Probabilities and Quantum Mechanics

First Year Undergraduate Research Project (M1R)

Imperial Mathematics

2026

Project Lead:
Ryan Barnett

Today: Research themes 1&3

Research Theme 1: Quantum Games

Games (board games, etc) often involve probability and random numbers

Can cool stuff happen when we put in quantum?

What if we allow for dice superposition or even entanglement?

We will start with the **Quantum Penny Flip Game** (D. Meyer) and consider variations and extensions.

Tools: Linear algebra, statistics, creativity(!)

Quantum Penny Flip Game

Classical game:

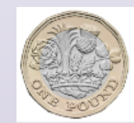
- Alice vs Bob
- Coin starts out heads. Nobody can look at coin until final step of game.
- Alice goes first. She either turns over the coin or does nothing. Bob is unaware what Alice does.
- Bob goes next. He either turns over the coin or does nothing. Alice is unaware of what Bob does.
- Alice goes last. She either turns over the coin or does nothing.
- Finally the coin is looked at. If it is heads, Alice wins. If it is tails, Bob wins.

With optimal play the chances of Alice winning/losing is 50-50. Bob can randomly choose to either turn over or not on his turn.

D. A. Meyer, “Quantum Strategies” Physical Review Letters 82, 1052 (1999)

Quantum Penny Flip Game

Let’s describe the classical game using quantum maths

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 

Start: $|\psi\rangle = |0\rangle$

Turn over: apply $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Do nothing: apply $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

“Look” $P_{\text{heads}} = |\langle 0|\psi\rangle|^2$

Quantum Penny Flip Game

To make the game more quantum/interesting, give Alice a new skill: Hadamard gate. She can now apply X, H, or do nothing.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Hadamard is actually a superpower here. She can win 100% of time. Suppose she applies Hadamard on her first turn:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |\psi'\rangle$$

This state is immune to whatever Bob does because it is an eigenstate of X:

$$X|\psi'\rangle = |\psi'\rangle$$

Alice always gets back $|\psi'\rangle$

Then she applies H again: $H|\psi'\rangle = H^2|0\rangle = |0\rangle$. Always heads!

Quantum Penny Flip Game

One can consider several variations/extensions. For instance:

What if Bob also gets a new ability: Z Pauli gate, to counter Alice?

A major variation: what if they play with a two-qubit entangled state that is initially $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Can new/interesting strategies emerge from *entanglement*

The *CHSH game

The rules of this game may seem a bit contrived, but it illustrates a quantum strategy based on entanglement

Alice and Bob go to isolated booths and are on the same team. The dealer presents them each with a classical 0/1 bit chosen randomly. There is a 25% chance of each of the following combos: 00, 01, 10, 11 from the dealer. Alice then writes a 0 or 1 on a piece of paper and hands it to dealer (call this a). Bob does the same (call this b).

Winning conditions

- $a = b$ if classical bits are 00, 01,or 10
- $a \neq b$ if classical bits are 11

Best strategy: Alice and Bob choose $a = b = 0$ (or $a = b = 1$) every time. They win 75% of time.

Game based on: Clauser, Horne, Shimony, Holt "Proposed Experiment to Test Local Hidden-Variable Theories" Physical Review Letters 23, 880–884 (1969)
A nice write up is on IBM's website.

The CHSH game

The interesting addition. Suppose they each have a qubit of an entangled state: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. But they still cannot communicate.

The punchline: they can use this to get better odds

The CHSH game

The protocol:

If Alice gets bit 0 from dealer she measures her qubit in the $\{ |\phi_+\rangle, |\phi_-\rangle \}$ basis while if she gets 1 she measures in the $\{ |\tilde{\phi}_+\rangle, |\tilde{\phi}_-\rangle \}$ basis.

If Bob gets bit 0 from dealer he measures his qubit in the $\{ |\xi_+\rangle, |\xi_-\rangle \}$ basis while if he gets 1 he measures in the $\{ |\tilde{\xi}_+\rangle, |\tilde{\xi}_-\rangle \}$ basis.

The states: $|\phi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\tilde{\phi}_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|\xi_+\rangle = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$, $|\tilde{\xi}_+\rangle = \begin{pmatrix} \cos(\alpha) \\ -\sin(\alpha) \end{pmatrix}$ where $\alpha = \pi/8$.

They report back their measurement outcome to dealer in 0/1 format.

The CHSH game

The math is tedious but only makes use of the Born rule.

For instance, if the dealer gives 00 the probabilities are:

$$P_{\text{win}} = |\langle \phi_+ \xi_+ | \psi \rangle|^2 + |\langle \phi_- \xi_- | \psi \rangle|^2$$

$$P_{\text{lose}} = |\langle \phi_+ \xi_- | \psi \rangle|^2 + |\langle \phi_- \xi_+ | \psi \rangle|^2$$

Need to work out all cases…

The result:

$$P_{\text{win}} = \cos^2(\pi/8) \approx 85.4\,\%$$

The CHSH game

This example is sufficiently involved that there may not be time for other directions (if you choose to pursue it). But you could answer insightful questions like: why are the measurement axes chosen that way?

Build your own game

(Wikimedia Commons)

But bear in mind: you won't be able to describe a complex game during your poster presentation. Focus on the essential bits relying quantum measurement and probability. Maybe have QR code that give detailed rules.

Research Theme 3: Entanglement

Is this state entangled? Meaning will Alice's measurement influence Bob's state?

$$|\psi\rangle = (-0.320706 - 0.0241544i)|00\rangle + (0.0560778 + 0.200581i)|01\rangle$$
$$+(0.610594 + 0.434295i)|10\rangle + (0.390946 - 0.372708i)|11\rangle$$

It is not obvious.

How do we quantify entanglement? Is there a geometrical way to do it? What about *qutrits*?

Tools: Linear algebra (slightly more advanced), geometry

Entanglement

The workhorse: singular value decomposition

The singular value decomposition of square matrix A is: $A = U\Lambda V^\dagger$ where U and V are unitary matrices and Λ is a real diagonal matrix with nonnegative entries.

A general form for a two-qubit state is: $|\psi\rangle = \sum_{nm} A_{nm}|n, m\rangle$. Now apply SVD to get $|\psi\rangle = \sum_n \lambda_n |\phi_n, \xi_n\rangle$ where $\{|\phi_0\rangle, |\phi_1\rangle\}$ and $\{|\xi_0\rangle, |\xi_1\rangle\}$ orthonormal single-qubit bases.

The state is not entangled when exactly one $\lambda_n = 1$ (rest are zero). Otherwise it is entangled. Note that normalisation means $\sum_n \lambda_n^2 = 1$

Entanglement

Quantifying entanglement. Entanglement entropy: $S = -\sum_n \lambda_n^2 \log(\lambda_n^2)$.

Research question: what are the maximally entangled two-qubit states?

It is very useful to understand single qubit states as points living on the **Bloch sphere**.

How can this be used for two-qubit states? Hint: decompose $|\psi\rangle$ into parts that are symmetric and antisymmetric under interchanging qubits.

Can we develop a geometric construction to tell us how entangled our state is?

Planning

The rest of our meetings will be workshop style.
Let me know how things are going!