# Probabilities and Quantum Mechanics

First Year Undergraduate Research Project (M1R)

Imperial Mathematics

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**Today:** We will discuss background info for project and sub-projects

# Start by considering simple statistics problem

Coin flip!

$P(0) = p_0$  (heads)

$P(1) = p_1$  (tails)

$P(n)$ is the probability mass* function

$$\sum_{n=0}^{1} P(n) = 1$$ (normalisation)

This is the full story

*Some people just call it the probability function

# Density matrix

Let’s encode the probabilities in the diagonal components of matrix:

$$\rho = \begin{pmatrix} p_0 & * \\ * & p_1 \end{pmatrix}$$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 

Rule for computing probabilities: $p_0 = \langle 0 | \rho | 0 \rangle \,, p_1 = \langle 1 | \rho | 1 \rangle$

Off diagonal elements are irrelevant for classical coin

Same (but less efficient) formulation as that on previous slide

# Density matrix: going quantum

The following properties are imposed on matrix $\rho$ :

- It is **Hermitian**: $\rho = \rho^\dagger$. The dagger symbol $\dagger$ means take transpose and complex conjugate.
- It is positive semi-definite: $\langle \phi | \rho | \phi \rangle \geq 0, \forall | \phi \rangle$
- It traces to one: $\mathrm{Tr}(\rho) = 1$

Additionally: a dm is called **pure** if $\rho^2 = \rho$. Otherwise it is **mixed**.

# Info on states

$|\phi\rangle$ denotes a column vector

$\langle\phi|$ is a row vector we get by taking the conjugate transpose of $|\phi\rangle$

We further require normalisation: $\langle\phi|\phi\rangle = 1$

$|\phi\rangle$ is often called a **state**

# Info on Hermitian matrices

$$A^\dagger = A$$

Its eigenvalues are real

Its eigenvectors can be taken to be orthonormal

$$A\,|\,\phi_n\rangle = \lambda_n\,|\,\phi_n\rangle \qquad \langle\phi_n\,|\,\phi_m\rangle = \delta_{nm}$$

# Problem!

$$\rho = \begin{pmatrix} p_0 & \alpha \\ \beta & p_1 \end{pmatrix}$$

Constraints on $\alpha$ and $\beta$ ?

Ans: $\beta^* = \alpha$ and $p_1 p_2 \geq |\alpha|^2$ . For pure state, $p_1 p_2 = |\alpha|^2$

# Why the extra trouble?

Because in quantum, states like $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ (ie superposition of heads and tails) are just as natural as $|0\rangle$ and $|1\rangle$.

We can **measure** the quantum coin with respect to *arbitrary* **orthonormal** basis $\{|\phi_0\rangle, |\phi_1\rangle\}$

**Born rule**: probability to measure the system to be in state $|\phi_n\rangle$ is $p_n = \langle\phi_n|\rho|\phi_n\rangle$

**Collapse**: After the measurement, the density matrix will become the state that is observed.

**Off diagonal elements are necessary to encode all this info**

# Instructive example

$$\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Encodes classical uncertainty

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\rho_2 = |\psi\rangle\langle\psi| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Encodes quantum uncertainty (pure state)

Convince yourself: we cannot distinguish between these two systems by only measuring in the heads / tails basis. But we can distinguish if we use another orthonormal basis. Can you think of one?

# More on pure states

They can be written as:

$$\rho = |\psi\rangle\langle\psi|$$

Clearly, $\rho^2 = \rho$

Often we work with $|\psi\rangle$ itself rather than $\rho$ when we have pure state

**Born rule**: $p_n = |\langle\phi_n|\psi\rangle|^2 = \langle\phi_n|\rho|\phi_n\rangle$

**More standard quantum language…**

# Two quantum coins

$\rho$ will be a 4x4 matrix.

Conventional basis: $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$

Same Born rule: $P(n,m) = \langle n,m | \rho | n,m \rangle$

But what if only subsystem 1 is measured?

Ans: $P(n) = \sum_m P(n,m)$

# Two quantum coins

Suppose "n" is measured for state of subsystem 1.

DM after measurement: $\rho' = \dfrac{P_n \rho P_n}{\mathrm{Tr}(P_n \rho P_n)}$ where

$$P_n = \sum_m |n,m\rangle\langle n,m|$$

"Collapse"

What then is probability of second state being "m"?

Born rule: $P(m|n) = \langle n,m|\rho'|n,m\rangle$

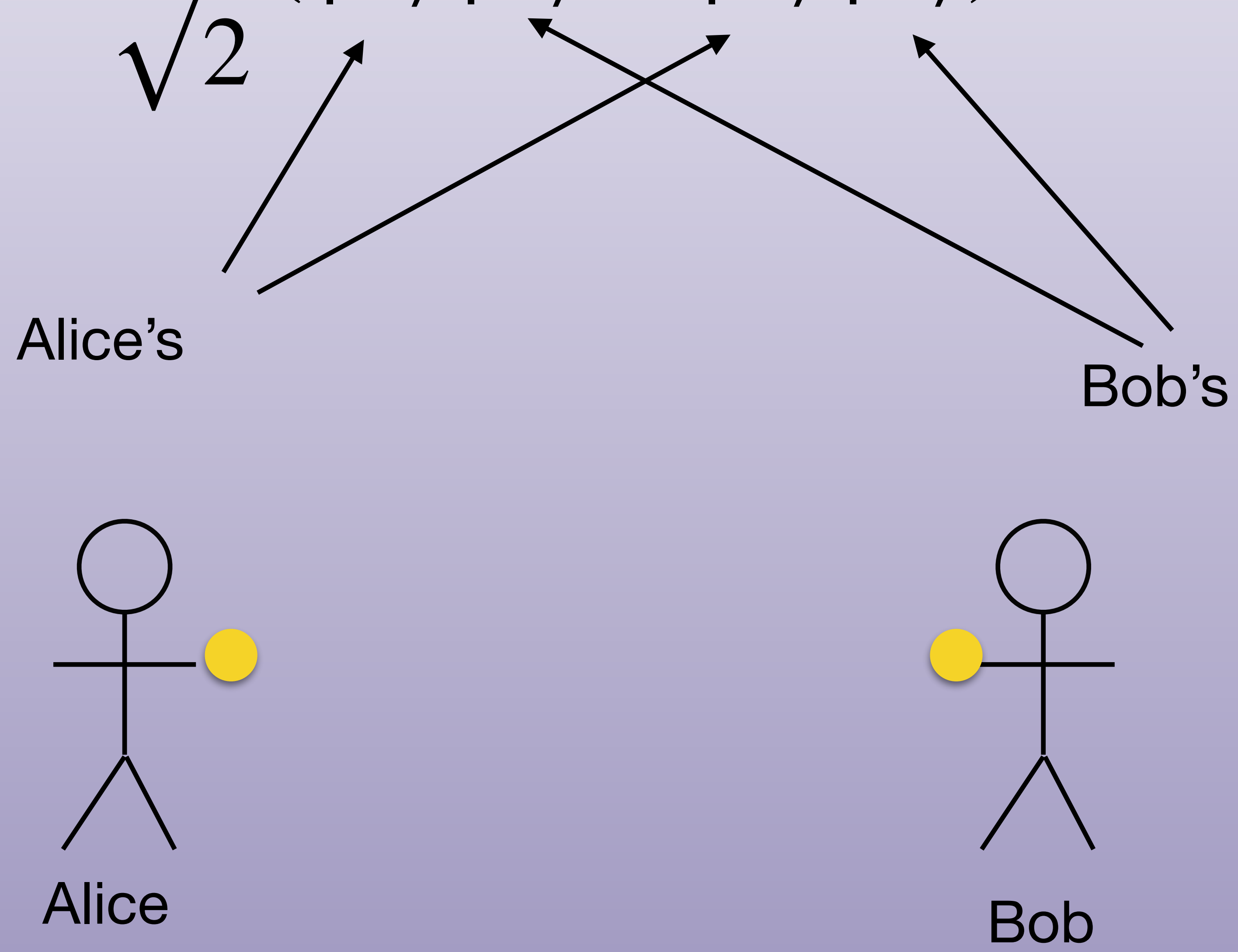
With a bit of work we see: $P(m|n) = P(n,m)/P(n)$

Bayes theorem! (Excellent sanity check)

# Entanglement

Let’s consider two coins. Alice has one, Bob has the other. Take the coins to be in the **entangled state:**

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$



# Entanglement


Alice looks at her coin first. Then Bob looks at his.

Probability that Bob's coin is heads provided Alice saw heads: 100%
Probability that Bob's coin is tails provided Alice saw tails: 100%

Alice's measurement outcome influenced Bob's coin!

**Try working through this using machinery on previous few slides**

# The Schrödinger Equation

Important but not main character of project

It describes how states evolve

$$i\frac{d}{dt}|\psi > = H|\psi >$$

$H$ is a Hermitian matrix determined by some physical system

Need $H$ to be Hermitian for normalisation of $|\psi\rangle$ to hold for later times

# Superposition states are easily obtained

Start with $|0\rangle$

Evolve according to:

$$i\frac{d}{dt}|\psi> = H|\psi> \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solve and get: $|\psi> = \cos(t)|0\rangle - i\sin(t)|1\rangle$

# Planning

This is a lot of info. Take it away and experiment with it. Try to derive results and explore. Look things up, talk with friends, talk with me, etc if confused!

I want to leave plenty of time in these meetings for unstructured discussion. This meeting included!

This lecture was crash course on general background info. Next ones will focus on projects.

Pursue what you find interesting. Deviations (within reason) from suggested directions are encouraged.