Transformations

We will now introduce another arbitrary coordinate system $(u', v', w')$ and consider how to transform quantities between this and the $(u, v, w)$ coordinate system. We will refer to these as the primed and unprimed coordinate systems. In the primed coordinate system, we find basis vectors using the same procedure as before: $\mathbf{e}_{a'} = \frac{\partial \mathbf{r}}{\partial u^{a'}}$ and $\mathbf{e}^{a'} = \nabla u^{a'}$. Using the chain rule, we can find how the basis vectors transform under a change of coordinates:

$$\mathbf{e}_a = \frac{\partial \mathbf{r}}{\partial u^a} = \frac{\partial \mathbf{r}}{\partial u^{b'}}\frac{\partial u^{b'}}{\partial u^a} = J_a^{b'}\mathbf{e}_{b'}$$

where $J_a^{b'} = \frac{\partial u^{b'}}{\partial u^a}$ is the **Jacobian matrix**. In a similar way, we can find the transformation rule for $\mathbf{e}^a$:

$$\mathbf{e}^a = \nabla u^a = \frac{\partial u^a}{\partial x^b}\hat{\mathbf{x}}^b = \frac{\partial u^a}{\partial u^{c'}}\frac{\partial u^{c'}}{\partial x^b}\hat{\mathbf{x}}^b = J_{c'}^a \mathbf{e}^{c'} = J_{b'}^a \mathbf{e}^{b'}.$$

With these coordinate systems, there are multiple ways to write a vector $\mathbf{X}$

$$\mathbf{X} = X^a \mathbf{e}_a = X_a \mathbf{e}^a = X^{a'} \mathbf{e}_{a'} = X_{a'} \mathbf{e}^{a'}.$$

Inserting the transformation rules for the basis vectors into the above determines how the components of $\mathbf{X}$ transform. One finds

$$X^{a'} = J_b^{a'} X^b \quad \text{and} \quad X_{a'} = J_{a'}^b X_b. \tag{22}$$

Finally, we note that the chain rule gives a useful result:

$$J_{c'}^a J_b^{c'} = \frac{\partial u^a}{\partial u^{c'}}\frac{\partial u^{c'}}{\partial u^b} = \frac{\partial u^a}{\partial u^b} = \delta_b^a.$$

## 2.3 Surfaces in Euclidean Space

We will now consider surfaces in three-dimensional Euclidean space. Most of the results for Euclidean space established previously can be seen to carry over to this case. We take the surface to be given parametrically by

$$\mathbf{r}(u,v) = x(u,v)\hat{\mathbf{x}} + y(u,v)\hat{\mathbf{y}} + z(u,v)\hat{\mathbf{z}}.$$

As before, we obtain the natural basis vectors $\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial v}$. These vectors evaluated at a particular point $P$ on the surface are tangent to the surface at this point. The collection of vectors tangent to the surface at $P$ form a vector space which we denote as $T_P$. Our vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ can be taken as a basis for this tangent space.

As before, we define the metric as $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$. We take $g^{ab}$ to be the inverse of $g_{ab}$: $g^{ac} g_{cb} = \delta_b^a$. Finally we take $\mathbf{e}^a = g^{ab}\mathbf{e}_b$ (note the difficulty in defining $\mathbf{e}^a$ as we did in the previous section). Note with this $\mathbf{e}^a \cdot \mathbf{e}_b = g^{ac}\mathbf{e}_c \cdot \mathbf{e}_b = g^{ac} g_{cb} = \delta_b^a$.

**Example**: The unit sphere. We take the unit sphere to be parametrised by $\theta = u = u^1$, $\phi = v = u^2$ as

$$\mathbf{r}(\theta,\phi) = \sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}}.$$

By differentiating this expression we have

$$\begin{aligned}\mathbf{e}_1 &= \cos(\theta)\cos(\phi)\hat{\mathbf{x}} + \cos(\theta)\sin(\phi)\hat{\mathbf{y}} - \sin(\theta)\hat{\mathbf{z}} \\ \mathbf{e}_2 &= -\sin(\theta)\sin(\phi)\hat{\mathbf{x}} + \sin(\theta)\cos(\phi)\hat{\mathbf{y}}\end{aligned}$$

With these, we can evaluate the metric. The line element can be found to be

$$(ds)^2 = d\mathbf{r}\cdot d\mathbf{r} = g_{ab}du^a du^b = (d\theta)^2 + \sin^2(\theta)(d\phi)^2.$$

## 2.4 General Tensors and Manifolds

Euclidean space and two-dimensional surfaces in Euclidean space are both examples of a more general entity called a **manifold**. An $N$-dimensional manifold is a space which is locally isomorphic to Euclidean space $\mathbb{R}^N$. For instance a small region about any point on the unit sphere is locally isomorphic to $\mathbb{R}^2$. Points on an $N$-dimensional manifold are labelled by $N$ coordinates $(x^1, x^2, \ldots, x^N)$ in such a way that there is a one-to-one correspondence between points in the manifold and coordinate values. It is not generally possible to describe an entire manifold in such a way with a single set of coordinates. Taking our example of the unit sphere, there is not a one-to-one correspondence between coordinate values and points in the manifold at the north and south poles of the sphere. That is, when $\theta = 0$ or $\theta = \pi$, all values of $\phi$ describe the same point. For such situations, one needs to use multiple coordinate systems to cover different regions of the manifold.

Suppose we have a region of a manifold described by two coordinate systems $(x^1, x^2, \ldots, x^N)$ and $(x^{1'}, x^{2'}, \ldots, x^{N'})$. Assuming the one-to-one correspondence described above, the $x^a$ coordinates can be written as functions of the $x^{a'}$ coordinates and vice versa. We can thus define the Jacobian matrices as before: $J^a_{b'} = \frac{\partial x^a}{\partial x^{b'}}$ and $J^{a'}_b = \frac{\partial x^{a'}}{\partial x^b}$ which satisfy, through the chain rule, $J^a_{b'} J^{b'}_c = \delta^a_c$. We carry over the transformation rules for contravariant and covariant vectors arrived at earlier to the present case: given a contravariant vector $X^a$ and a covariant vector $Y_a$, we have $X^{a'} = J^{a'}_b X^b$ and $Y_{a'} = J^b_{a'} Y_b$.[7]

Contravariant and covariant vectors are examples of a more general quantity called a **tensor**. Suppose at a particular point in the manifold, we have $N^{n+m}$ quantities that transform under a change of coordinates as

$$T^{a'_1 \ldots a'_n}_{b'_1 \ldots b'_m} = J^{a'_1}_{c_1} \ldots J^{a'_n}_{c_n} J^{d_1}_{b'_1} \ldots J^{d_m}_{b'_m} T^{c_1 \ldots c_n}_{d_1 \ldots d_m}.$$

Then $T^{a_1 \ldots a_n}_{b_1 \ldots b_m}$ is a type $(n, m)$ tensor. A tensor field defined over some region of the manifold associates a tensor of the same rank to every point in the region.

We will only consider manifolds which have an associated metric $g_{ab}$ (which are known as **Riemannian manifolds**). We take the metric to be a type $(0, 2)$ tensor field. As before we take $g^{ab}$ to be the inverse of $g_{ab}$. It can be shown (exercise) that $g^{ab}$ is then a type $(2, 0)$ tensor field. As before, $g^{ab}$ and $g_{ab}$ can be used to raise and lower indices. For instance, for a $(2, 0)$ tensor $T^{ab}$, $T^{ab} g_{bc} = T^a_{\ c}$ (it can also be checked that these quantities have the correct transformation properties). Also, we see that $g^a_b = \delta^a_b$.

> **Example**: Take $T^a_{bc}$ to be a type $(1, 2)$ tensor. Define $R_a = T^b_{ab} = \sum_b T^b_{ab}$. The $b$ indices are said to be **contracted**. In another coordinate system,
>
> $$R_{a'} = T^{b'}_{a'b'} = J^{b'}_c J^d_{a'} J^e_{b'} T^c_{de} = \delta^e_c J^d_{a'} T^c_{de} = J^d_{a'} T^e_{de} = J^d_{a'} R_d.$$
>
> So we see that $R_a$ is a tensor. Generally, one can obtain an $(n-1, m-1)$ tensor from an $(n, m)$ tensor through contraction.

[7] We are being loose with terminology here. Strictly speaking, $X^a$ and $Y_a$ are *components* of vectors and not vectors. This distinction is commonly dropped in textbooks on GR. We will adopt this convention and refer to $X^a$ itself as a contravariant vector.

## 2.5 Calculus of Variations and Geodesics

In relativity, particles follow **geodesics**. A geodesic is the generalisation of the notion of a straight line to a curved manifold. In the following, we will learn how to compute geodesics.

### 2.5.1 Calculus of Variations

Suppose we want to find curves on the manifold parametrised by $t$, $x^a(t)$, which extremise

$$S[x^a] = \int_{t_1}^{t_2} dt L(x^a, \dot{x}^a, t)$$

subject to having fixed endpoints, say $x(t_1) = x_1$ and $x(t_2) = x_2$. To do this write $\bar{x}^a = x^a + \delta x^a$ where $\delta x^a(t_1) = \delta x^a(t_2) = 0$. We insert $\bar{x}^a$ into $S$ and expand in $\delta x^a$. In order for $x^a$ to describe an extremum, the first order term must vanish. To first order in $\delta x^a$, we have

$$\begin{aligned} S[\bar{x}^a] &= \int_{t_1}^{t_2} dt L(x^a + \delta x^a, \dot{x}^a + \delta \dot{x}^a, t) \\ &= \int_{t_1}^{t_2} dt \left( L(x^a, \dot{x}^a, t) + \frac{\partial L}{\partial x^a} \delta x^a + \frac{\partial L}{\partial \dot{x}^a} \delta \dot{x}^a \right). \end{aligned}$$

Note that a summation over $a$ is implied for the linear terms in $\delta x^a$. Next, we integrate by parts to obtain

$$S[\bar{x}^a] = S[x^a] + \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial x^a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^a} \right) \delta x^a.$$

We require the linear piece to vanish for arbitrary $\delta x^a$. Therefore,

$$\frac{\partial L}{\partial x^a} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^a} \tag{23}$$

which are the **Euler-Lagrange** equations.

### 2.5.2 Geodesics

As we saw before, the length of a curve between two points $P_1$ and $P_2$ is

$$\ell = \int_{P_1}^{P_2} ds = \int_{t_1}^{t_2} dt \sqrt{g_{ab} \dot{x}^a \dot{x}^b}$$

The geodesics are given by the Euler-Lagrange equations of $L = \sqrt{g_{ab} \dot{x}^a \dot{x}^b}$. Before diving into computing these, we introduce

$$K = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b. \tag{24}$$

Working with $K$ is easier since it doesn't involve square roots. For $K$ we have

$$\begin{aligned} \frac{\partial K}{\partial x^a} &= L \frac{\partial L}{\partial x^a} \\ \frac{d}{dt} \frac{\partial K}{\partial \dot{x}^a} &= \frac{d}{dt} \left( L \frac{\partial L}{\partial \dot{x}^a} \right) = \ddot{s} \frac{\partial L}{\partial \dot{x}^a} + L \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^a} \end{aligned}$$

since $L = \frac{ds}{dt} = \dot{s}$. By the Euler-Lagrange equations for $L$ we thus have

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{x}^a} = \frac{\partial K}{\partial x^a} + \ddot{s}\frac{\partial L}{\partial \dot{x}^a}.$$

Now we note that we have the freedom to choose a convenient parametrisation of the curve. If we choose it to be related to $s$ as $t = \alpha s + \beta$ where $\alpha \neq 0$ and $\beta$ are constants, then $\ddot{s} = 0$. Our parameter $t$ is then called an **affine parameter**. Choosing $t$ to be an affine parameter, the Euler-Lagrange equations become

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{x}^a} = \frac{\partial K}{\partial x^a}. \tag{25}$$

This is the easiest way to proceed.

Evaluating Eq. 25 with Eq. 24, gives (exercise)

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \tag{26}$$

where

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc}\right) \tag{27}$$

(we will begin using the shorthand $\frac{\partial}{\partial x^a} = \partial_a$). We will call Eq. 26 the **geodesic equation**. $\Gamma^a_{bc}$ is known as a **Christoffel symbol**. From Eq. 27 we see that the Christoffel symbols have the symmetry $\Gamma^a_{bc} = \Gamma^a_{cb}$.

**Example**: Obtaining the Christoffel symbols from the Euler-Lagrange equations. This example will present a trick for determining Christoffel symbols which often proves to be the most efficient way of computing him. To be concrete, let's focus on the case of the unit sphere. For this, we have

$$K = \frac{1}{2}\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right).$$

The Euler-Lagrange equations are

$$\ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\phi}^2 = 0$$
$$\ddot{\phi} + 2\cot(\theta)\dot{\theta}\dot{\phi} = 0.$$

By comparison with the geodesic equation, Eq. 26, we deduce that the only non-vanishing Christoffel symbols are

$$\Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta)$$
$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot(\theta).$$

We can check that direct evaluation of Eq. 27 gives the same results, but the method of this example is much less tedious. Also, note that we are writing $\Gamma^\theta_{\phi\phi}$ instead of $\Gamma^1_{22}$, etc. This helps us keep track of variables (was the $x^1$ variable $\theta$ or $\phi$?).

### 2.5.3 Comparison with Newtonian Mechanics

In **Lagrangian mechanics**, Newton's second law is arrived at through a variational principle. The **Lagrangian** of a Newtonian particle under an external potential $V$ is defined to be the particle's kinetic energy minus its potential energy $V$. For example, let's consider a particle in one spatial dimension. Then the Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - V(x)$. It is readily verified that the Euler-Lagrange equations give Newton's second law: $m\ddot{x} = -\frac{\partial V}{\partial x}$. Classical particles move in ways which extremize the **action** $S$ where $S = \int_{t_1}^{t_2} dtL$ .

The results of the previous section have an interesting connection with classical mechanics. First note that $K = \frac{1}{2}\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt} = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$ is the kinetic energy (and Lagrangian) of a unit-mass particle under no external forces moving on a manifold. Newton's second law for this particle will be given by the Euler-Lagrange equations for $K$, Eq. 26. Conservation of energy tells us that for a solution of Eq. 26 we will have $E = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b = \text{const.}$[8] $E = \frac{1}{2}(ds/dt)^2$ then tells us that $t$ is an affine parameter ($\ddot{s} = 0$). Thus the classical trajectories will also extremize the arc length. From this, we conclude that the allowable trajectories of the classical particle will follow geodesics of the manifold. The allowable curves traced out by classical particles on a manifold are determined by the geometry of the manifold alone.

## 2.6 Parallel Transport

Let's start by considering surfaces in Euclidean space. Let $\mathbf{X}_P$ be a vector tangent to the surface at a particular point $P$. Now let $x^a(t)$ describe a curve on the manifold starting at point $P$ at $t = 0$. How can we move the vector along the curve, keeping it in the tangent space of the manifold along the curve, and also keeping it "as parallel as possible"? If our manifold was all of Euclidean space (and not a surface) the answer is more or less clear: we take $\mathbf{X}$ to be the same at every point along the curve. However, this cannot generally be accomplished for a surface since different points along the curve will have different tangent spaces. Instead, we take our condition for parallel transport to be

$$\mathcal{P}\cdot\frac{d\mathbf{X}(x^a(t))}{dt} = 0 \tag{28}$$

where $\mathcal{P} = \mathbf{e}_a\mathbf{e}^a = \mathbf{e}^a\mathbf{e}_a$ projects into the tangent space.[9] Note that $d\mathbf{e}_a/dt$ need not be in the tangent space. We also require that $\mathbf{X}$ always is in the tangent space along the curve: $\mathcal{P}\cdot\mathbf{X} = \mathbf{X}$. From Eq. 28 one finds that

$$\dot{X}^a + \Gamma^a_{bc}X^b\dot{x}^c = 0 \tag{29}$$

[8] We can arrive at conservation of energy directly from the Euler-Lagrange equations. Taking the time derivative of $K$,

$$\frac{dK}{dt} = \frac{\partial K}{\partial x^a}\dot{x}^a + \frac{\partial K}{\partial \dot{x}^a}\ddot{x}^a = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}^a}\right)\dot{x}^a + \frac{\partial K}{\partial \dot{x}^a}\ddot{x}^a = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}^a}\dot{x}^a\right)$$

so $\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}^a}\dot{x}^a - K\right) = 0$. We can verify that $K = \frac{\partial K}{\partial \dot{x}^a}\dot{x}^a - K$.

[9] Notice the absence of the dot product in the definition of $\mathcal{P}$. $\mathcal{P}$ dotted with a vector on either the left or right produces another vector. $\mathcal{P}$ is a projector since $\mathcal{P}\cdot\mathcal{P} = \mathcal{P}$. Also, note for any vector $\mathbf{Y}$ in the tangent space at a particular point, $\mathcal{P}\cdot\mathbf{Y} = \mathbf{Y}$. Also, for a vector $\hat{\mathbf{n}}$ normal to the surface, $\mathcal{P}\cdot\hat{\mathbf{n}} = 0$.

where $\Gamma^a_{bc} = \mathbf{e}^a \cdot \partial_b \mathbf{e}_c$.

We need to check that $\Gamma^a_{bc}$ is consistent with our previous definition. First we lower the $a$ superscript: $\Gamma_{abc} = g_{ad}\Gamma^d_{bc} = \mathbf{e}_a \cdot \partial_b \mathbf{e}_c$. Then using $\partial_a \mathbf{e}_b = \partial_a \partial_b \mathbf{r} = \partial_b \partial_a \mathbf{r} = \partial_b \mathbf{e}_a$ one can verify that

$$\begin{aligned}\Gamma_{abc} = \mathbf{e}_a \cdot \partial_b \mathbf{e}_c &= \frac{1}{2}\left(\partial_b(\mathbf{e}_a \cdot \mathbf{e}_c) + \partial_c(\mathbf{e}_a \cdot \mathbf{e}_b) - \partial_a(\mathbf{e}_b \cdot \mathbf{e}_c)\right) \\ &= \frac{1}{2}\left(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}\right).\end{aligned}$$

So

$$\Gamma^a_{bc} = g^{ad}\Gamma_{dbc} = \frac{1}{2} g^{ab}\left(\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc}\right)$$

which is consistent with our previous definition.

We arrived at Eq. 29 by considering surfaces, but we take it to extend to arbitrary manifolds. We call it the **parallel transport equation**.

Parallel transport will give us a way of characterising the curvature of a manifold. For a flat manifold, you would imagine that a vector parallel transported about a closed loop would produce the same vector. For a curved manifold, this isn't the case.

It is also interesting to consider the special case of the parallel transport equation, Eq. 29, where $X^a$ is a vector tangent to the curve: $X^a = \frac{dx^a}{dt}$. Then the parallel transport equation reduces to the geodesic equation. That is, the tangent vector $\dot{x^a}$ is parallel transported along a geodesic. This gives an alternative way of thinking about geodesics.

> **Example**: Parallel transport for the unit sphere.[10] Consider parallel transporting a vector along the curve $\theta = \text{const}$, $\phi = 2\pi t$ for $0 \le t \le 1$. For simplicity, we take the curve to reside in the northern hemisphere: $\omega \equiv \cos(\theta) > 0$. What is the angle between the initial and final vectors?
>
> Using the Christoffel symbols obtained from a previous example, we find the parallel transport equations
>
> $$\begin{aligned}\dot{X}^\theta - \sin(\theta)\cos(\theta) X^\phi \dot{\phi} &= 0 \\ \dot{X}^\phi + \cot(\theta) X^\theta \dot{\phi} &= 0\end{aligned}$$
>
> These have the solution
>
> $$\begin{aligned}X^\theta &= A\cos(\omega\phi) + B\sin(\omega\phi) \\ X^\phi &= \frac{-A}{\sin(\theta)}\sin(\omega\phi) + \frac{B}{\sin(\theta)}\cos(\omega\phi)\end{aligned}$$
>
> where $A$ and $B$ are constants. The initial vector at $t = 0$ will fix $A$ and $B$.
>
> The angle $\alpha$ between the initial final vectors is given by
>
> $$\cos(\alpha) = \frac{\mathbf{X}(0) \cdot \mathbf{X}(1)}{|\mathbf{X}(\mathbf{0})||\mathbf{X}(\mathbf{1})|}.$$
>
> Using the above results we find
>
> $$\cos(\alpha) = \frac{A(A\cos(2\pi\omega) + B\sin(2\pi\omega)) + B(-A\sin(2\pi\omega) + B\cos(2\pi\omega))}{A^2 + B^2} = \cos(2\pi\omega).$$

[10] This example is relevant for describing the Foucault pendulum `https://en.wikipedia.org/wiki/Foucault_pendulum`.

## 2.7 Covariant Differentiation

We want to define a sensible way of differentiating tensor fields which results in tensor fields. For instance given a contravariant vector field $X^a$, we want some differential operator $\nabla_a$ such that $\nabla_b X^a$ is a type $(1,1)$ tensor field. First let's consider regular partial differentiation. We find that $\partial_b X^a$ transforms as

$$\partial_b X^a = J_b^{c'} J_{d'}^a \partial_{c'} X^{d'} + J_b^{c'} J_{c'd'}^a X^{d'}$$

where we define $J_{b'c'}^a = \frac{\partial^2 x^a}{\partial x^{b'} x^{c'}}$. Due to the presence of the last term in the above we see that $\partial_b X^a$ is not a tensor. This result is perhaps not surprising since when investigating the change of a vector, one must account for changes both in the components and basis vectors.

We continue looking for a suitable differential operator. Let's reinstate basis vectors and consider

$$\partial_b \mathbf{X} = \partial_b X^a \mathbf{e}_b + X^a \partial_b \mathbf{e}_a.$$

Since $\partial_a \mathbf{e}^b$ may have components not in the tangent space, let's project these out (like in the section on parallel transport). This gives

$$\mathcal{P} \cdot \partial_b \mathbf{X} = (\nabla_b X^a)\mathbf{e}_a.$$

where

$$\nabla_b X^a = \partial_b X^a + \Gamma_{bc}^a X^c. \tag{30}$$

Let's check the transformation properties of $\nabla_b X^a$. Start in the primed coordinate system and transform back:

$$(\nabla_{b'} X^{a'})\mathbf{e}_{a'} = \mathcal{P} \cdot \partial_{b'} \mathbf{X} = J_{b'}^c \mathcal{P} \cdot \partial_c \mathbf{X} = J_{b'}^c (\nabla_c X^d)\mathbf{e}_d.$$

Dotting the far left and far right expressions in this equation with $\mathbf{e}^{e'}$ gives

$$(\nabla_{b'} X^{e'}) = J_{b'}^c (\mathbf{e}^{e'} \cdot \mathbf{e}_d)(\nabla_c X^d).$$

Note that $\mathbf{e}^{e'} \cdot \mathbf{e}_d = \mathbf{e}^{e'} \cdot \mathbf{e}_{f'} J_d^{f'} = J_d^{e'}$. Using this, and relabelling indices, we see that

$$\nabla_{b'} X^{a'} = J_{b'}^c J_d^{a'} \nabla_c X^d.$$

So $\nabla_b X^a$ is a type (1,1) tensor! Eq. 30 tells us how to take the **covariant derivative** of a contravariant vector field.

What about other types of tensors? In extending this to arbitrary tensors, we require the covariant derivative to

1. obey the Leibniz rule (aka product rule)
2. reduce to partial differentiation when acting upon a scalar field
3. be a linear operation
4. result in tensor fields

In the second condition, we take a **scalar field** to be a type $(0,0)$ tensor field. Let's apply these conditions to determine the covariant derivative of a covariant vector field. Take $Y^a$ and $Z_a$ to be arbitrary tensor fields. Note that $Y^a Z_a$ is an invariant scalar field since it transforms trivially under a change of coordinates. By the second condition, $\nabla_b(Y^a Z_a) = \partial_b(Y^a Z_a)$. Then using the first condition,

$$(\nabla_b Y^a)Z_a + Y^a(\nabla_b Z_a) = (\partial_b Y^a)Z_a + (Y^a \partial_b Z_a).$$

Finally using Eq. 30, we find

$$Y^a(\nabla_b Z_a - \partial_b Z_a + \Gamma^c_{ba} Z_c) = 0.$$

This holds for arbitrary $Y^a$. Therefore,

$$\nabla_b Z_a = \partial_b Z_a - \Gamma^c_{ba} Z_c.$$

So we have learned how to compute the covariant derivative of covariant vectors.

This procedure can be repeated for more complicated tensors. For instance, for a type (2,2) tensor field $T^{ab}{}_{cd}$ we consider $\nabla_e(T^{ab}{}_{cd} X_a Y_b U^c W^d) = \partial_e(T^{ab}{}_{cd} X_a Y_b U^c W^d)$ for arbitrary $X, Y, U, W$ vectors to find (after much algebra)

$$\nabla_e T^{ab}{}_{cd} = \partial_e T^{ab}{}_{cd} + \Gamma^a_{ef} T^{fb}{}_{cd} + \Gamma^b_{ef} T^{af}{}_{cd} - \Gamma^f_{ec} T^{ab}{}_{fd} - \Gamma^f_{ed} T^{ab}{}_{cf}.$$

The general pattern is clear.

Let's consider the covariant derivative of some familiar tensors. First, the metric tensor. Using our rules,

$$\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd} = \partial_a g_{bc} - \Gamma_{cab} - \Gamma_{bac}.$$

Then using expressions for the Christoffel symbols in terms of the metric tensor, we find

$$\nabla_a g_{bc} = 0.$$

This result will greatly simplify calculations. Next, let's consider the Kronecker delta, which is a (1,1) tensor. Using our rules, one finds

$$\nabla_a \delta^b_c = 0.$$

**Example**: Simplify $g_{bc} \nabla_a R^{bc}$ where $R^{bc}$ is a tensor field. Using the above relations, we find $g_{bc} \nabla_a R^{bc} = \partial_a R$ where we note that $R \equiv R^a_{\ a}$ is a scalar field.

## 2.8 Geodesics in Spacetime

We will now take a short break from tensor mathematics since we already have enough to tell half of the story. The line element of a massive particle in special relativity, $(ds)^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2(d\tau)^2$, generalises naturally in general relativity as $(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2(d\tau)^2$. We see that the proper time $\tau$ is an affine parameter. Particles in general relativity under no external forces are taken to follow spacetime geodesics. So given a metric, since $\tau$ is affine, we have that a massive particle will satisfy the equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0.$$

The definitions we had before for timelike, spacelike, and null vectors generalise similarly by replacing $\eta_{\mu\nu} \to g_{\mu\nu}$. A photon's four-momentum will still be a null vector.

In this section, we will check that the above requirement, that particles follow spacetime geodesics, is a sensible one. In particular, we will check that the geodesic equation reduces to Newton's second law for a particle under a gravitational field in the limit where relativistic effects are small. Before doing this, we will give a short overview (or review) of Newtonian gravitation.

### 2.8.1 Newtonian Gravitation

We take $\rho(\mathbf{r})$ to give the mass-density of some distribution of matter. The gravitational potential $\Phi(\mathbf{r})$ is related to $\rho$ through Poisson's equation

$$\vec{\nabla}^2\Phi = 4\pi G\rho$$

where $G \approx 6.7Nm^2/kg^2$ is the gravitational constant. To avoid confusion with the covariant derivative, we will begin using $\vec{\nabla}$ for the gradient operator. For simplicity, we take the centre of mass of the massive object to be at the origin. We take the object to be spherically symmetric and localised in the sense that $\rho = 0$ everywhere beyond some distance $R$. With the boundary condition $\Phi \to 0$ as $r \to \infty$, one can solve Poisson's equation to find $\Phi(\mathbf{r}) = -\frac{GM}{r}$ for $r > R$ where $M$ is the mass of the object. A particle of mass $m$ (take $m \ll M$) placed in this field will move according to Newton's second law:

$$m\frac{d^2\mathbf{r}}{dt^2} = -m\vec{\nabla}\Phi.$$

### 2.8.2 Weak Field Limit of the Geodesic Equation

Given a metric we can compute the equations of motion for a particle. This is analogous to the above: given $\Phi$, we have $\frac{d^2\mathbf{r}}{dt^2} = -\vec{\nabla}\Phi$. We still need a generalisation of Poisson's equation. That is, given a mass-energy distribution, how do we compute the metric? This is the second part of the story, which we will look at soon.

We now look at the weak-field limit. We consider slow velocities:

$$\left|\frac{1}{c}\frac{dx^i}{dt}\right| \ll 1$$

for $i = 1, 2, 3$. We will start using indices from the middle of the alphabet $ijk\ldots$ to denote spatial variables. We also take the metric to be time-independent and nearly flat:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $|h_{\mu\nu}| \ll 1$.

Dividing $c^2(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu$ by $(dt)^2$, from the slow-velocity requirement, we have

$$c^2\left(\frac{d\tau}{dt}\right)^2 \approx g_{00}c^2 \approx c^2$$

and so $dt/d\tau \approx 1$ as we would expect (proper time should be close to coordinate time in this limit). Using this and again our condition of slow velocities, we have

$$\Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{dt}\frac{dx^\sigma}{dt}\left(\frac{dt}{d\tau}\right)^2 \approx c^2\Gamma^\mu_{00}.$$

Let's look at these components of the Christoffel symbol. Since the metric is taken to be time independent,

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}\left(\partial_0 g_{0\nu} + \partial_0 g_{0\nu} - \partial_\nu g_{00}\right) = -\frac{1}{2}g^{\mu\nu}\partial_\nu g_{00} \approx -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}.$$

Inserting this result into the geodesic equation, we have, in this limit,

$$\frac{d^2x^\mu}{d\tau^2} - \frac{1}{2}c^2\eta^{\mu\nu}\partial_\nu h_{00} = 0.$$

The $\mu = 0$ equation gives $\frac{d^2t}{d\tau^2} = 0$ which is consistent with what we found before. For the spatial components, we have

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}c^2\eta^{ij}\partial_j h_{00} = -\frac{1}{2}c^2\partial_i h_{00}.$$

Finally, using $dt/d\tau \approx 1$, we have

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}c^2\partial_i h_{00}$$

If we identify $h_{00} = \frac{2}{c^2}\Phi$, then this equation reproduces the Newtonian result: $\frac{d^2\mathbf{r}}{dt^2} = -\vec{\nabla}\Phi$.

> **Example:** Rindler coordinates. We can work out using our previous expression for Rindler coordinates (dropping the $B$ subscript) that
>
> $$(ds)^2 = c^2(dt)^2 - dx^2 - dy^2 - dz^2 = X^2\left(\frac{a}{c^2}\right)^2 c^2(dT)^2 - dX^2 - dY^2 - dZ^2$$
>
> where we put back in the other spatial dimensions as $Y = y$ and $Z = z$.
> From this, we can read off the Rindler metric, $g_{\mu\nu} = \mathrm{diag}(X^2\left(\frac{a}{c^2}\right)^2, -1, -1, -1)$. Now let's expand this metric about $\bar{X} = \frac{c^2}{a}$. Define $\xi$ through $X = \bar{X} + \xi$ and require $|\xi| \ll |\bar{X}|$. Then we have to first order in $\xi$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where
>
> $$h_{\mu\nu} = \mathrm{diag}(\frac{2a\xi}{c^2}, 0, 0, 0).$$
>
> We thus make the identification $\Phi = \frac{c^2}{2}h_{00} = a\xi$. This is the potential for a uniform Newtonian gravitational field (with gravitational acceleration $a$).

## 2.9 The Riemann Curvature Tensor

We will now develop the remaining machinery we need to describe the Einstein field equations. Covariant derivatives, unlike partial derivatives, do not generally commute. Using the rules for covariant differentiation, one finds that for a general contravariant vector field $X^a$ (exercise),

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) X^a = R^a_{\ bcd} X^b$$

where

$$R^a_{\ bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce} \Gamma^e_{bd} - \Gamma^a_{de} \Gamma^e_{bc}$$

is the **Riemann curvature tensor**.

The Riemann curvature tensor satisfies a number of symmetries:

$$\begin{aligned} R_{abcd} + R_{adbc} + R_{acdb} &= 0 \\ R_{abcd} = -R_{abdc} = -R_{bacd} &= R_{cdab} \end{aligned} \tag{31}$$

which are not all obvious from the above definition. We will later establish these symmetries by using locally flat coordinates. The above symmetries reduce the number of independent quantities of $R^a_{\ bcd}$ to $N^2(N^2-1)/12$ where $N$ is the dimension of the manifold. So, for spacetime, we have 20 independent quantities while for two-dimensions there is only one (the Gaussian curvature).

### 2.9.1 Parallel Transporting About Small Loops

The Riemann curvature tensor emerges naturally in the following context. Consider parallel transporting a vector $X^a$ about a small loop in the manifold $x^a(t)$ for $0 \le t \le T$ around a point $P$. We ask: by how much do the initial $X^a(0)$ and final $X^b(T)$ vectors differ?

Introduce $\xi^a$ through $x^a = (x^a)_P + \xi^a$ where $(x^a)_P$ denotes the coordinates of the point. Generally, any quantity inside $()_P$, will be taken to be evaluated at $P$. The parallel transport equation written in integral form is

$$X^a(t) = X^a(0) - \int_0^t dt' \Gamma^a_{bc}(t') X^b(t') \dot{x^c}(t').$$

From this we obtain

$$X^a(t) = X^a(0) - \int_0^t dt' \Gamma^a_{bc}(t') X^b(0) \dot{x^c}(t') + \int_0^t dt' \Gamma^a_{bc}(t') \dot{x^c}(t') \int_0^{t'} dt''\ \Gamma^b_{de}(t'') X^d(t'') \dot{x^e}(t'').$$

Everything is exact to this point. The above procedure can be repeated to generate a series expansion for $X^a(t)$. We seek an expression that is accurate to second order in distances from $P$. To this accuracy, we can use:

$$X^a(t) \approx X^a(0) - \int_0^t dt' \Gamma^a_{bc}(t') X^b(0) \dot{x^c}(t') + \int_0^t dt' \Gamma^a_{bc}(t') \dot{x^c}(t') \int_0^{t'} dt''\ \Gamma^b_{de}(t'') X^d(0) \dot{x^e}(t'').$$

Let's work on individual terms in this expression. Put $t = T$. To second order accuracy:

$$
\begin{aligned}
-\int_0^T dt' \Gamma^a_{bc}(t') X^b(0) \dot{x^c}(t') &\approx -\int_0^T dt' [(\Gamma^a_{bc})_P + (\partial_d \Gamma^a_{bc})_P \xi^d(t')] X^b(0) \dot{x^c}(t') \\
&= -\int_0^T dt' (\partial_d \Gamma^a_{bc})_P \xi^d(t') X^b(0) \dot{\xi^c}(t') \\
&= -(\partial_d \Gamma^a_{bc})_P X^b(0) \oint d\xi^c \xi^d
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T dt' \Gamma^a_{bc}(t') \dot{x^c}(t') \int_0^{t'} dt'' \, \Gamma^b_{de}(t'') X^d(0) \dot{x^e}(t'') &\approx (\Gamma^a_{bc})_P (\Gamma^b_{de})_P X^d(0) \int_0^T dt' \dot{x^c}(t') \int_0^{t'} dt'' \, \dot{x^e}(t'') \\
&= (\Gamma^a_{bc})_P (\Gamma^b_{de})_P X^d(0) \int_0^T dt' \dot{x^c}(t') (x^e(t') - x^e(0)) \\
&= (\Gamma^a_{bc})_P (\Gamma^b_{de})_P X^d(0) \int_0^T dt' \dot{x^c}(t') x^e(t') \\
&= (\Gamma^a_{bc})_P (\Gamma^b_{de})_P X^d(0) \oint d\xi^c \xi^e \\
&= (\Gamma^a_{ce})_P (\Gamma^e_{bd})_P X^b(0) \oint d\xi^c \xi^d.
\end{aligned}
$$

So we have

$$
\delta X^a \equiv X^a(T) - X^a(0) = (-\partial_d \Gamma^a_{bc} + \Gamma^a_{ce} \Gamma^e_{bd})_P \, X^b(0) \oint d\xi^c \xi^d.
$$

Now let's look at $f^{ab} \equiv \oint d\xi^a \xi^b$. Through an integration by parts, this can be rewritten as $f^{ab} = \frac{1}{2} \oint (d\xi^a \xi^b - d\xi^b \xi^a)$, from which we see that $f^{ab}$ is antisymmetric: $f^{ab} = -f^{ba}$. Because of this, we can antisymmetrise over $cd$ the quantity in parenthesis in the above expression. The curvature tensor appears! That is, we obtain

$$
\delta X^a = \frac{1}{2} X^b(0) \, (R^a{}_{bcd})_P \, f^{cd}.
$$

Now let's specify a particular curve. Let's hold constant all of our coordinates except for two, $x^{\bar{c}}$ and $x^{\bar{d}}$. Let's integrate counterclockwise along a rectangle centred at point $P$ in the $\bar{c}\bar{d}$-plane. Take the width and height of this rectangle to be $\Delta x^{\bar{c}}$ and $\Delta x^{\bar{d}}$ respectively. Then through Green's theorem, we obtain

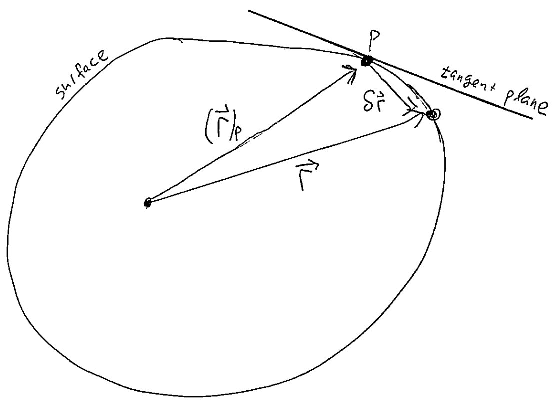
$$
f^{\bar{c}\bar{d}} = -f^{\bar{d}\bar{c}} = \frac{1}{2} \oint (d\xi^{\bar{c}} \xi^{\bar{d}} - d\xi^{\bar{d}} \xi^{\bar{c}}) = -\int d\xi^{\bar{c}} d\xi^{\bar{d}} = -\Delta x^{\bar{c}} \, \Delta x^{\bar{d}}
$$

where all other components of $f$ vanish. Finally, we obtain

$$
\delta X^a = -X^b(0) \left( R^a{}_{b\bar{c}\bar{d}} \right)_P \Delta x^{\bar{c}} \, \Delta x^{\bar{d}}
$$

(no summation over $\bar{c}$ and $\bar{d}$).

Thus the Riemann curvatures tensor tells us how the initial and final vectors will differ. We call a manifold **flat** if the Riemann curvature tensor vanishes everywhere. Otherwise, we call the manifold **curved**.




## 2.10 Locally Flat Coordinates

Our aim is to construct a coordinate system in which at a particular point $P$, the Christoffel symbols vanish and the metric tensor is the identity matrix (or the Minkowski metric for spacetime). Motivation for this includes:

- Locally flat coordinates will help us establish important symmetries of the Riemann curvature tensor
- Locally flat coordinates are demanded by the equivalence principle. That is, for general relativity to locally reduce to special relativity, in light of the geodesic equation, we must be able find a frame in which the Christoffel symbols vanish at an event. In the context of spacetime, locally flat coordinates are often referred to as **locally inertial frames** or **freely falling frames**.

As we have done a couple of times before, we will consider surfaces to develop our intuition. Following the picture, we introduce the $y$-coordinates through

$$y^a \equiv \delta\mathbf{r} \cdot (\mathbf{e}^a)_P$$

where $\delta\mathbf{r} = \mathbf{r} - (\mathbf{r})_P$. Expanding $\mathbf{r}$ about $P$ we obtain $\delta\mathbf{r} \approx (\mathbf{e}_a)_P \delta x^a + \frac{1}{2}(\partial_b \mathbf{e}_a)_P \delta x^a \, \delta x^b$ where $\delta x^a = x^a - (x^a)_P$. Then to second order in distances from $P$ we have

$$y^a = \delta x^a + \frac{1}{2} \, (\Gamma^a_{bc})_P \, \delta x^b \delta x^c.$$

Inverting this equation we find

$$\delta x^a = y^a - \frac{1}{2} \, (\Gamma^a_{bc})_P \, y^b y^c$$

which again is second-order accurate. From this we can determine an expression for the Jacobian which is accurate at first order:

$$\frac{\partial x^a}{\partial y^b} = \delta^a_b - (\Gamma^a_{bc})_P \, y^c.$$

We use this to find the metric in the $y$-coordinate system $\tilde{g}_{ab}$ (we will use tildes to denote quantities in the $y$-coordinate system) as

$$\tilde{g}_{ab} = \frac{\partial x^c}{\partial y^a}\frac{\partial x^d}{\partial y^b} g_{cd} \approx g_{ab} - (\Gamma_{abc})_P y^c - (\Gamma_{bac})_P y^c = g_{ab} - (\partial_c g_{ab})_P y^c$$

which is accurate at first order. In the second equality, we have used the definition of the Christoffel symbols. From this we find

$$\left(\frac{\partial \tilde{g}_{ab}}{\partial y^c}\right)_P = \left(\frac{\partial x^d}{\partial y^c}\frac{\partial g_{ab}}{\partial x^d}\right)_P - (\partial_c g_{ab})_P = 0.$$

Note that at point $P$, $\frac{\partial x^d}{\partial y^c} = \delta^d_c$. From the definition of the Christoffel symbols, from this we conclude that all Christoffel symbols in the $y$-coordinate system will also vanish at $P$:

$$\left(\tilde{\Gamma}^a_{bc}\right)_P = 0.$$

We can go further. Let's introduce a $z$-coordinate system which is related to the $y$-coordinate system as

$$y^a = S^a_{\ b} z^b.$$

In this, we take the components of $S$ to be constants. We will use double-tildes to denote quantities in the $z$-coordinate system. For the metric tensor,

$$\tilde{\tilde{g}}_{ab} = S^c_{\ a} S^d_{\ b} \tilde{g}_{cd}.$$

We wish to choose $S$ such that $\tilde{\tilde{g}}$ becomes the identity matrix. Let's proceed by writing matrix equations. Since $\tilde{g}$ is a symmetric metric, we can diagonalise it with an orthogonal matrix: $O^T \tilde{g} O = D$ where $D$ is a diagonal matrix composed of the eigenvalues of $\tilde{g}$. For Riemannian manifolds, $\tilde{g}$ will have all positive eigenvalues (while for spacetime it will have one positive and three negative). Taking all positive eigenvalues, we will have $D^{-1/2} O^T \tilde{g} O D^{-1/2} = \mathbb{1}$. We thus choose $S = OD^{-1/2}$. Through the transformation rule for the Christoffel symbols (practice problems), we see that the Christoffel symbols will also vanish in the $z$-coordinate system.

In summary (dropping tildes and going back to $x^a$ for coordinates), for a manifold with positive definite metric, we can find a coordinate system which is flat about a point $P$ in the sense that

$$g_{ab} \approx \mathbb{1}_{ab} + \frac{1}{2}(\partial_c \partial_d g_{ab})_P \delta x^c \delta x^d$$

and

$$(\Gamma^a_{bc})_P = 0.$$

Very similar arguments can be given for spacetime. Here a locally flat coordinate system will give

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{2}(\partial_\sigma \partial_\rho g_{\mu\nu})_P \delta x^\sigma \delta x^\rho$$

and

$$(\Gamma^\mu_{\nu\sigma})_P = 0.$$

### 2.10.1 Symmetries of the Riemann Tensor

In locally flat coordinates, the Riemann curvature tensor is

$$(R^a_{\ bcd})_P = (\partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{cb})_P.$$

Recalling that first-order derivatives of the metric tensor vanish at point $P$, we find (check),

$$(R_{abcd})_P = \frac{1}{2}(\partial_b \partial_c g_{ad} + \partial_a \partial_d g_{bc} - \partial_b \partial_d g_{ac} - \partial_a \partial_c g_{bd})_P.$$

In comparison with the general expression of the curvature tensor in terms of the metric tensor, the above expression is vastly simplified. From this we can directly verify the symmetries of the Riemann curvature tensor written earlier at point $P$ in our locally flat coordinate system.

However, since $R^a_{\ bcd}$ is a tensor, it is enough to establish the symmetries Eq. 31 in a single coordinate system. For instance, we have for locally flat coordinates, $(R_{abcd})_P = -(R_{bacd})_P$. We can use Jacobian matrices to transform to another coordinate system (which need not be locally flat) and find $(R_{a'b'c'd'})_P = -(R_{b'a'c'd'})_P$. The final point to note is that $P$ is arbitrary.

## 2.11 Tensors Derived From the Riemann Curvature Tensor and the Einstein Field Equation

For later use, we will now introduce tensors which follow from the Riemann curvature tensor. The **Ricci tensor** $R_{ab}$ is defined as

$$R_{ab} = R^c_{\ acb}.$$

From the symmetries of the Riemann tensor (how?), we see that $R_{ab}$ is symmetric: $R_{ab} = R_{ba}$. The **Ricci scalar** $R$ follows from the Ricci tensor as

$$R = R^a_{\ a} = g^{ab} R_{ba}.$$

Finally, the **Einstein tensor** $G_{ab}$ is defined as

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R.$$

The **Einstein field equations** are

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where $T_{\mu\nu}$ is the **energy-momentum tensor** which we will look at soon. For a particular energy momentum tensor, we solve the Einstein field equations to determine the metric. The analogy with Newtonian gravity is: given a mass distribution $\rho(\mathbf{r})$, we solve Poisson's equation to determine the gravitational potential. In due course we will verify that Einstein's field equations reduce to Poisson's equation in the non-relativistic limit. Then with this metric, we can solve the geodesic equation to determine the trajectory of a particle in this field. J. A. Wheeler succinctly described this process as "Spacetime tells matter how to move; matter tells spacetime how to curve."

## 2.12 The Bianchi Identity and the Contracted Bianchi Identity

We will need a couple more relations for the Riemann curvature tensor. The Riemann curvature tensor satisfies the **Bianchi identity**:

$$\nabla_e R^a{}_{bcd} + \nabla_d R^a{}_{bec} + \nabla_c R^a{}_{bde} = 0. \tag{32}$$

Like the symmetries of Riemann tensor established earlier, the Bianchi identity is easiest to establish by using locally flat coordinates. For such a coordinate system (since the Christoffel symbols vanish)

$$(\nabla_e R^a{}_{bcd})_P = (\partial_e R^a{}_{bcd})_P = (\partial_e \partial_c \Gamma^a_{bd} - \partial_e \partial_d \Gamma^a_{cb})_P.$$

From this, we can directly verify Eq. 32.

From the Bianchi identity, we can derive the **contracted Bianchi identity** which reads

$$\nabla_a G^{ab} = 0. \tag{33}$$

First, contract $a$ and $c$ in the Bianchi identity to find

$$\nabla_e R_{bd} - \nabla_d R_{be} + \nabla_a R^a{}_{bde} = 0.$$

For the second term, we have used Eq. 31: $\nabla_d R^a{}_{bea} = \nabla_d(g^{af} R_{fbea}) = -\nabla_d(g^{af} R_{fbae}) = -\nabla_d R^a{}_{bae} = -\nabla_d R_{be}$. Since $\nabla_c g^{ab} = 0$, for a tensor field $X^a$ we have $g^{ac}\nabla_b X_a = \nabla_b X^c$. With this in mind, we can "raise indices behind the covariant derivative operator". Raise $b$ and contract with $d$ to find

$$\nabla_e R - \nabla_b R^b{}_e + \nabla_a R^{ab}{}_{be} = 0.$$

Let's work on the last term on the left-hand side:

$$\begin{aligned}\nabla_a R^{ab}{}_{be} &= g^{af} g^{bh} \nabla_a R_{fhbe} = -g^{af} g^{bh} \nabla_a R_{hfbe} \\ &= -g^{af} \nabla_a R^b{}_{fbe} = -g^{af} \nabla_a R_{fe} = -\nabla_a R^a{}_e.\end{aligned}$$

So the above becomes

$$\nabla_e R - 2\nabla_b R^b{}_e = 0.$$

Relabelling indices,

$$\nabla_a (R^a_b - \frac{1}{2}\delta^a_b R) = 0.$$

Finally, raising $b$ we obtain $\nabla_a(R^{ab} - \frac{1}{2} g^{ab} R) = 0$ which is Eq. 33.

## 2.13 Relativistic Fluids and the Energy-Momentum Tensor

The remaining piece of Einstein's field equation which we need to address is the energy-momentum tensor $T^{\mu\nu}$. For most situations in this course, we will take the energy-momentum tensor for a perfect fluid which reads

$$T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu - pg^{\mu\nu}.$$

In our discussion of special relativity, we introduced the four-velocity of a particle. We now generalise $u^\mu$ to a vector field so that $(u^\mu)_P$ describes the average four-velocity of particles at point $P$ in spacetime. We take $\rho$ to be the proper density (mass density measured in a momentarily co-moving frame) of the matter. The pressure, which also can depend on position is $p$. Both $\rho$ and $p$ are scalars.

We take the equations of motion of the relativistic fluid to be given by

$$\nabla_\mu T^{\mu\nu} = 0.$$

Note the consistency between this and the field equations as a result of the contracted Bianchi identity. We will follow our usual practice of showing that these equations of motion reduce to something sensible in the non-relativistic limit. We will now work on writing these equations of motion in a more intuitive way.

First some useful relations. Since $u^\mu u_\mu = c^2$, we have that $u^\nu \nabla_\mu u_\nu = u_\nu \nabla_\mu u^\nu = 0$. For our perfect fluid we also have $T^{\mu\nu}u_\nu = \rho c^2 u^\mu$. Multiply our equations of motion by $u_\nu$ (and sum as usual over $\nu$):

$$0 = u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu(T^{\mu\nu}u_\nu) - T^{\mu\nu}\nabla_\mu u_\nu = \nabla_\mu(c^2\rho u^\mu) + p\nabla_\mu u^\mu.$$

So

$$\nabla_\mu(\rho u^\mu) + \frac{p}{c^2}\nabla_\mu u^\mu = 0. \tag{34}$$

To obtain our other fluid equation, we use the projection tensor

$$P^\mu_\nu = \delta^\mu_\nu - \frac{1}{c^2}u^\mu u_\nu.$$

Note that $P^\mu_\nu u_\mu = 0$ and $P^\mu_\sigma P^\sigma_\nu = P^\mu_\nu$. This enables us to look at components of the fluid equation orthogonal to the four-velocity field. With this,

$$\begin{aligned} 0 = P^\sigma_\nu \nabla_\mu T^{\mu\nu} &= P^\sigma_\nu \left((\rho + p/c^2)u^\mu \nabla_\mu u^\nu - g^{\mu\nu}\partial_\mu p\right) \\ &= (\rho + p/c^2)u^\mu \nabla_\mu u^\sigma - \partial_\mu p(g^{\mu\sigma} - u^\mu u^\sigma/c^2). \end{aligned}$$

Relabelling indices,

$$(\rho + p/c^2)u^\mu \nabla_\mu u^\nu = \partial_\mu p(g^{\mu\nu} - u^\mu u^\nu/c^2). \tag{35}$$

Eqns. 34, 35 give a more explicit description of the equations of motion of relativistic fluids. They reduce to the continuity and Euler equations from fluid mechanics[11] in the non-relativistic limit. In particular, we consider the limit where $\rho \gg p/c^2$, $g_{\mu\nu} = \eta_{\mu\nu}$, and $u^\mu =$

[11] A description of classical ideal fluids is outside the scope of this course. If you haven't seen these equations, I recommend *The Feynman Lectures on Physics* for an accessible introduction.

$(c, \mathbf{v})$ where $\mathbf{v}$ is the velocity field of the fluid. In this limit, Eq. 34 becomes (exercise)

$$\partial_t \rho = -\vec{\nabla} \cdot (\rho \mathbf{v})$$

which is the continuity equation. The $\mu = 1, 2, 3$ components of Eq. 35 become (exercise)

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \vec{\nabla})\mathbf{v}) = -\vec{\nabla} p$$

which is the Euler equation.

**Example:** Dust in Special Relativity. A relativistic fluid in the absence of pressure is referred to as **dust**. In this example we will look at dust in special relativity where $g_{\mu\nu} = \eta_{\mu\nu}$ and hence $\nabla_\mu = \partial_\mu$. Using $u^\mu = \gamma(c, \mathbf{v})$ and writing out the spatial, temporal, and mixed components of the energy-momentum tensor impels us to introduce $\tilde{\rho} = \gamma^2 \rho$. We then have

$$\begin{aligned} T^{00} &= \tilde{\rho} c^2 \\ T^{0i} = T^{i0} &= \tilde{\rho} c v_i \\ T^{ij} &= \tilde{\rho} v_i v_j \end{aligned}$$

where $\mathbf{v}$ is the coordinate velocity (or three-velocity) field. Note that since $v_i$ is not a tensor we have the unusual placement of indices. $\tilde{\rho}$ is in fact the frame-dependent density of the fluid. The combined effect of length contraction and relativistic enhancement of mass provides the factor of $\gamma^2$. It can be verified that the equations of motion $\partial_\mu T^{\mu\nu} = 0$ can be written, without approximation, as (exercise)

$$\partial_t \tilde{\rho} = -\vec{\nabla} \cdot (\tilde{\rho} \mathbf{v})$$

and

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \vec{\nabla})\mathbf{v} = 0.$$

Unlike Eqns. 34 and 35, the above are not tensor equations. These equations provide perhaps the clearest connection with classical ideal fluids.

## 2.14 Weak Field Limit of the Einstein Field Equations

We need to show that the Einstein Field Equations reduce to Poisson's equation in the non-relativistic limit. To do this, it is helpful to write the field equations in an alternative way. Let $\kappa = \frac{8\pi G}{c^4}$. Then the field equations are

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu}.$$

Lowering $\nu$ and contracting with $\mu$ and noting that $g^\mu_\mu = \delta^\mu_\mu = 4$, we have

$$R = -\kappa T$$

where $T \equiv T^{\mu}_{\mu}$. Putting this back into the field equations we find

$$R^{\mu\nu} = \kappa(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T) \tag{36}$$

or, lowering indices,

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{37}$$

which is an alternative way of writing the field equations. As a special case when $T_{\mu\nu} = 0$ this reduces to

$$R_{\mu\nu} = 0$$

which are the **vacuum field equations**.

We consider the limiting case of small velocities $v/c \ll 1$. The metric is taken to be time independent and nearly flat: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$. We further require that $p/c^2 \ll \rho$. With these requirements, the 00 component of the energy momentum tensor will be dominant. We have $T_{00} \approx T^{00} \approx \rho c^2$ and $T = T^{\mu}_{\mu} = (\rho + p/c^2)c^2 - 4p \approx \rho c^2$. With this, we find

$$R_{00} = \kappa(T_{00} - \frac{1}{2}g_{00}T) \approx \frac{\kappa}{2}\rho c^2. \tag{38}$$

Now let's consider the 00 component of the Ricci tensor in this limit. To first order in $h_{\mu\nu}$,

$$R^{\mu}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho}$$

and so to this order,

$$R_{00} = R^{\mu}_{0\mu 0} = \partial_\mu \Gamma^{\mu}_{00} - \partial_0 \Gamma^{\mu}_{0\mu} = \partial_\mu \Gamma^{\mu}_{00}$$

(recall we are taking the metric to be time-independent). Recalling some results from 2.8.2, we have to first order in $h_{\mu\nu}$

$$R_{00} = \partial_\mu \Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu h_{00} = \frac{1}{2}\vec{\nabla}^2 h_{00} = \frac{1}{c^2}\vec{\nabla}^2\Phi.$$

Inserting this into Eq. 38 we find Poisson's equation:

$$\vec{\nabla}^2\Phi = 4\pi G\rho.$$

# 3 General Relativity Applied

## 3.1 The Schwarzschild Geometry

It is now time to put the theory of general relativity uncovered during the last chapter to work.

We seek to find the spacetime metric corresponding to a spherically-symmetric massive body of total mass $M$. We will solve the vacuum field equations

$$R_{\mu\nu} = 0$$

which are valid outside of the massive body's interior. We take the metric to be spherically symmetric, time-independent, and asymptotically flat. The following ansatz satisfies these three conditions:

$$(ds)^2 = c^2 e^{2A(r)}(dt)^2 - e^{2B(r)}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2(\theta)(d\phi)^2.$$

where we require $A, B \to 0$ as $r \to \infty$ so that the metric reduces to the Minkowski metric at large distances from the body.[12] From this, we can compute the Ricci tensor[13]. One finds the non-vanishing components are

$$\begin{aligned} R_{00} &= e^{2(A-B)}(A'^2 + \frac{2}{r}A' - A'B' + A'') \\ R_{11} &= -A'^2 + \frac{2}{r}B' + A'B' - A'' \\ R_{22} &= e^{-2B}(-1 + e^{2B} - rA' + rB') \\ R_{33} &= \sin^2(\theta)R_{22}. \end{aligned}$$

In the above, primes denote derivatives with respect to $r$. Combining the equations for $R_{00}$ and $R_{11}$ gives $A' + B' = 0$. Using our boundary condition then gives $A + B = 0$. Inserting $A = -B$ into the equation for $R_{22}$ gives

$$2rA' + 1 = e^{-2A}.$$

Noting that we can write the LHS of this equation as $e^{-2A}\frac{d}{dr}(re^{2A})$, we can immediately solve to find $re^{2A} = r + k$ where $k$ is a constant of integration. With this we have

$$(ds)^2 = c^2(1 + \frac{k}{r})(dt)^2 - \frac{1}{1 + \frac{k}{r}}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2(\theta)(d\phi)^2.$$

How do we determine $k$? We require that the metric reduces to the correct weak-field limit for a gravitational potential $\Phi(r) = -\frac{GM}{r}$. Using results from the previous chapter, we require

$$g_{00} = 1 + \frac{k}{r} = 1 + h_{00} = 1 + \frac{2}{c^2}\Phi(r).$$

We then find $k = -2MG/c^2$. With this value for $k$, we have found the **Schwarzschild metric**.

[12] Actually, one can arrive at this metric by starting with a general spherically symmetric metric and applying certain variable changes.

[13] Read off the metric from $(ds)^2$. From the metric, compute the Christoffel symbols. From the Christoffel symbols compute the Riemann tensor. From the Riemann tensor, compute the Ricci tensor. This is a straightforward but computationally intensive task and is perhaps best left to a symbolic mathematics package like *Maple* or *Mathematica*.

## 3.2 Classical Kepler Motion

Before considering motion under the Schwarzschild metric, we will first, for comparison and to set some notation, consider the Newtonian motion of a particle under a gravitational potential $\Phi(r) = -\frac{MG}{r}$. We take the mass of the particle $m$ to be much smaller than $M$ so that we can avoid working with "reduced masses". That is, we can take the position of the larger mass $M$ to be fixed in space. Newton's second law reads

$$\frac{d^2\mathbf{r}}{dt^2} = -\overrightarrow{\nabla}\Phi = -\frac{MG}{r^2}\hat{\mathbf{r}}.$$

The angular momentum $\mathbf{L} = \mathbf{r} \times m\frac{d\mathbf{r}}{dt}$ is conserved:

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times m\frac{d\mathbf{r}}{dt} + \mathbf{r} \times \left(\frac{-mG}{r^2}\hat{\mathbf{r}}\right) = 0.$$

Since $\mathbf{L}$ is conserved and $\mathbf{r} \cdot \mathbf{L} = 0$, the motion will be confined to a plane. We use polar coordinates $(r, \phi)$ for this plane.

From classical mechanics, $\frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\hat{\boldsymbol{\phi}}$ and $\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\hat{\mathbf{r}}$. Using this,

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2}{dt^2}(r\hat{\mathbf{r}}) = (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\hat{\boldsymbol{\phi}} = \frac{-MG}{r^2}\hat{\mathbf{r}}.$$

So

$$\ddot{r} - r\dot{\phi}^2 = -\frac{MG}{r^2}, \text{ and } \frac{d}{dt}(r^2\dot{\phi}) = 0. \tag{39}$$

It is useful to introduce $u = 1/r$ and regard $r$ as a function of $\phi$ (compare with the computation of geodesics on the unit sphere). Let $h = r^2\dot{\phi}$ which is a constant of motion by the above equations. Then

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\phi}\dot{\phi} = -h\frac{du}{d\phi}$$

and

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\phi^2}\dot{\phi} = -h^2u^2\frac{d^2u}{d\phi^2}$$

Inserting this into Eq. 39 we find

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}$$

which is the **Binet equation**. It has the solution

$$u(\phi) = \frac{GM}{h^2}(1 + e\cos(\phi - \phi_0)).$$

Where $\phi_0$ and $e$ are constants. $e$ is called the **eccentricity**. We obtain circular, elliptic, parabolic, and hyperbolic motion for $e = 0$, $0 < |e| < 1$, $|e| = 1$, and $|e| > 1$ respectively. For instance, for $\phi_0 = 0$ and $|e| \neq 1$, we can write this solution as

$$(1 - e^2)\left(x + \frac{Re}{1 - e^2}\right)^2 + y^2 = \frac{R^2}{1 - e^2}$$

where $R = \frac{h^2}{GM}$.

## 3.3 Precision of the Perihelion

We will now consider the analogous problem in general relativity, of a massive particle in orbit.

It is convention to define $m = \frac{MG}{c^2}$. This quantity has units of length. With this, the Schwarzschild line element is

$$(ds)^2 = \left(1 - \frac{2m}{r}\right) c^2(dt)^2 - \left(1 - \frac{2m}{r}\right)^{-1} (dr)^2 - r^2(d\Omega)^2$$

where $(d\Omega)^2 = \sin^2(\theta)(d\phi)^2 + (d\theta)^2$. This is related to the proper time differential in the usual way for massive particles: $(ds)^2 = c^2(d\tau)^2$. Since $\tau$ is an affine parameter, the geodesic equation will be given by the Euler-Lagrange equations of $K = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$.

Let's first search for conserved quantities. Since the metric does not explicitly depend on $\phi$ or $t$, the corresponding Euler-Lagrange equations gives the conserved quantities

$$k = \left(1 - \frac{2m}{r}\right)\dot{t}$$
$$h = r^2 \sin^2(\theta)\dot{\phi}.$$

Additionally, $K = \frac{1}{2}\left(\frac{ds}{d\tau}\right)^2 = \frac{1}{2}c^2$ itself is conserved. As in the previous section, we consider planar motion and set $\theta = \pi/2$. Inserting our conserved quantities into $K$ and reintroducing $u = 1/r$ (see previous section), we find

$$c^2k^2 - h^2\left(\frac{du}{d\phi}\right)^2 = (c^2 + h^2u^2)(1 - 2mu).$$

Next we differentiate the above with respect to $\phi$ to find

$$\frac{d^2u}{d\phi^2} + u = \frac{c^2m}{h^2} + 3mu^2.$$

This is the **relativistic Binet equation**. The difference between this and the Binet equation from the previous section is the last term on the right-hand side, which makes the current equation a non-linear ode. Using the mass of the sun for $M$, we find $m = MG/c^2 \approx 1500$ m. This is well within the radius of the sun. Therefore, for typical planetary motion, $um = m/r$ will be a small number and the last term on the right-hand side of the relativistic Binet equation can be viewed as a small correction. Its role in solar system motion is typically small but sometimes measurable. This term will be largest for planets close to the sun.

In the following we will consider nearly circular motion and write $u = u_0 + \delta u$ where $u_0$ is a constant and $\delta u \ll u_0$. Putting this into Binet's equation, we have to zeroth and first order in $\delta u$,

$$u_0 = \frac{c^2m}{h^2} + 3mu_0^2$$
$$\frac{d^2\delta u}{d\phi^2} + \delta u = 6mu_0\delta u.$$

Solving the second equation, $\delta u = A\cos(\sqrt{1-6mu_0}(\phi-\phi_0))$ where $A$ and $\phi_0$ are constants. So,

$$u = u_0 + \delta u = u_0[1 + e\cos(\sqrt{1-6mu_0}\phi)]$$

where we have put $\phi_0 = 0$ for convenience.

The point of closest approach to the sun is referred to as the **perihelion**. Unlike Kepler motion, the above predicts that the perihelion will precess. The observation of this effect for the planet Mercury was one of the earliest observational tests of general relativity.

Let's take $e > 0$. Then the values of $\phi$ corresponding to perihelia are $\phi_n = 2\pi n/\sqrt{1-6mu_0} \approx 2\pi n(1+3mu_0)$ for integer $n$. From this we see that after every orbit, the perihelion will advance by

$$\Delta\phi = 6\pi m u_0.$$

Putting in approximate numbers for Mercury (89 days per orbit, $1/u_0 \approx 5.8 \times 10^{10}$ m), we find that after a century its perihelion will advance by $42$ arcseconds ($60^2$ arcseconds = $1°$). The observational value is $43.1 \pm .5$ arcseconds.[14]

## 3.4 General Orbits in the Schwarzschild Geometry

We will now broaden our discussion to include the motion of photons. From our discussion of photons in special relativity, we have the result $(ds)^2 = \eta_{\mu\nu}dx^\mu dx^\nu = (c^2 - v^2)(dt)^2 = 0$ from the first postulate. This result continues to hold in general relativity where $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$. We conclude this by appealing to locally flat coordinates. That is, we can find a coordinate system in which $g_{\mu\nu}$ locally reduces to $\eta_{\mu\nu}$. By the equivalence principle, we can find a frame in which special relativity locally holds. In this frame $(ds)^2 = 0$. However, since $(ds)^2$ is a scalar quantity, this must hold in all frames. Proper time is ill defined for photons. We therefore need to choose a different affine parameter, which we call $w$, with which to describe their motion.

In order to not produce fairly similar derivations for light and massive particles we write the following:

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \alpha c^2$$

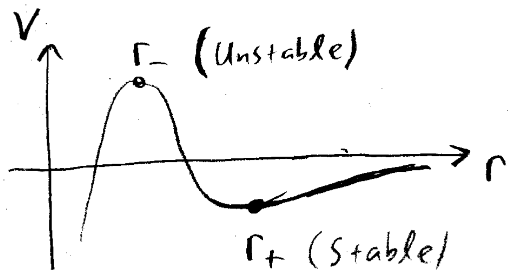
where $\alpha = 0$ and $\dot{x}^\mu = dx^\mu/dw$ for light while $\alpha = 1$ and $\dot{x}^\mu = dx^\mu/d\tau$ for massive particles. As before, for geodesic motion, we have the conserved quantities $h = r^2\dot{\phi}$ and $k = (1-2m/r)\dot{t}$ where we are still restricting to planar motion $\theta = \pi/2$. Inserting these into the above, we obtain

$$\frac{1}{2}\dot{r}^2 + V(r) = E$$

where $E = c^2(k^2-\alpha)/2$ and

$$V(r) = -\frac{mc^2}{r}\alpha + \frac{h^2}{2r^2} - \frac{mh^2}{r^3}.$$

[14] Actually, the perihelion of Mercury will also precess according to Newton's laws. This is the result of gravitational fields from sources other than those of Mercury and the sun. More precisely, the amount of *additional* advancement (not accounted for by Newton's laws) is $(43.1 \pm .5)$ arcseconds per century.




r

Taking the derivative of the above with respect to the affine parameter we have

$$\ddot{r} = -\partial_r V.$$

The above equations describe a Newtonian particle of mass $m$ in a potential $V$. Thus the advantage of the current formulation is that we may now use all of our intuition from Newtonian physics to understand the dynamics in at least a qualitative way. For a massive particle ($\alpha = 1$), we have

$$V(r) = -\frac{mc^2}{r} + \frac{h^2}{2r^2} - \frac{mh^2}{r^3}.$$

This will have a maximum and minimum when $\frac{h}{mc} > \sqrt{12}$ (see picture) occurring at $r_+$ and $r_-$ where

$$r_\pm = \frac{h^2}{2mc^2}\left(1 \pm \sqrt{1 - 12\left(\frac{mc}{h}\right)^2}\right).$$

This can be contrasted with the classical problem which can have stable orbits for $h \neq 0$. It can be seen that circular orbits at $r_+$ are stable while those at $r_-$ are unstable. That is, any small perturbation added to the circular orbit at $r_-$ generically will grow.

We now turn to the motion of light. For this we have

$$V(r) = \frac{h^2}{2r^2} - \frac{mh^2}{r^3}.$$

This potential has a maximum at $r = 3m$ which corresponds to an unstable circular orbit.

> **Example:** Proper time for orbits. Suppose we have a satellite in a circular (geodesic) orbit at radius $r$ about a planet. What is the proper time duration $(\Delta\tau)_{\text{orb}}$ for a complete orbit as measured by a passenger aboard the satellite? A second observer stays at a fixed location in the Schwarzschild coordinate system at $r$ (using, say, a rocket pack). What time does this observer measure $(\Delta\tau)_{\text{obs}}$ for an orbit of the satellite?
>
> Let's first try to find $(\Delta\tau)_{\text{orb}}$. For circular orbit, we have $u = 1/r = \text{const}$. Thus the Binet equation becomes
>
> $$u = \frac{c^2 m}{h^2} + 3mu^2.$$

We use $h = r^2\dot{\phi}$ to find

$$\dot{\phi}^2 = \frac{mc^2u^3}{1-3mu} = \frac{GM}{r^3(1-\frac{3m}{r})}.$$

For simplicity, we take the orbit to be counter-clockwise (so $\dot{\phi}$ is positive). Then can directly integrate to find

$$(\Delta\tau)_{\text{orb}} = 2\pi\sqrt{\frac{r^3(1-3m/r)}{GM}}.$$

Next, let's find the coordinate time duration $\Delta t$ for a complete orbit. We use $c^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ and put $\dot{r} = 0$ to find

$$c^2 = (1-2m/r)c^2\dot{t}^2 - r^2\dot{\phi}^2.$$

Inserting the above result for $\dot{\phi}$, we find

$$1 = \dot{t}^2(1-3m/r).$$

This can be directly integrated to give

$$\Delta t = \frac{1}{\sqrt{1-3m/r}}(\Delta\tau)_{\text{orb}} = 2\pi\sqrt{\frac{r^3}{GM}}.$$

In this, we have taken the positive square root so that this reduces to the correct result in the $m \to 0$ limit. Incidentally, this is the same result one obtains in Newtonian physics (Kepler's third law).

Now for $(\Delta\tau)_{\text{obs}}$. This observer *does not* follow a geodesic. Thus we cannot use the geodesic equation to describe his motion. However, we still may use $c^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ (our derivation of this said nothing of geodesics). Since this observer is at a fixed location, we have[15]

$$c^2 = (1-2m/r)c^2\dot{t}^2.$$

From this we see that

$$(\Delta\tau)_{\text{obs}} = \sqrt{1-2m/r}\,\Delta t = 2\pi\sqrt{\frac{r^3(1-2m/r)}{GM}}.$$

Putting everything together, we see that $(\Delta\tau)_{\text{orb}} < (\Delta\tau)_{\text{obs}} < \Delta t$.

[15] This relation suggests that people living at lower elevations will age slower. This effect was in fact measured by researchers at NIST using atomic clocks `http://www.nist.gov/public_affairs/releases/aluminum-atomic-clock_092310.cfm`

## 3.5 Bending of Light

We saw that light can undergo an unstable circular orbit at $r = 3m$. Now we consider the case where $r \gg m$ (recall for the sun, $m$ is only about 1.5 km). The derivation of Binet's equation for photons is similar to the derivation in Sec. 3.3. For this one finds (exercise)

$$u'' + u = 3mu^2.$$

The right-hand side of this equation is small for $r \gg m$. Let $u_0 = \bar{u}\sin(\phi)$. This corresponds to a straight line parallel to the $x$-axis with $y = 1/\bar{u}$. It satisfies the above Binet equation when $m = 0$.

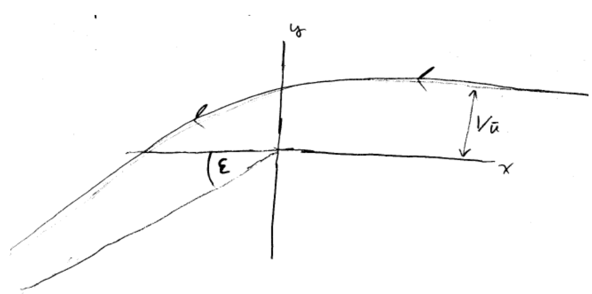
We now seek approximate solutions of the Binet equation for photons when $m\bar{u} \ll 1$. We take $u = u_0 + \delta u$ where $\delta u$ is regarded as small. Inserting this into the Binet equation, we find to zeroth order

$$u_0'' + u_0 = 0$$

while to first order

$$\delta u'' + \delta u = 3mu_0^2 = 3m\bar{u}^2 \sin^2(\phi).$$

We are assuming that $\delta u$ is on the order of $m\bar{u}^2$ (which we can verify later).




The second differential equation has the solution

$$\delta u = m\bar{u}^2(1 + \cos^2\phi + A\cos\phi + B\sin\phi).$$

where $A$ and $B$ are constants. So we have

$$u = \bar{u}\sin(\phi) + m\bar{u}^2(1 + \cos^2\phi + A\cos\phi + B\sin\phi).$$

We consider the photon trajectory shown in the picture. We impose the condition that at very early times, the photon travels parallel to the $x$-axis with $y = 1/\bar{u}$.

Therefore when $\phi = 0$, we require $u = u_0 + \delta u = 1/r = 0$. Using our solution, we find that this requires $A = -2$. Therefore,

$$u = \bar{u}\sin(\phi) + m\bar{u}^2(4\sin^4(\phi/2) + B\sin\phi).$$

Long before the light approaches $M$, we also require that the trajectory is approximately parallel to the $x$-axis with $y = 1/\bar{u}$. This requires $B = 0$. So with these conditions, we have

$$u = \bar{u}\sin(\phi) + 4m\bar{u}^2\sin^4(\phi/2).$$

Now we consider the deflection of light described by this equation. That is, we want to determine $\epsilon$ for which $u = 0$. Putting $u = 0$ and $\phi = \pi + \epsilon$ in the above equation, we find

$$0 = \bar{u}\sin(\pi + \epsilon) + m\bar{u}^2 4 \sin^4\left(\frac{\pi + \epsilon}{2}\right) \approx -\bar{u}\epsilon + m\bar{u}^2 4.$$

So to lowest order in $m\bar{u}$,

$$\epsilon = 4m\bar{u}.$$

This effect, observed during a solar eclipse in 1919 by Eddington and collaborators, provided an early verification of general relativity.

## 3.6 Spectral Shift

In our discussion of special relativity, we saw that the energy of photons depends on the reference frame in which they are observed. We will now consider how gravity affects the energy of photons.

We will derive our main result by carrying over a trick we learned back in our discussion of special relativity. Take an observer with four-velocity $(u^\mu)_{\text{obs}}$ and a photon with four-velocity $p^\mu$. Suppose that the photon barely misses the observer at a particular spacetime point. At this event, the observer assigns energy $h\nu = (u^\mu)_{\text{obs}} p_\mu$ to the photon. We established this result for special relativity, but, through the equivalence principle it also holds in general relativity. Note for our Schwarzschild coordinate system, $u^\mu = (c\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$.

We now consider an emitter which emits photons of energy $h\nu_E$ travelling radially outward towards a receiver. Both the emitter and the receiver are at fixed spatial coordinates in Schwarzschild spacetime. An observer at the emitter measures the energy of the photons to be $h\nu_E$. What energy, $h\nu_R$, does an observer at the receiver assign to the photons?

Take the emitter to have $r = r_E$ and the receiver to have $r = r_R$ where $r_E < r_R$. Then the two stationary observers will have four-velocities

$$(u^\mu)_E = (c\dot{t}, 0, 0, 0) = \left(\frac{c}{\sqrt{1 - 2m/r_E}}, 0, 0, 0\right)$$
$$(u^\mu)_R = \left(\frac{c}{\sqrt{1 - 2m/r_R}}, 0, 0, 0\right).$$

For the light, since $\frac{dx^\mu}{dw}$ is parallel to $p^\mu$, we can choose the affine parameter $w$ such that $p^\mu = \frac{dx^\mu}{dw}$. Using $k = (1 - 2m/r)\dot{t}$, and noting that the photon travels radially outward ($\dot{r} > 0$), we have have

$$p^\mu = \left(\frac{kc}{1 - 2m/r}, kc, 0, 0\right)$$

and

$$p_\mu = \left(kc, -\frac{kc}{1 - 2m/r}, 0, 0\right).$$

To find $p_\mu$ we have, as usual, used $p_\mu = g_{\mu\nu}p^\nu$.

Now we can directly evaluate the energies of the light as measured by observers at the emitter and receiver:

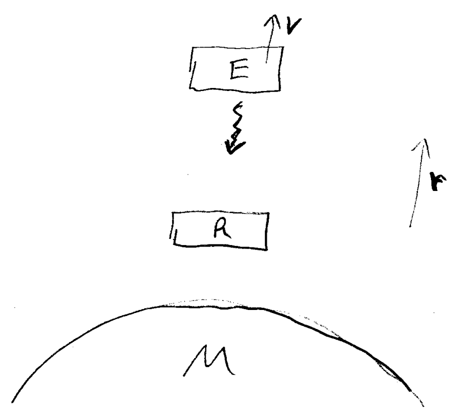
$$h\nu_E = (u^\mu)_R p_\mu = \frac{kc^2}{\sqrt{1-2m/r_E}}$$
$$h\nu_R = (u^\mu)_R p_\mu = \frac{kc^2}{\sqrt{1-2m/r_R}}.$$

Dividing these results, we arrive at the spectral shift formula

$$\frac{\nu_E}{\nu_R} = \frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1-2m/r_R}{1-2m/r_E}}$$

For the case where the photons travel radially inwards (and $r_R < r_E$), one obtains an identical expression (check). So we see that photons "falling" under a gravitational field become more energetic (or blue shifted).

## 3.7 The Pound-Rebka Experiment




We will now consider an example which nicely combines the Doppler shift (which we saw in special relativity) with the gravitational spectral shift. As shown in the figure, photons are emitted and travel radially inward to a receiver. Through the previous analysis, the photons will become more energetic, or blue shifted, as they "fall" under earth's gravitational field (we take $M$ for this example to be the mass of the earth). On the other hand, when the emitter is moving radially outward with velocity $v = \frac{dr_E}{dt} > 0$, we expect, through the Doppler effect, that this will decrease, or red shift, the energy of the light. We ask: for which $v$ will the energy of the emitted and received photons (as determined by observers situated at the emitter and receiver) be the same? This result is relevant for the Pound-Rebka experiment.[16] In contrast to the other "classical" tests of general relativity we discussed previously, namely

[16] Pound, R. V.; Rebka Jr. G. A. (November 1, 1959). "Gravitational Red-Shift in Nuclear Resonance". Physical Review Letters 3 (9): 439

the bending of light and the precession of the perihelion, the work of Pound and Rebka was terrestrial experiment: $\Delta r = r_E - r_R$ is the distance from the roof to the basement of a building.

To keep our expressions manageable, we will work to first order in $m/r_E$, $m/r_R$, and $v/c$ which are all small numbers for the experiment under consideration. For the earth, $m$ is only around a centimetre. We can use our expression for $(u^\mu)_R$ obtained in the previous section:

$$(u^\mu)_R = \left( \frac{c}{\sqrt{1 - 2m/r_R}}, 0, 0, 0 \right) \approx (c(1 + m/r_R), 0, 0, 0)$$

For the light, we have

$$p_\mu = \left( kc, \frac{kc}{1 - 2m/r}, 0, 0 \right).$$

This is for light travelling radially inward (compare to $p_\mu$ from the previous section).

For $(u^\mu)_E$, we have

$$(u^\mu)_E = (c\dot{t}, \dot{r}, 0, 0).$$

Using $\dot{r} = \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}$ we have

$$(u^\mu)_E = c\dot{t}(1, v/c, 0, 0).$$

Using our requirement $c^2 = u^\mu u_\mu$, gives

$$c^2 = c^2\dot{t}^2 \left( 1 - 2m/r_E - \frac{v^2/c^2}{1 - 2m/r_E} \right) \approx c^2\dot{t}^2(1 - 2m/r_E).$$

So $\dot{t} \approx 1 + m/r_E$ and

$$(u^\mu)_E \approx (c(1 + m/r_E), v, 0, 0)$$

Now we these expressions for the four-vectors, we obtain

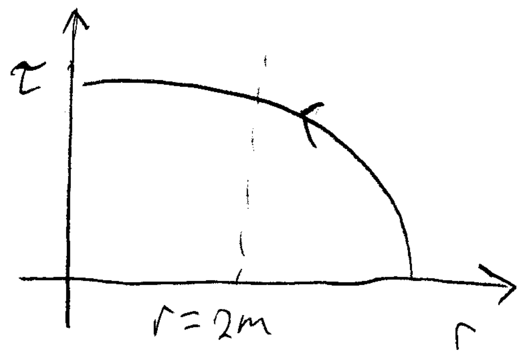
$$\begin{aligned} h\nu_E &= kc^2(1 + m/r_E + v/c) \\ h\nu_R &= kc^2(1 + m/r_R). \end{aligned}$$

The energy of the emitted and received photons will be the same when

$$\frac{v}{c} = \frac{m}{r_R} - \frac{m}{r_E} \approx \frac{m\Delta r}{R^2}$$

where $R$ is the radius of the earth. Using $g = \frac{GM}{R^2} \approx 10$ m/s$^2$, we can write this as

$$v = \frac{g\Delta r}{c}.$$




r

## 3.8 Black Holes

### 3.8.1 Gravitational Collapse

We have been mostly concentrating on examples where general relativity provides small corrections to classical dynamics. We will now turn our attention to black holes where the role of general relativity is more profound.

In our derivation of the Schwarzschild metric, we concentrated on regions outside a massive body's interior where $T^{\mu\nu} = 0$. Though we will not perform the calculation, one can solve Einstein's equations

$$G^{\mu\nu} = \kappa T^{\mu\nu}$$

using a perfect fluid for $T^{\mu\nu}$ for the interior regions of a spherically-symmetric star. Taking a uniform density of the star: $\rho = \text{const}$ (though this approximation is not entirely physically sensible) one finds an interesting result: if the radius of the star is less than $\frac{9}{4}m$, then the pressure will diverge somewhere in the interior of the star. This signifies an instability: **gravitational collapse**.

**Black holes** are expected to form after very massive stars collapse. In this course, we will only consider Schwarzschild black holes, which are black holes with no angular momentum or charge.

### 3.8.2 Radial free fall of a massive particle into a black hole

Let's start by considering the radial free fall of a massive particle into a black hole. From $c^2 = (ds/d\tau)^2$ we have

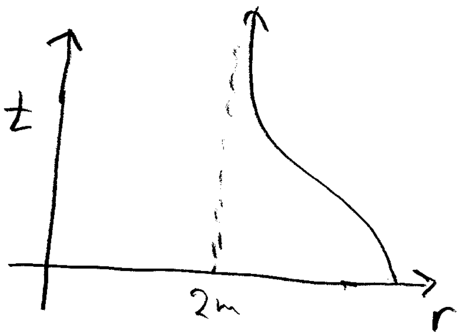
$$c^2 = \frac{k^2c^2}{1-2m/r} - \frac{\dot{r}}{1-2m/r}.$$

Suppose the particle starts from rest at $r = \bar{r}$. This then fixes $k^2 = 1 - 2m/\bar{r}$. So

$$\dot{r}^2 = c^2(2m/r - 2m/\bar{r}).$$

For convenience, let's take $\bar{r} \to \infty$. Then

$$\dot{r} = -c\sqrt{2m/r}.$$




r

From this, we can plot the trajectory as a function of proper time (this trajectory actually isn't any different from the Newtonian one).

Now let's consider parametrising in terms of coordinate time. Starting from rest at infinity, gives $k = (1 - 2m/r)\dot{t} = 1$. So

$$\dot{r} = \frac{dr}{dt}\frac{dt}{d\tau} = \frac{dr}{dt}\frac{1}{1 - 2m/r}.$$

With this we find

$$\frac{dt}{dr} = \frac{-1}{c(1 - 2m/r)\sqrt{2m/r}}.$$

The corresponding **world line** is shown to the right. We obtain qualitatively different results! For the present case, the trajectory does not seem to pass $r = 2m$, the **Schwarzschild radius**.

This can be traced to the **coordinate singularity** of the Schwarzschild metric at $r = 2m$. Such singularities can be removed by choosing a different coordinate system.[17] Evidence that this is not a physical singularity can be seen from the scalar quantity

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48m^2}{r^6}$$

which is well-behaved at $r = 2m$, but singular at $r = 0$.

**Example** Bob is falling radially into a black hole (again starting from rest at infinity for simplicity). He carries with him a laser which emits light of frequency $\nu$ (according to the factory specs). He shines this laser radially outward to his friend Alice who stays at a fixed location $\bar{r}$ on Bob's world line. What does Alice measure for the energy of the photons which were emitted when Bob was at radial location $r$?

[17] For example, let's use cylindrical coordinates to describe the unit sphere. Then (with $\rho = \sqrt{x^2 + y^2}$) we have

$$(ds)^2 = \frac{(d\rho)^2}{1 - \rho^2} + \rho^2(d\phi)^2.$$

This has a coordinate singularity at $\rho = 1$. Of course there is nothing singular happening on the equator of the unit sphere. The above singularity is a result of the coordinate system we chose.

Bob's four-velocity is:

$$(u^\mu)_{\text{Bob}} = \left( \frac{c}{1 - 2m/r}, -c\sqrt{\frac{2m}{r}}, 0, 0 \right).$$

Alice's four-velocity is:

$$(u^\mu)_{\text{Alice}} = \left( \frac{c}{\sqrt{1 - 2m/\bar{r}}}, 0, 0, 0 \right).$$

The four-momentum of the light is:

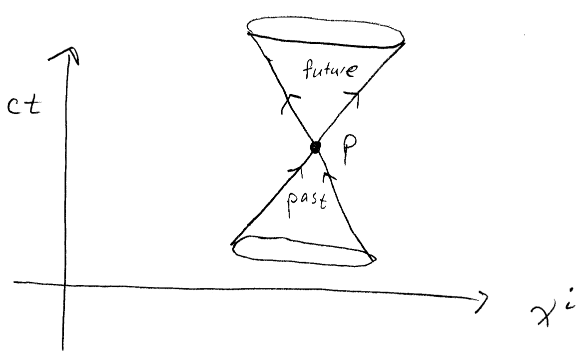
$$p_\mu = \left( kc, \frac{-kc}{1 - 2m/r}, 0, 0 \right).$$

Using $h\nu = p_\mu (u^\mu)_{\text{Bob}}$ and $h\nu_{\text{obs}} = p_\mu (u^\mu)_{\text{Alice}}$, we find that the frequency observed by Alice is

$$\nu_{\text{obs}} = \nu(1 - \sqrt{2m/r}) \frac{1}{\sqrt{1 - 2m/\bar{r}}}.$$

As Bob approaches the Schwarzschild radius, the corresponding light once it reaches Alice becomes substantially redshifted.

### 3.8.3 Radial motion of light in the vicinity of a black hole




It is instructive to draw the world lines of the allowable trajectories of light in the Schwarzschild geometry. Before doing this, we will introduce the useful concept of a light cone. We first consider light cones in special relativity. A light cone is the path a flash of light from a single spacetime point $P$ would take through spacetime. We also extend these null geodesics to times before $P$. The future and past light cones of an observer located at spacetime point $P$ are shown in the diagram. Events outside the light cone cannot be causally related to this observer. Arrows in this diagram indicate the direction light will propagate. Trajectories of massive particles are timelike. This means that a massive particle starting at point $P$ will only explore the region within the future light cone.

Now on to radial null geodesics in the Schwarzschild geometry. For this we have

$$0 = (ds)^2 = c^2(1 - 2m/r)(dt)^2 - (1 - 2m/r)^{-1}(dr)^2$$

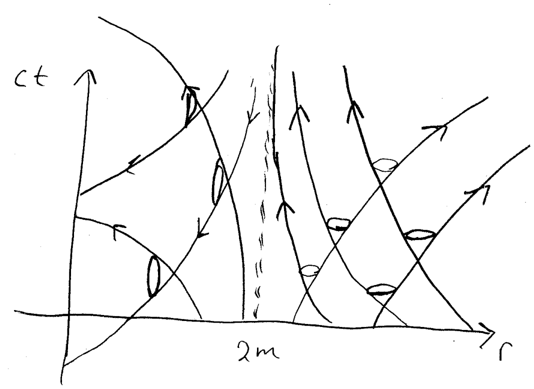
so

$$c\frac{dt}{dr} = \pm\frac{1}{1 - 2m/r}.$$

Solving this differential equation gives

$$ct + A = \pm(r + 2m \log|r - 2m|)$$

where $A$ is an integration constant and the $\pm$ solutions give ingoing and outgoing null geodesics for $r > 2m$.




This family of curves is plotted in the figure above. We also place arrows on these curves indicating the direction light propagates, and indicate the corresponding future light cones. For $r > 2m$, the light cone structure is clear. That is, for large $r$, the geometry is flat, the light rays make $45°$ angles with the horizontal axis, and the future light cones point up as in special relativity. All of the light rays for $r > 2m$ can be traced to their large-$r$ limit. While the light cone structure in the figure is correct for $r < 2m$, with the current analysis it is not obvious why this is so. The problem is that we have a coordinate singularity at $r = 2m$

To deduce the light cone structure, for $r < 2m$, it is useful to find a coordinate system which removes the coordinate singularity at $r = 2m$. Let's see if we can write the Schwarzschild line element in a revealing way (still considering radial motion):

$$\begin{aligned}(ds)^2 &= c^2(1 - 2m/r)(dt)^2 - (1 - 2m/r)^{-1}(dr)^2 \\ &= (1 - 2m/r)\left(cdt + (1 - 2m/r)^{-1}dr\right)\left(cdt - (1 - 2m/r)^{-1}dr\right) \\ &= (1 - 2m/r)\left(cdt + \frac{2m}{r - 2m}dr + dr\right)\left(cdt - \frac{2m}{r - 2m}dr - dr\right).\end{aligned}$$

Let's introduce $c\bar{t} = ct + 2m \log |r - 2m|$ so that $cd\bar{t} = cdt + \frac{2m}{r-2m} dr$. Putting this into $(ds)^2$ we find

$$(ds)^2 = (1 - 2m/r)\,(cd\bar{t} + dr)\left(cd\bar{t} - \frac{1 + 2m/r}{1 - 2m/r} dr\right).$$

Multiplying this out, we find that, unlike our original coordinates, nothing blows up at the Schwarzschild radius.

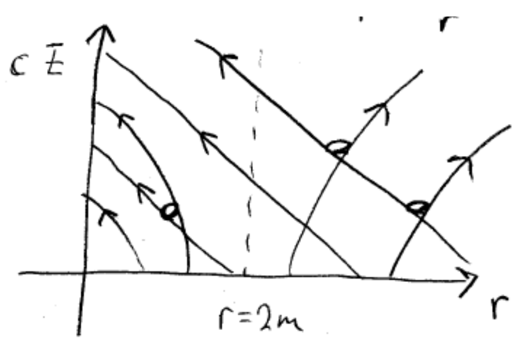
Setting the two terms in parenthesis to zero, gives the null geodesics. From $cd\bar{t} = -dr$,

$$c\bar{t} = -r + \text{const.}$$

Integrating $c\frac{d\bar{t}}{dr} = \frac{1+2m/r}{1-2m/r}$ gives

$$c\bar{t} = r + 4m \log |r - 2m| + \text{const.}$$

The family of curves for our new coordinate system are plotted below. We can deduce the propagation directions since the orientations of the light cones will change continuously through $r = 2m$ in this coordinate system.




We see that all null trajectories inside the Schwarzschild radius will eventually go to the singularity at $r = 0$ (we can use this to convince ourselves that the previous drawing is correct). Since timelike curves are bounded by these cones, once a massive particle is within the Schwarzschild radius there is no escape.

### 3.8.4 Maximising survival time inside a black hole.

Suppose a rocket ship finds itself inside the Schwarzschild radius of a black hole. What is the longest time the ship can survive, according to the proper time determined by the passengers? We allow ourselves the freedom the specify the velocity of the ship at an initial time, to maximise this survival time. The rocket has an engine and so does not need to follow a geodesic.

Let's start with our relation for any timelike curve:

$$c^2(d\tau)^2 = c^2(1 - 2m/r)(dt)^2 - (1 - 2m/r)^{-1}(dr)^2 - r^2(d\Omega)^2.$$

Noting the signs of the terms in the above expression for $r < 2m$, we have

$$c^2(d\tau)^2 \leq -(1-2m/r)^{-1}(dr)^2.$$

This provides an upper bound for $d\tau$ (note that $d\tau$ is positive).

Now let's consider a radial geodesic. The relevant equation is

$$c^2 = \frac{k^2c^2 - \dot{r}^2}{1-2m/r}.$$

Let's further choose initial conditions so that $k = 0$. This corresponds to starting from rest at the Schwarzschild radius. Then we find

$$c^2(d\tau)^2 = -(1-2m/r)^{-1}(dr)^2.$$

for this trajectory. Thus this radial geodesic attains the upper bound we found previously. To maximise the survival time, the pilot should not use the engines. He should just allow the ship to fall radially to the singularity with $k = 0$. This result is certainly counterintuitive. The comparison to fish near a waterfall breaks down.

If the rocket starts from rest slightly beneath the Schwarzschild radius, this corresponds to the longest possible survival time. For this we obtain

$$c\Delta\tau = -\int_{2m}^{0} dr\sqrt{\frac{r}{2m-r}} = 2m\int_0^1 dx\sqrt{\frac{x}{1-x}} = m\pi.$$

## 3.9 Cosmology

### 3.9.1 The Cosmological Principle

We will now move on to consider motion over much larger length scales and much longer time scales, and motivate a metric aimed at describing the geometry of the universe. In doing so we will be guided by the **cosmological principle**:

- At every epoch, the universe looks the same from every spatial point except for local irregularities.

The principle implies both that there are no privileged points in the universe and that there are no privileged directions about any points when viewed over sufficiently large length scales. This means that the universe is homogeneous and isotropic. This will greatly reduce the number of candidate metrics we can use to describe the universe.

We take spatial coordinates which are comoving with the universe and write

$$(ds)^2 = c^2(dt)^2 - (S(t))^2\tilde{g}_{ij}dx^i dx^j$$

where $S$ is a dimensionless scale factor depending only on $t$. Since coordinates are comoving, $t$ measures the proper time at any fixed location in space. The cosmological principle requires that the spatial metric $\tilde{g}_{ij}$ describes a **maximally symmetric manifold**. A maximally symmetric $N$-dimensional manifold is a manifold having the maximum number, $N(N+1)$, of independent Killing vector fields.[18]

[18]The practice problems contain a question introducing Killing vector fields.

### 3.9.2 Maximally Symmetric Three-Dimensional Manifolds

We will now deduce the spatial portion of our cosmological metric. We drop tildes and restrict to three-dimensions in what follows. The Riemann curvature tensor of a maximally symmetric manifold can be written as

$$R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk})$$

where $\kappa$ is a constant.[19] Note that this satisfies all of the required symmetries of the curvature tensor. The corresponding Ricci tensor can also be seen to be constant. That is, we can work out (for three dimensions)

$$R_{ij} = 2\kappa g_{ij}, \quad R = 6\kappa. \tag{40}$$

Maximal symmetry of course requires spherical symmetry. A general spherically symmetric three-dimensional metric can be written as

$$(ds)^2 = e^{\lambda(r)}(dr)^2 + r^2((d\theta)^2 + \sin^2\theta(d\phi)^2)$$

where $\lambda$ is an arbitrary function of $r$.

From this we can compute (using, say, *Mathematica*)

$$R = \frac{2}{r^2}e^{-\lambda}(-1 + e^{\lambda} + r\lambda')$$

where primes denote derivatives with respect to $r$. Setting $R = 6\kappa$ and solving this differential equation gives

$$e^{-\lambda(r)} = 1 - \kappa r^2 + A/r$$

where $A$ is a constant of integration. $R_{ab} = 2\kappa g_{ab}$ further requires $A = 0$. The resulting line element

$$(ds)^2 = \frac{1}{1-\kappa r^2}(dr)^2 + r^2(d\Omega)^2$$

has Riemann curvature tensor of the form 40.

To gain some intuition about these metrics, it is useful to embed the manifold in a four-dimensional space.

<u>$\kappa > 0$ / positive curvature / closed</u>: Let $r = \frac{1}{\sqrt{\kappa}}\sin(\chi)$. Then

$$(ds)^2 = \frac{1}{\kappa}\left((d\chi)^2 + \sin^2\chi(d\Omega)^2\right).$$

Introduce the new variables

$$\begin{aligned} w &= R\cos(\chi) \\ x &= R\sin(\chi)\sin(\theta)\cos(\phi) \\ y &= R\sin(\chi)\sin(\theta)\sin(\phi) \\ z &= R\sin(\chi)\cos(\theta) \end{aligned}$$

[19] We are not showing this result, but hopefully it is plausible. See IX.6 from Zee's book for a derivation.

where $R = 1/\sqrt{\kappa}$. Then $w^2 + x^2 + y^2 + z^2 = R^2$ and

$$(ds)^2 = (dw)^2 + (dx)^2 + (dy)^2 + (dz)^2.$$

Thus we can think of this manifold as a three-dimensional sphere embedded in four-dimensional Euclidean space. This manifold is **closed** in the sense that it has finite volume.

$\kappa = 0$ / zero curvature / flat: When $\kappa = 0$, the metric clearly describes three-dimensional Euclidean space.

$\kappa < 0$ / negative curvature / open: Let $r = \frac{1}{\sqrt{|\kappa|}} \sinh(\chi)$. Then

$$(ds)^2 = \frac{1}{\kappa} \left( (d\chi)^2 + \sinh^2 \chi (d\Omega)^2 \right).$$

Introduce the new variables

$$\begin{aligned} w &= R \cosh(\chi) \\ x &= R \sinh(\chi) \sin(\theta) \cos(\phi) \\ y &= R \sinh(\chi) \sin(\theta) \sin(\phi) \\ z &= R \sinh(\chi) \cos(\theta) \end{aligned}$$

where $R = 1/\sqrt{|\kappa|}$. Then $w^2 - x^2 + y^2 + z^2 = R^2$ and

$$(ds)^2 = -(dw)^2 + (dx)^2 + (dy)^2 + (dz)^2.$$

Thus we can think of this manifold as a three-dimensional hyperboloid embedded in a flat Minkowski space with +2 signature. This manifold is **open** in the sense that it has infinite volume.

### 3.9.3 The Friedmann-Robertson-Walker Metric

With these three possibilities in mind, we now return to spacetime. Things will be simpler if we scale the $r$ variable. For nonzero $\kappa$, let $r \rightarrow r/\sqrt{|\kappa|}$ while for $\kappa = 0$, we leave $r$ unchanged. Additionally, let $R(t) = \frac{S(t)}{\sqrt{|\kappa|}}$ for non-zero $\kappa$ and $R(t) = S(t)$ when $\kappa = 0$. With this we obtain the line element

$$(ds)^2 = c^2(dt)^2 - (R(t))^2 \left( \frac{1}{1 - kr^2} (dr)^2 + r^2 (d\Omega)^2 \right). \tag{41}$$

In this, $k$ is either 1,0, or -1 which respectively corresponds to a closed, flat, or open universe. Eq. 41 defines the **Friedmann-Robertson-Walker (FRW) metric**.

### 3.9.4 A Perfect Comoving Fluid

We will now fill the universe up with a perfect comoving fluid with $u^\mu = (c, 0, 0, 0)$ and

$$T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu - pg^{\mu\nu}.$$

The pressure $p$ and the proper density $\rho$ are taken to be functions of $t$ only. This form of the energy-momentum tensor is again motivated by the cosmological principle. First we consider the equations of motion $\nabla_\mu T^{\mu\nu} = 0$. In Sec. 2.13 we found the following relativistic fluid equations

$$\nabla_\mu(\rho u^\mu) + \frac{p}{c^2}\nabla_\mu u^\mu = 0 \tag{42}$$

and

$$(\rho + p/c^2)u^\mu \nabla_\mu u^\nu = \partial_\mu p(g^{\mu\nu} - u^\mu u^\nu/c^2). \tag{43}$$

Let's plug $u^\mu = (c, 0, 0, 0)$ into these equations.

The right-hand side of Eq. 43 is zero. This is because $p$ depends only on time, and $g^{00} = 1$ for the FRW metric. For the left-hand side we consider

$$u^\mu \nabla_\mu u^\nu = c\nabla_0 u^\nu = c\Gamma^\nu_{0\sigma}u^\sigma = c^2\Gamma^\nu_{00}.$$

For the FRW metric, one can see that $\Gamma^\nu_{00} = 0$ for all $\nu$ and so $u^\mu \nabla_\mu u^\nu = 0$. Therefore Eq. 43 gives no information. For Eq. 42, we need to evaluate the divergence of the four-velocity field. We find

$$\nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\mu_{\mu\nu}u^\nu = \Gamma^\mu_{\mu 0}c.$$

Using the definition of the Christoffel symbol, we have in general (check)

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\sigma}\partial_\nu g_{\mu\sigma}.$$

Using the FRW metric, we find

$$\Gamma^\mu_{\mu 0} = \frac{1}{2}g^{\mu\sigma}\partial_0 g_{\mu\sigma} = 3\frac{\dot{R}}{cR}$$

where $\dot{R} = dR/dt$. Thus

$$\nabla_\mu u^\mu = 3\dot{R}/R.$$

Finally, inserting this expression for the $\nabla_\mu u^\mu$ into Eq. 42, we find

$$\frac{d}{dt}(\rho R^3) + \frac{p}{c^2}\frac{d}{dt}R^3 = 0. \tag{44}$$

Thus the relativistic fluid equations reduce to the relatively simple equation above for the comoving perfect fluid.

### 3.9.5 Cosmological Field Equations

Now we consider the Einstein field equations for our FRW metric with the energy-momentum tensor of the previous section. The 00 component of the Einstein tensor is computed to be

$$G_{00} = 3\frac{k + \dot{R}^2/c^2}{R^2}.$$

Noting that $T_{00} = \rho c^2$, $G_{00} = \frac{8\pi G}{c^4} T_{00}$ becomes

$$\dot{R}^2 + kc^2 - \frac{8\pi}{3} G\rho R^2 = 0 \tag{45}$$

which is typically called the Friedmann equation. The other components of the Einstein field equations do not produce any additional independent equations.

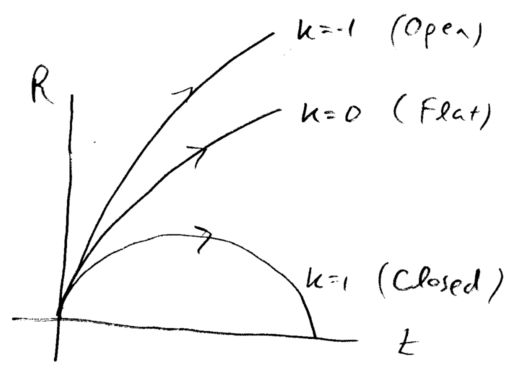
The relevant equations governing the dynamics of the universe are 44, 45, and the equation of state which relates $\rho$ to $p$ (more on this later). With initial data, these three equations can be solved to determine the dynamics of the universe (under the assumptions of the model).

### 3.9.6 Matter-Filled Universe

Consider a universe composed only of dust (matter) where $p = 0$. Then from Eq. 44 we find $\rho \propto 1/R^3$. Inserting this into the Friedmann equation, we find

$$\dot{R}^2 = \frac{A}{R} - kc^2$$

where $A$ is a positive constant. This equation actually has the same form as the energy of a Newtonian particle in a classical gravitational field. With the initial condition of the Big Bang, $R(0) = 0$, the solutions for closed, flat, and open universes are sketched below. Note that for the closed universe ($k = 1$), the universe will eventually reach a time when $\dot{R} = 0$. For this case, the model predicts the universe to end in a "big crunch".




For perhaps philosophical reasons, Einstein favoured a closed, static universe. However, each of the cases investigated above predict a non-constant $R(t)$. To remedy this, Einstein introduced a **cosmological constant** $\Lambda$ into his field equations as

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4}(T^{\mu\nu})_{\text{m}} \tag{46}$$

where $(T^{\mu\nu})_{\rm m}$ is the energy-momentum tensor for matter. Since $\nabla_\mu g^{\mu\nu} = 0$, the additional term does not spoil the zero-divergence condition of the field equations. This extra term actually can be absorbed into the energy-momentum tensor. That is, we let

$$T^{\mu\nu} = (T^{\mu\nu})_{\rm m} + (T^{\mu\nu})_\Lambda$$

where

$$(T^{\mu\nu})_\Lambda = (\rho_\Lambda + p_\Lambda/c^2)u^\mu u^\nu - p_\Lambda g^{\mu\nu}.$$

Next we take $p_\Lambda = -\rho_\Lambda c^2$ so that $(T^{\mu\nu})_\Lambda = \rho_\Lambda c^2 g^{\mu\nu}$. Finally, if we choose $\rho_\Lambda = \frac{c^2}{8\pi G}\Lambda$, Eq. 46 can be rewritten as $G^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}$.

The fluid equation of motion 44 for this $T^{\mu\nu}$ gives (denoting the dust-density by $\rho_m$)

$$\rho = \rho_m + \rho_\Lambda = B/R^3 + \rho_\Lambda$$

where $B$ is a positive constant. Inserting this into Friedmann equation, we find

$$\frac{1}{2}\dot{R}^2 + V(R) = -\frac{1}{2}kc^2$$

where

$$V(R) = -CR^2 - D/R$$

($C$ and $D$ are positive constants). These equations describe a Newtonian particle in the potential $V(R)$. Since $V(R)$ has an extremum for which $V < 0$, it is possible to obtain a static universe when $k = 1$. However, shortly after the development of these ideas, experimental data appeared showing the red shift of distant stars and thus indicating that the universe is in fact expanding ($\dot{R} > 0$) and not static.

### 3.9.7 Multi-component cosmological fluid

More complete cosmological models allow the matter in the universe to consist of several components. We take three components: radiation, matter, and vacuum density. We write the energy-momentum tensor, density, and pressure as a sum over components:

$$T^{\mu\nu} = \sum_i (T^{\mu\nu})_i$$
$$\rho = \sum_i \rho_i$$
$$p = \sum_i p_i$$

We further take the fluid components to be non-interacting, and thus take

$$\nabla_\mu (T^{\mu\nu})_i = 0$$

for each component. The density and pressure of each component are related by an **equation of state**:

$$p_i = w_i \rho_i c^2.$$

For matter, vacuum density, and radiation, $w_i$ is 0, -1, and 1/3 respectively.

Knowledge of the density and the scale factor $R(t)$ of a particular fluid component at a particular time $\bar{t}$, Eq. 44 can be used to predict future (and past) values of the corresponding density:

$$\rho_i(t) = \rho_i(\bar{t}) \left( \frac{R(\bar{t})}{R(t)} \right)^{3(1+w_i)}. \tag{47}$$

This relation tells us that the early universe shortly after the big bang was dominated by radiation.

Current estimates give that the universe is 30% matter and 70% vacuum density. The mysterious vacuum-density component also is known as **dark energy**, which, within these models, is needed to explain more recent observations that the universe is expanding at an *accelerating* rate, which is known as inflation. There is currently no satisfactory theoretical understanding of this component. Furthermore, most (80%) of the matter in the universe is thought to not be baryonic (baryonic matter is matter described by the Standard Model of particle physics). The non-baryonic matter is called **dark matter** and is also not well understood. There is still quite a bit to be understood!

### 3.9.8 de Sitter Spacetime

Motivated by the previous section, let's consider a universe composed entirely of dark energy, and also take the universe to be flat ($k = 0$). The Friedmann equation is

$$\dot{R}^2 = \frac{8\pi}{3} G \rho R^2.$$

From this we find

$$H^2 = (\dot{R}/R)^2 = \frac{8\pi}{3} G \rho = \frac{\Lambda}{3c^2} = \text{const}$$

where we have introduced the Hubble constant $H = \dot{R}/R$. This has the solution $R(t) = R(0)e^{Ht}$. Our FRW metric then becomes

$$(ds)^2 = c^2(dt)^2 - e^{2Ht}((dx)^2 + (dy)^2 + (dz)^2).$$

This is the so-called de Sitter spacetime. In this model, the universe is expanding exponentially fast.