Probabilities and Quantum Mechanics

First Year Undergraduate Research Project (M1R)

Imperial Mathematics

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Today: Recap of last time. Advice for posters. Dig into research themes.

Recap from last time

- Classical coin fully described by p_0 and p_1
- Quantum coin described by density matrix ρ
- Classical coin described by one parameter: p_1 (note $p_1 = 1 - p_0$). Quantum problem described by three real parameters (two if it is pure state)
- Why does quantum require more? Because we need to be able to answer questions that don't make sense classically
- Born rules
- State collapse from measurement
- Quantum coin is a qubit

Info on two-qubit notation

Conventional orthonormal basis: $\{\,|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

The important thing is that these are orthogonal

It is conventional to use **tensor product** formulation. It works like this:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}.$$

Then:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Direct generalisation to N qubits…

Info on exponentiating matrices

Let A be a square matrix. The exponential of A is defined through the expansion

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

The poster

My two cents…

Include clear background info and references. You should be able to explain what you did in a way that is understandable to your peers. Include a short list of relevant references on your poster.

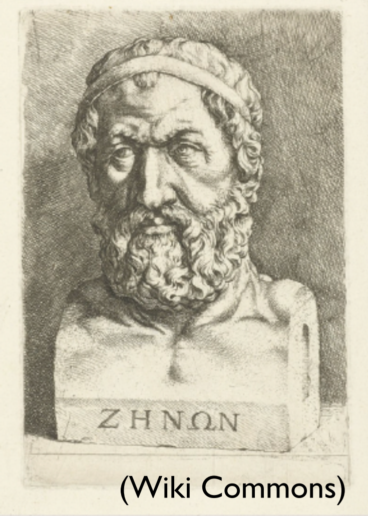
Make it engaging. Have a standard pitch (why is this interesting, what have you done), but also be ready to enter into discussions.

Now for the research themes

Research Theme 4: Quantum Zeno

The ancient Greek philosopher's paradox where you can never reach your destination seems to be resolved since

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$$



(Wiki Commons)

However, suppose during the journey, we perform several "where am I?" quantum measurements? Do Zeno's concerns persist?

Tools: density matrices (we will learn), numerical methods

Quantum Zeno

There are many contexts in which this could arise. The key is to elucidate interplay between smooth **unitary** dynamics of Schrödinger equation and **projective measurements**.

Let's start with situation we've considered before:

Start with $|0\rangle$

Evolve according to:

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solve and get:

$$|\psi\rangle = \cos(t)|0\rangle - i\sin(t)|1\rangle$$

Quantum Zeno

$$|\psi(t)\rangle = \cos(t)\,|0\rangle - i\sin(t)\,|1\rangle$$

The system starts at $|0\rangle$ and reaches $|1\rangle$ at $t = \pi/2$ (probability of measuring $|1\rangle$ at $t = \pi/2$ is 100%. This is our undisturbed journey.

We can also consider opposite journey: start at $|1\rangle$ and evolve to $|0\rangle$. The relevant solution here is:

$$|\tilde{\psi}(t)\rangle = \cos(t)\,|1\rangle - i\sin(t)\,|0\rangle$$

Quantum Zeno

Now let's consider a variation. After every period of time τ the system is measured in the 0/1 basis.

Right before the first measurement, the density matrix (pure) is:

$$\rho(\tau) = |\psi(\tau)\rangle\langle\psi(\tau)|$$

Right after the first measurement, the density gets mixed (using Born rules):

$$\rho'(\tau) = |0\rangle\langle 0|\langle 0|\rho(\tau)|0\rangle + |1\rangle\langle 1|\langle 1|\rho(\tau)|1\rangle$$

(Classical uncertainty)

Now evolve this via Schrödinger for another time period τ:

$$\rho(2\tau) = |\psi(\tau)\rangle\langle\psi(\tau)|\langle 0|\rho'(\tau)|0\rangle + |\tilde{\psi}(\tau)\rangle\langle\tilde{\psi}(\tau)|\langle 1|\rho'(\tau)|1\rangle$$

Quantum Zeno

Then measure

$$\rho'(2\tau) = |0\rangle\langle 0|\langle 0|\rho(2\tau)|0\rangle + |1\rangle\langle 1|\langle 1|\rho(2\tau)|1\rangle$$

And so on…

Can we find a general expression for $\rho'(N\tau)$?

Can we find values for which the evolution gets “stuck” on heads, i.e.

$\langle 0|\rho'(N\tau)|0\rangle \approx 1$ (for large N)

Quantum Zeno

Possible extensions:

What if we measure in another basis (e.g. basis of H)? What do we learn?

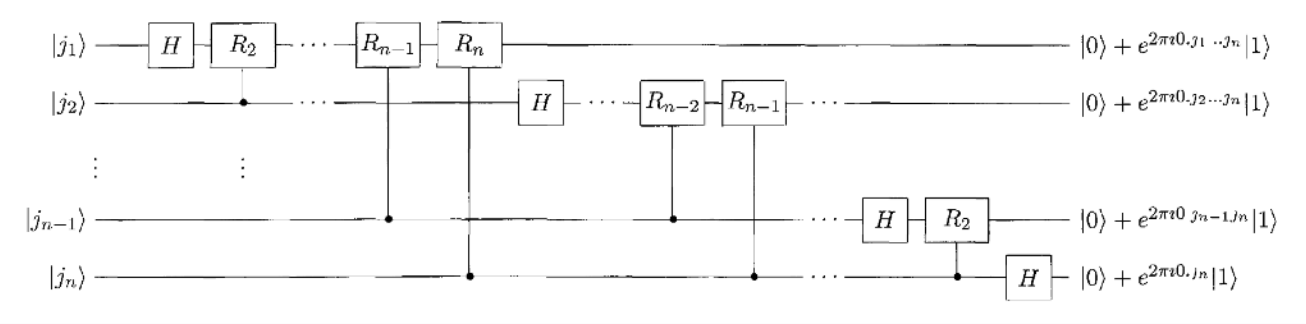
What if instead of a binary quantum coin, we have more states involved?

For instance: $H = -\sum_n (|n\rangle\langle n+1| + |n+1\rangle\langle n|)$

Connection to entanglement

Research Theme 2: Engineering quantum gates

To build a universal quantum computer, you need to have a certain collection of primitive gates at your disposal.



(Nielsen and Chuang, Cambridge Press)

Quantum Fourier Transform

This theme will explore how to achieve these gates from **physical** Hamiltonians like:

$$\mathscr{H} = \frac{\varepsilon}{2} Z + \gamma X \cos(\omega t)$$

Which combo of parameters will lead to the gates we need?

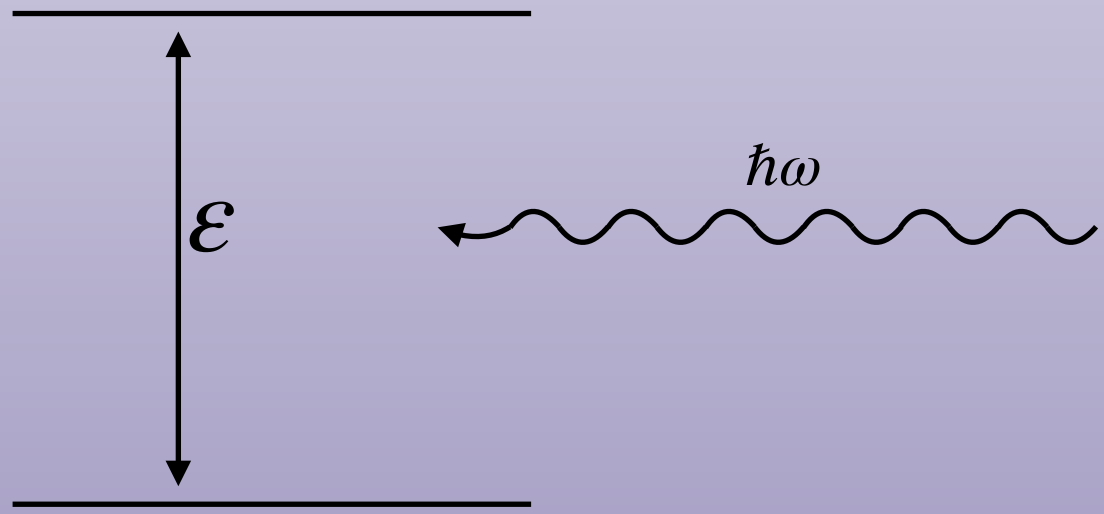
Tools: Linear algebra, differential equations, numerical methods

Engineering quantum gates

Notation: Pauli matrices: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathscr{H} = \frac{\varepsilon}{2} Z + \gamma X \cos(\omega t)$$

The $\mathscr{H}$ we wrote is fairly generic. It describes the physics of different qubit platforms. It describes a driven two-level system.



ε : energy level spacing, ω : driving frequency, γ : coupling strength

Engineering quantum gates

To build a universal quantum computer it is sufficient to have three primitive gates, two one-qubit gates and one two-qubit gate.

The goal here is to realise these two one-qubit gates from $\mathscr{H}$.

Knobs to turn: τ and γ (coupling strength).

Gates sought:

$$T = e^{-i\phi Z}$$ (Phase gate)

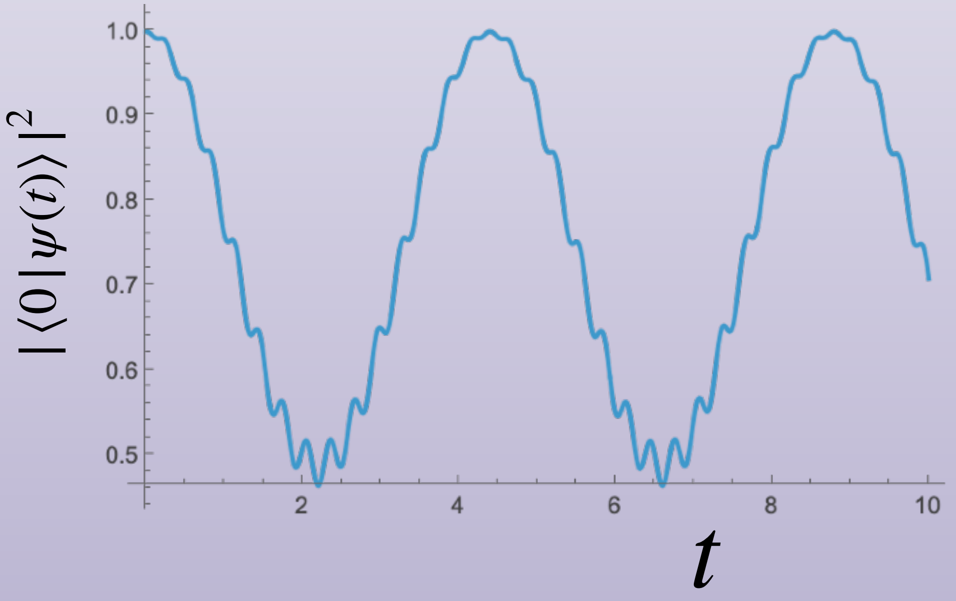
$$H = \frac{1}{\sqrt{2}}(X + Z)$$ (Hadamard)

Engineering quantum gates

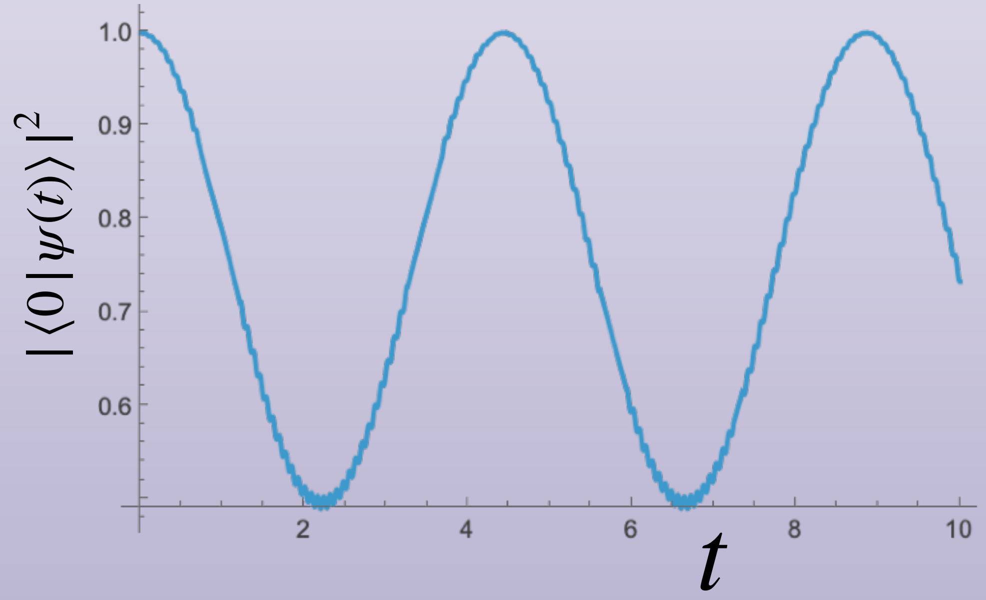
Understanding the Hamiltonian

Some numerical experiments…

Start with $|0\rangle$



$$\varepsilon = 11, \omega = 10, \gamma = 1$$



$$\varepsilon = 41, \omega = 40, \gamma = 1$$

Oscillations + fast micro-motion that becomes negligible when $\sqrt{(\varepsilon - \omega)^2 + \gamma^2} \ll \omega$. This regime is where experiments most often operate.

Engineering quantum gates

To understand, it is customary to work entirely within the frame rotating at ω:

$$|\psi\rangle = e^{-iZ\omega t/2}|\psi'\rangle$$

Subbing in you will find:

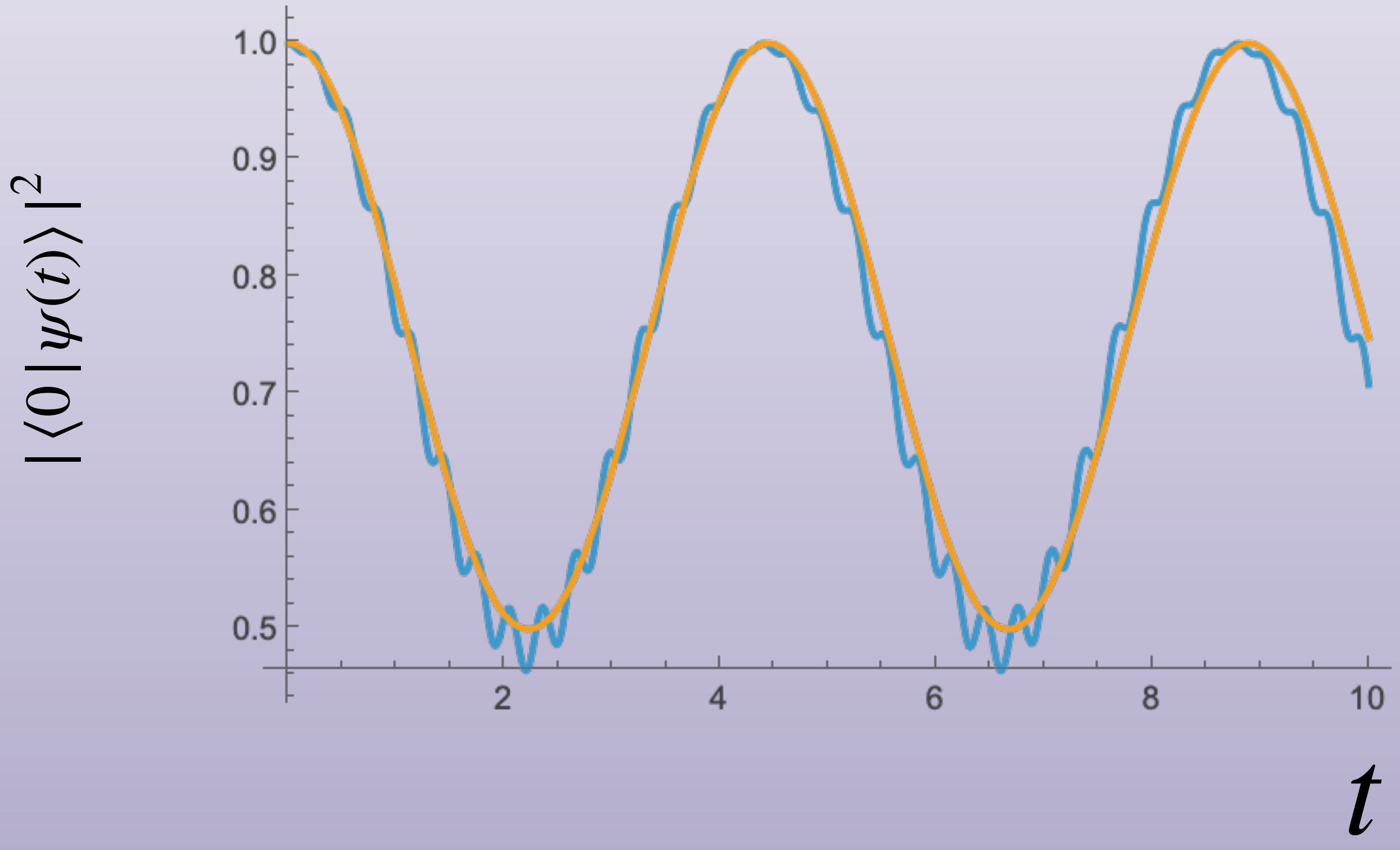
$$H_{\text{rot}} = \frac{1}{2}(\varepsilon - \omega)Z + \frac{\gamma}{2}X + V(t)$$

$$V(t) = \frac{\gamma}{2}[X\cos(2\omega t) - Y\sin(2\omega t)]$$

The rotating wave approximation is to drop $V(t)$

$$H_{\text{RWA}} = \frac{1}{2}(\varepsilon - \omega)Z + \frac{\gamma}{2}X$$

Engineering quantum gates



$$\varepsilon = 11, \omega = 10, \gamma = 1$$

Orange curve is RWA solution

Engineering quantum gates

Knobs to turn: τ and γ (coupling strength).

Realise sought gates as $U = e^{-i\mathscr{H}_{\mathrm{RWA}}\tau}$.

Hadamard, T, along with CNOT (two qubit) are sufficient for building universal quantum computer.

Potential project directions:

Clean my reasoning up.

What values do you need for τ and γ to obtain sought gates?
Why does RWA work?
How do you get other single qubit gates from T and H?
How could you incorporate corrections beyond RWA?
Other ideas…