On the radius of Gaussian free field excursion clusters

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Abstract

We consider the Gaussian free field φ on $\mathbb{Z}^d$, for $d \geq 3$, and give sharp bounds on the probability that the radius of a finite cluster in the excursion set $\{\varphi \geq h\}$ exceeds a large value N, for any height $h \neq h_*$, where h_* refers to the corresponding percolation critical parameter. In dimension $d = 3$, we prove that this probability is sub-exponential in N and decays as $\exp\{-\frac{\pi}{6}(h-h_*)^2 \frac{N}{\log N}\}$ as $N \to \infty$ to principal exponential order. When $d \geq 4$, we prove that these tails decay exponentially in N. Our results extend to other quantities of interest, such as truncated two-point functions and the two-arms probability for annuli crossings at scale N.

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1 Introduction

This article investigates the percolative properties of excursion sets $\{\varphi \geq h\}$ of the Gaussian free field φ on $\mathbb{Z}^d$ in dimensions $d \geq 3$, for varying height parameter $h \in \mathbb{R}$. This model, the rigorous study of which was initiated in [2], and more recently re-instigated in [19], serves as a benchmark example of a (non-planar) percolation model with strong, algebraically decaying correlations. One of its appealing features is the rich interplay with potential theory for the underlying random walk, which is beneficial to its study. A central role is thus played by electrostatic notions such as capacity, see e.g. [1], [5], and more recently [20], [16], [15], [3], [21], [4], which will also feature prominently in the present work.

Our main focus concerns the radii of finite clusters in the excursion sets $\{\varphi \geq h\}$, which we set out to introduce. Under a suitable probability $\mathbb{P}$, the field φ is the centered Gaussian field with covariance $\mathbb{E}[\varphi_x \varphi_y] = g(x,y)$, for $x, y \in \mathbb{Z}^d$, where $g(\cdot,\cdot)$ denotes the Green function of the simple random walk on $\mathbb{Z}^d$, see (2.1). The critical parameter for percolation of the associated excursion sets $\{\varphi \geq h\} := \{x \in \mathbb{Z}^d : \varphi_x \geq h\}$ is defined as

$$h_* = h_*(d) := \inf\{h \in \mathbb{R} : \lim_N \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N] = 0\}, \tag{1.1}$$

where, with hopefully obvious notation, the event in (1.1) refers to a (nearest-neighbor) path in $\{\varphi \geq h\}$ connecting 0 and ∂B_N, where $B_N = B_N(0)$, $B_N(x) := \{y \in \mathbb{Z}^d : |y - x|_\infty \leq N\}$ for all $x \in \mathbb{Z}^d$, $N \geq 1$, and ∂K refers to the inner (vertex) boundary of a set $K \subset \mathbb{Z}^d$. It is known that $0 < h_* < \infty$ for all $d \geq 3$, see [2], [19], [9], and that the infinite cluster, when existing, is almost surely unique. Auxiliary parameters $\bar{h}$ and h_{**} satisfying $\bar{h} \leq h_* \leq h_{**}$ were frequently used in the past, respectively characterizing a phase of 'well-behavedness' for the infinite cluster and a strongly subcritical regime, in which connectivities decay rapidly. Recently it was proved in [11] that

$$\bar{h} = h_* = h_{**}. \tag{1.2}$$

As a consequence, one knows the following: there exists $c_1 = c_1(d,h) > 0$ and $c = c(h,d) > 0$ such that, for all $N \geq 1$

$$\mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N] \leq e^{-c\,N^{c_1}}, \text{ if } h > h_*(d) \ (= h_{**}(d)), \tag{1.3}$$

$$\mathbb{P}\left[\mathrm{LocUniq}(N,h)^c\right] \leq e^{-c\,N^{c_1}}, \text{ if } h < h_*(d) \ (= \bar{h}(d)), \tag{1.4}$$

where the 'local uniqueness' event in (1.4) is defined as

$$\mathrm{LocUniq}(N,h) = \{\{\varphi \geq h\} \text{ has a } \textit{unique} \text{ connected component crossing } B_{2N} \setminus B_N\}. \tag{1.5}$$

Here and in the sequel, a set $S \subset \mathbb{Z}^d$ is said to cross $V \setminus U$, for $U \subset V \subset \mathbb{Z}^d$ if S has a connected component intersecting both U and ∂V. The estimate (1.4) is inherited from the bounds for the 'existence' and 'uniqueness' events usually appearing in the definition of $\bar{h}$, see e.g. (1.10)–(1.11) in [8], which assert that for all $h < \bar{h}\,(= h_*)$ and $N \geq 1$,

$$\begin{aligned}&\mathbb{P}\left[\begin{array}{c}\text{there exists a connected component in}\\ \{\varphi \geq h\} \cap B_N \text{ with diameter at least } N/5\end{array}\right] \leq e^{-c\,N^{c_1}} \text{ and}\\ &\mathbb{P}\left[\begin{array}{c}\text{any two clusters in } \{\varphi \geq h\} \cap B_N \text{ having diameter at}\\ \text{least } N/10 \text{ are connected to each other in } \{\varphi \geq h\} \cap B_{2N}\end{array}\right] \leq e^{-c\,N^{c_1}},\end{aligned} \tag{1.6}$$

where the diameter of a set is with respect to the sup-norm. Indeed, (1.4) follows by straightforward gluing arguments, combining the events in (1.6) at a fixed number of scales commensurate with N. Stretched exponential bounds such as (1.3), (1.4) and (1.6) typically arise as a by-product of certain static renormalization methods, see e.g. [19] regarding (1.3), which exemplifies this phenomenon. Little is otherwise known about the true order of decay for the probabilities in (1.3) and (1.4). To date, the best available results are due to [17, 18], which *solely* concern the subcritical regime and yield that (1.3) holds with $c_1(d,h) = 1$ if $d \geq 4$ and $h > h_*$, along with an upper bound for $\mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N]$ of exponential order $N/(\log N)^{3+\varepsilon}$, for any $\varepsilon > 0$, when $d = 3$ and $h > h_*$.

Our findings address these matters. Our main results are most easily formulated in terms of a 'truncated one-arm event'. We refer to the discussion following the statement of Theorem 1.2 below regarding extensions of (1.7) and (1.8) to other quantities of interest. Upper bounds in the spirit of those obtained below for $h > h_*$ have also been derived in [7] for the so-called metric graph associated to $\mathbb{Z}^d$. Contrary to what is suggested in Section 1.3 of [7], the logarithmic factor in dimension three is *not* an artefact.

Theorem 1.1. *If $d = 3$, then for every $h \neq h_*$ one has*

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N,\, 0 \overset{\varphi \geq h}{\not\leftrightarrow} \infty] = -\frac{\pi}{6}(h - h_*)^2. \tag{1.7}$$

In higher dimensions, we have the following:

Theorem 1.2. *For every $d \geq 4$ and $h \neq h_*$, there exist $C = C(d,h)$, $c = c(d,h) > 0$ such that*

$$e^{-CN} \leq \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N,\, 0 \overset{\varphi \geq h}{\not\leftrightarrow} \infty] \leq e^{-cN}. \tag{1.8}$$

Theorems 1.1 and 1.2 follow immediately by combining the results of Theorems 3.1 and 5.1 below, which separately deal with the corresponding lower and upper bounds, respectively. In fact, as asserted in these two theorems, (1.7) and (1.8) continue to hold when $h < h_*$ if one replaces the event in question by $\text{LocUniq}(N,h)^c$, see (1.5). Together with the disconnection upper bound from [20, Theorem 5.5], which yields that disconnecting B_N from ∂B_{2N} decays exponentially at scale N^{d-2} when $h < h_*$, this is easily seen to imply that

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[\text{2-arms}(N,h)] = -\frac{\pi}{6}(h - h_*)^2, \text{ if } h < h_* \text{ and } d = 3 \tag{1.9}$$

along with a statement similar to (1.8) when $d \geq 4$, where '2-arms' refers to the existence of two disjoint crossing clusters of $\{\varphi \geq h\} \cap (B_{2N} \setminus B_N)$.

Concerning the truncated two-point function, defined as

$$\tau_h^{\text{tr}}(x,y) = \mathbb{P}[x \xleftrightarrow{\varphi \geq h} y,\, x \overset{\varphi \geq h}{\not\leftrightarrow} \infty], \text{ for } x, y \in \mathbb{Z}^d \text{ and } h \in \mathbb{R}, \tag{1.10}$$

which is symmetric in x and y and satisfies $\tau_h^{\text{tr}}(x,y) = \tau_h^{\text{tr}}(0, y - x)$ by translation invariance of the set $\{\varphi \geq h\}$, our results readily imply (cf. the proof of Theorem 3.1 below) that the asymptotics (1.7) and (1.8) also hold for $\tau_h^{\text{tr}}(0, Ne_1)$, $h \neq h_*$, where e_1 denotes the unit vector in a coordinate direction of $\mathbb{Z}^d$. More generally, with $|\cdot|$ denoting the Euclidean distance, one may also expect that for arbitrary $x, y \in \mathbb{Z}^d$,

$$\lim_{|x-y|\to\infty} \frac{\log|x-y|}{|x-y|} \log \tau_h^{\mathrm{tr}}(x,y) = -\frac{\pi}{6}(h-h_*)^2, \text{ when } d=3. \tag{1.11}$$

We refer to Remarks 2.3 and 5.17, 2) below regarding the (technical) modifications to our argument needed to prove (1.11) and compelling evidence for its truthfulness. Finally, let us emphasize that, while (1.2) leads to a form of Theorem 1.1 indicating that h_* is approached at the *same* rate both as $h \searrow h_*$ and $h \nearrow h_*$, a version of our findings could be stated in terms of $\bar{h}$ and h_{**} only, much as in [20], [15], [3], thus yielding (1.7) and (1.8) upon applying (1.2). This is the sole place where (1.2) is used.

We now highlight some ideas behind the proofs. One is immediately struck by the discrepancy in the strength of the above results. This is closely related to the fact that the random walk does not 'see' one-dimensional sets (such as bounded off-critical percolation clusters) when $d \geq 4$. Our proofs witness this structural difference between the cases $d = 3$ and $d \geq 4$ very clearly. To see this, first observe that (see Lemma 2.2 for precise statements) as $N \to \infty$,

$$\begin{aligned} &\mathrm{cap}\big(([0,\dots,N]\cap\mathbb{Z})\times\{0\}^{d-1}\big) \sim \frac{\pi}{3}\frac{N}{\log N}, && \text{when } d=3, \text{ whereas} \\ &\mathrm{cap}\big(([0,\dots,N]\cap\mathbb{Z})\times\{0\}^{d-1}\big) \asymp N, && \text{when } d\geq 4. \end{aligned} \tag{1.12}$$

Now, the coarse-graining described in more detail below (from which we eventually deduce the upper bounds in (1.7), (1.8)), yields a sum two terms for the probability in question. One of them corresponds to a truncated version of φ (a local field, independent at large scales), for which a corresponding one-arm event decays exponentially in N, regardless of the dimension d. The other term, which carries the long-range dependence, stems from the behavior of the harmonic field in a collection of well-separated boxes, and will turn out to behave in a manner proportional to $\mathrm{cap}\big(([0,\dots,N]\cap\mathbb{Z})\times\{0\}^{d-1}\big)$ to leading exponential order. In view of (1.12), this means that the harmonic term clearly dominates in dimension 3, whereas the two terms live at the same exponential scale in dimension four and higher (and in fact the local term is typically larger).

The lower bounds derived in Section 3 further reflect this disparity. For $d = 3$, in the subcritical regime, we use a change of measure argument in order to draw a finite path in $\{\varphi \geq h\}$ in a thin horizontal tube. The supercritical regime requires a more delicate treatment, as discussed below. Intuitively, the field shifts itself by the right amount in a suitable region as to make the event in question typical, cf. Lemma 3.2 for a general result in this direction, which is of independent interest. The limit on the right-hand side of (1.7) thereby emerges in the corresponding Radon-Nikodym derivative as half of the leading order pre-factor for the capacity of the shifted region, which is close to that of a line of length N, see (2.12), times the square of the height gap. Similar arguments have been used in the study of hard wall conditions for φ, see [1], and disconnection probabilities for supercritical excursion sets, see [20]. Importantly, the monotonicity of the events in question (common to these references) is absent for the one in (1.7) when $h < h_*$, which requires that we 'insulate' the path, i.e. build an interface in $\{\varphi < h\}$ to shield it away from ∞. This makes the implementation of our lower bound strategy relatively involved in the supercritical regime and forces us to introduce Dirichlet boundary conditions to decorrelate constituents of opposite monotonicity. In sharp contrast, the lower bounds in (1.8) follow by 'FKG-type' arguments, which do not witness the critical parameter h_* at all, see (3.3).

Most of our work goes into proving the upper bounds required for Theorems 1.1 and 1.2, summarized in Theorem 5.1 below. A stepping stone towards this is a certain coarse-graining scheme for paths, developed in Section 4 (see in particular Proposition 4.3 below), which we now briefly describe. Roughly speaking, for a path γ of linear size N, the coarse-graining of γ, formalized in Definition 4.2, only retains the trace of γ in a system of 'well-separated' boxes at scale $L \ll N$. Importantly, the scheme walks the fine line of operating at a preferential entropic cost (parametrized by a function $\Gamma(\cdot)$, see (4.13), (4.14)), while retaining a sufficiently 'large' piece of path when measured in terms of capacity. This latter property, ensured by Proposition 4.3, see (4.16), is crucial for the precise estimates we aim at.

In the subcritical phase $h > h_*$, the above scheme is used to cascade a connection event such as $\{0 \stackrel{\varphi \geq h}{\longleftrightarrow} \partial B_N\}$ from scale N down to scale $L(\ll N)$. For each of the boxes at scale L in the resulting collection, the occurrence of a crossing in that box is split into a similar event for a localized field with good decorrelation properties as the box is varied, and the occurrence of an atypical behavior for the corresponding harmonic average, see (5.5)-(5.8). The leading-order contribution is thereby carried by the harmonic field in all but a small fraction of L-boxes, which we control by means of state-of-the-art estimates developed in [20], cf. Lemma 4.1 below. The strength of these estimates hinges on a suitable capacity lower bound for the underlying collection of boxes, which Proposition 4.3 provides.

The resulting two-scale estimate for the one-arm event can then be applied iteratively, see Proposition 5.2 below, to boost an a-priori bound such as (1.3) (but see Remarks 5.3 and 5.6 below to accommodate much weaker a-priori bounds) to the desired decay in a finite number of steps, if L is carefully chosen as a function of N (as will turn out, L needs to grow poly-logarithmically in N). In fact, two steps suffice if one starts from (1.3).

The derivation of the desired upper bounds in the supercritical regime, see (5.2) and (5.4) in Theorem 5.1, is considerably more involved. When $h < h_*$, connections become typical and the cost displayed in (1.7) and (1.8) measures the difficulty to avoid the infinite cluster. Our approach revolves around an event G_N, see (5.32), ensuring roughly speaking that any macroscopic path at scale N will have a_N 'contact points' in each of b_N interfaces all of which are connected to infinity in $\{\varphi \geq h\}$. These contact points are in fact local areas at a microscopic scale L_0 in which a certain insertion tolerance property holds (which the model does not possess as such due to the strength of the correlations), thus yielding a small i.i.d. cost to avoid connecting to the infinite cluster. This property is conveniently defined in terms of a 'mid-point' extension of φ that was used in [8], see (2.37) and (5.30) (incidentally, we also take advantage of this extension to deal with competing monotonicity properties of the path and the insulating interface when deriving the lower bounds for $d \geq 4$ and $h < h_*$).

An upper bound on the key quantity $\mathbb{P}[G_N^c]$ is then derived using a bootstrapping scheme, see Proposition 5.14 below, which works roughly as follows. Starting from a certain (localized) good event $\mathcal{G}_z$ at base scale $L \gg L_0$, comprising a local uniqueness property at that scale and a number a_L ($= 1$ to begin with) of contact points to the ambient cluster for any large path, see Definition 5.12, for which a suitable a-priori estimate is available (cf. Lemma 5.16), the scheme does one of two things: i) in intermediate steps, it re-produces the same event $\mathcal{G}_z$ at larger scale N, improving on both its likelihood and the number a_N of contact points (eventually we need $a_N b_N$ to grow linearly with N when $d \geq 4$ and sub-linearly but with $a_N b_N \gg N/\log N$ when $d = 3$); ii) in the final step, the scheme generates the target event G_N, creating multiple interfaces by stacking good boxes at scale L. In either case, the scheme witnesses this improvement on a certain event, see (5.51), defined in terms of the coarse-graining from Proposition 4.3, and for which a dichotomy (involving local fields and harmonic averages) holds, see (5.65)-(5.67). The proofs of the desired upper bounds then follow somewhat similarly as in the subcritical case.

We now briefly describe the organization of this article. Section 2 gathers several preliminary results that will be used in subsequent sections. Section 3 proves the lower bounds corresponding to Theorems 1.1 and 1.2, see Theorem 3.1. Section 4 supplies the coarse-graining scheme for paths, see Proposition 4.3, which will be instrumental in deriving the upper bounds. The proof differs depending on whether $d = 3$ or $d \geq 4$, which are dealt with separately in Sections 4.1 and 4.2. The desired upper bounds are then derived in Section 5. The sub- and supercritial phases are considered separately in Sections 5.1 and 5.2.

Our convention regarding constants is the following. Throughout, $c, c', C, C', \dots$ denote positive constants that may change from place to place. Numbered constants are defined the first time they appear and remain fixed thereafter. All constants may depend implicitly on the dimension d. Their dependence on other parameters will be made explicit.

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2 Preliminaries and capacity estimates for tubes

In this section, we gather several ingredients that will be used in the sequel. We first introduce some more notation and state a topological condition on paths yielding the existence of blocking interfaces, see Lemma 2.1 below. We proceed to recall certain aspects of potential theory for the random walk on $\mathbb{Z}^d$ and supply suitably precise capacity estimates for 'tubular' sets, including 'porous' versions thereof, see Lemmas 2.2–2.5 below. Finally, we discuss important properties of the free field φ, including a certain mid-point extension of φ.

We consider $\mathbb{Z}^d$, $d \geq 3$, endowed with the usual nearest-neighbor graph structure. We write $x \sim y$ if x and y neighbors, i.e. if $x, y \in \mathbb{Z}^d$ and $|x - y| = 1$. We use $|\cdot|$ to denote the Euclidean and $|\cdot|_\infty$ the ℓ^∞-norm in $\mathbb{Z}^d$ as well as $d(\cdot,\cdot)$ and $d_\infty(\cdot,\cdot)$ to denote the corresponding distances between sets. Recall that $B_N(x)$ denotes the box of radius N around x with respect to $|\cdot|_\infty$, and let $B_N(U) \coloneqq \bigcup_{x \in U} B_N(x)$ for $U \subset \mathbb{Z}^d$. For $U \subset \mathbb{Z}^d$, $\partial U \coloneqq \{x \in U : \exists y \notin U \text{ s.t. } y \sim x\}$ is the *inner (vertex) boundary* of U and $U^c = \mathbb{Z}^d \setminus U$ is the complement of U in $\mathbb{Z}^d$. We also define the outer boundary of a set $U \subset \mathbb{Z}^d$ as $\partial_{\text{out}} U = \partial(U^c)$. For $U, V \subset \mathbb{Z}^d$, we write $U \subset\subset V$ to indicate that U has finitely many elements. A *path* γ in $\mathbb{Z}^d$ is a map $\gamma : \{0, \dots, k\} \to \mathbb{Z}^d$ for some integer $k \geq 0$ such that $|\gamma(i+1) - \gamma(i)| = 1$ for all $0 \leq i < k$. A $*$*-path* is defined similarly, with $|\cdot|_\infty$ replacing $|\cdot|$. A ($*$-)connected set $U \subset \mathbb{Z}^d$ is a set such that any points $x, y \in U$ can be joined by a ($*$-)path whose range is contained in U. Throughout, we use the words *connected component* and *cluster* interchangeably to refer to maximal connected sets.

We now state a useful criterion for the existence of 'dual' surfaces separating two sets, which is interesting in its own right. In the sequel for any $U \subset\subset \mathbb{Z}^d$, let U^c_∞ denote the (unique) connected component of U^c having infinite cardinality, and define $\partial_{\text{ext}} U = \partial(U^c_\infty)$, the exterior boundary of U. For any two finite sets $U_1, U_2 \subset \mathbb{Z}^d$, we say U_1 is *surrounded* by U_2, denoted as $U_1 \preceq U_2$, if U_1 is contained in some finite connected component of $\mathbb{Z}^d \setminus U_2$. Notice that the relation '$\preceq$' is in fact a partial order.

Lemma 2.1 (Existence of blocking interfaces)**.** *Let $V \subset \mathbb{Z}^d$ be a box and $U \subset V$. Also let $\Sigma \subset V \setminus U$ be such that any $*$-path between U and ∂V intersects Σ in at least $k \geq 1$ points. Then there exist $*$-connected subsets $O_1, \dots, O_k$ of Σ such that $S \preceq O_1 \preceq \dots \preceq O_k$.*

Proof of Lemma 2.1. An obvious consequence of the hypothesis of the lemma is that U is not $*$-connected to ∂V in $V \setminus \Sigma \supset U$. It then follows e.g. by [6, Lemma 2.1] that the exterior boundary of the $*$-connected component $\mathscr{C}_U^*$ of U in $V \setminus \Sigma$ is itself $*$-connected, which we pick as O_1. Notice that $U \preceq O_1$ and $O_1 \subset \Sigma$ by definition. Now observe that the hypothesis of the lemma still holds with $k-1$, $\mathscr{C}_U^* \cup O_1$ – which is a $*$-connected set – and $\Sigma \setminus (C_U^* \cup O_1)$ substituting for k, U and Σ respectively. Thus, by iterating the same argument k times we deduce the lemma. $\square$

We now review various aspects of potential theory on $\mathbb{Z}^d$ which will be used in the sequel. We denote by P_x the canonical law of the discrete-time (symmetric) simple random walk on $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$. We write $(X_n)_{n\geq 0}$ for the corresponding canonical process and $(\theta_n)_{n\geq 0}$ for the canonical time shifts. For $U \subset \mathbb{Z}^d$, we introduce the following stopping times: the entrance time $H_U \coloneqq \inf\{n \geq 0 : X_n \in U\}$ in U, the exit time $T_U \coloneqq H_{\mathbb{Z}^d \setminus U}$ from U and the hitting time $\widetilde{H}_U \coloneqq \inf\{n \geq 1 : X_n \in U\}$ of U. We write

$$g_U(x,y) \coloneqq \sum_{n\geq 0} P_x[X_n = y,\, n < T_U], \quad \text{for } x, y \in \mathbb{Z}^d \tag{2.1}$$

for the Green function of the walk killed outside U. By [14], Theorem 1.5.4, with $g = g_{\mathbb{Z}^d}$, one has the asymptotic formula

$$g(x) \coloneqq g(0,x) \sim c_2 |x|^{2-d}, \qquad \text{as } |x| \to \infty, \tag{2.2}$$

(where $\sim$ means that the ratio of both sides tends to 1 in the given limit), for an explicit constant $c_2 = c_2(d) \in (0,\infty)$ with $c_2(3) = \frac{3}{2\pi}$. For $K \subset\subset U \subset \mathbb{Z}^d$, we introduce the equilibrium measure of K relative to U,

$$e_{K,U}(x) \coloneqq P_x[\widetilde{H}_K > T_U] 1_{x \in \partial K} \tag{2.3}$$

and its total mass

$$\mathrm{cap}_U(K) \coloneqq \sum_x e_{K,U}(x), \tag{2.4}$$

the capacity of K (relative to U). We will omit U from all notation whenever $U = \mathbb{Z}^d$. One has the last-exit decomposition, see, e.g. [14, Lemma 2.1.1] for a proof, valid for all $K \subset\subset U \subset \mathbb{Z}^d$,

$$P_x[H_K < T_U] = \sum_y g_U(x,y) e_{K,U}(y), \text{ for all } x \in \mathbb{Z}^d. \tag{2.5}$$

Summing (2.5) over $x \in K$, one immediately sees that

$$\frac{|K|}{\max_{x\in K} \sum_{y\in K} g_U(x,y)} \leq \mathrm{cap}_U(K) \leq \frac{|K|}{\min_{x\in K} \sum_{y\in K} g_U(x,y)}. \tag{2.6}$$

One also has the following sweeping identity (see for instance (1.12) of [20] when $U = \mathbb{Z}^d$):

$$e_{K,U}(y) = P_{e_{K',U}}[H_K < T_U,\ X_{H_K} = y], \text{ for every } K \subset K' \subset\subset U \text{ and } y \in \mathbb{Z}^d. \tag{2.7}$$

Summing over y in (2.7) gives

$$\text{cap}_U(K) = \text{cap}_U(K')P_{\overline{e}_{K',U}}[H_K < T_U], \tag{2.8}$$

where $\overline{e}_{K',U}(\cdot) = e_{K',U}(\cdot)/\text{cap}_U(K')$ is the normalized equilibrium measure. In particular, it follows immediately from (2.8) that $\text{cap}_U(K)$ is increasing in K. Note also that $\text{cap}_U(K)$ is decreasing in U for fixed K. We will also use the following variational characterization of the capacity: for $K \subset\subset U \subset \mathbb{Z}^d$,

$$\text{cap}_U(K) = \frac{1}{\inf_\nu E_U(\nu)}, \text{ where } E_U(\nu) = \sum_{x,y} \nu(x) g_U(x,y) \nu(y) \tag{2.9}$$

and the infimum runs over all probability measures supported on K.

We now give precise bounds on the capacity of certain sets of interest. The capacity of a ball classically satisfies

$$cN^{d-2} \le \text{cap}(B_N) \le CN^{d-2}, \text{ for all } N \ge 0, \tag{2.10}$$

see, e.g., [14, (2.16)]. We are typically going to work in certain (cylindrical) *'tube domains'*, which we introduce now. Given $L \le N$, the tube of length N and width L, which we denote by $T_N(L)$, is defined as the L-neighborhood of the N-line segment $[0, N] \times \{0\}^{d-1}$. Formally,

$$T_N(L) := ([-L, N + L] \cap \mathbb{Z}) \times ([-L, L] \cap \mathbb{Z})^{d-1}. \tag{2.11}$$

We abbreviate $T_N(0) = T_N$, which is a line of length N, and routinely omit the intersection with $\mathbb{Z}$ from our notation below. We now derive certain capacity estimates for tube domains which will be useful in the sequel. We start with the line.

Lemma 2.2 (Capacity of lines)**.** *For $d = 3$, one has*

$$\text{cap}(T_N) \sim \frac{\pi}{3} \frac{N}{\log N}, \quad \textit{as } N \to \infty, \tag{2.12}$$

whereas for $d \ge 4$, there exists $c_3(d) \in (0, 1)$ such that for all $N \ge 1$,

$$c_3 N \le \text{cap}(T_N) \le c_3^{-1} N. \tag{2.13}$$

Proof. Using (2.2) with the precise value of $c_2(3)$, we obtain

$$\sum_{y \in T_N} g(x, y) \le 2 \sum_{y \in x + T_N} g(x, y) \sim \frac{3}{\pi} \log N, \text{ for all } x \in T_N.$$

Substituting this into (2.6) with the choice $K = T_N$ and $U = \mathbb{Z}^d$ yields the asserted lower bound in (2.12). By a similar argument, using (2.6) and noting that the Green function is summable along one-dimensional sets when $d \ge 4$, one obtains *both* upper and lower bound in (2.13).

It remains to show the upper bound in (2.12). For $\delta \in (0, 1)$, letting

$$T_N^- = T_N^-(\delta) := T_N \setminus (([0, N^{1-\delta}] \cup [N - N^{1-\delta}, N]) \times \{0\}^2),$$

one bounds the equilibrium measure by 1 to obtain

$$\text{cap}(T_N) \le 2(1 + N^{1-\delta}) + \sum_{x \in T_N^-} e_{T_N}(x). \tag{2.14}$$

To take care of the sum over T_N^- on the right-hand side, one sums (2.5) for $K = T_N$ and $U = \mathbb{Z}^d$ over $x \in T_N$ and foregoes the terms with $y \in T_N \setminus T_N^-$. Together with (2.14) this yields

$$\text{cap}(T_N) \le 2(1 + N^{1-\delta}) + \frac{N+1}{\inf_{y\in T_N^-} \sum_{x\in T_N} g(x,y)}. \tag{2.15}$$

Now by definition of T_N^- and using (2.2) again we obtain for any $x \in T_N^-$,

$$\sum_{y\in T_N} g(x,y) \ge 2 \sum_{y\in x+T_{N^{1-\delta}}} g(x,y) \sim (1-\delta)\frac{3}{\pi}\log N.$$

Plugging this into (2.15), we get that $\limsup_{N\to\infty} \frac{\text{cap}(T_N)}{\frac{\pi}{3}\frac{N}{\log N}} \le \frac{1}{1-\delta}$, whereupon the upper bound in (2.12) follows by taking $\delta \to 0$. $\square$

Remark 2.3 (Rotational invariance of asymptotic capacity for lines). Let $\mathrm{u} \in \mathbb{R}^3$ with $|\mathrm{u}| = 1$ be any unit vector. Then the asymptotic expression in (2.12) remains valid if one replaces T_N by the line segment joining 0 and $N\mathrm{u}$ discretized in the following manner. For any $x \in \mathbb{R}^3$, let $[x]$ denote a point in $\mathbb{Z}^3$ achieving the minimum distance between x and $\mathbb{Z}^3$. Now let $T_{N,\mathrm{u}} \subset \mathbb{Z}^3$ consist of the points $[j\sqrt{3}\mathrm{u}]$ for all integers j between 0 and $\lceil N/\sqrt{3}\rceil$. Notice that it is always possible to choose the points in such a way that they are distinct. By this construction and the triangle inequality we have, for any $x, y \in \mathbb{R}^3$ such that $[x], [y] \in T_{N,\mathrm{u}}$,

$$|x-y| - \sqrt{3} \le |[x]-[y]| \le |x-y| + \sqrt{3}$$

and consequently $g(x,y) \sim g([x],[y])$ as $|x-y| \to \infty$. The asymptotics on the right-hand side of (2.12) now follow for $\text{cap}(T_{N,\mathrm{u}})$ by the exact same arguments as in the proof of Lemma 2.2. Indeed, the additional $1/\sqrt{3}$ factor appearing in the numerator in (2.6) owing to reduced cardinality compared to T_N gets canceled by the $1/\sqrt{3}$ factor appearing in the denominator because of the increased separation between successive points in $T_{N,\mathrm{u}}$. In fact, the asymptotics (2.12) should hold for any 'reasonable' discretisation of the line segment between 0 and $N\mathrm{u}$.

We will need the following upper bound on the escape probability from a sufficiently dense subset of the line in order to derive capacity estimates for thicker tube regions. We will also use this result in Section 4 while proving Lemma 4.6, which will involve porous versions of these sets (i.e. containing holes).

Lemma 2.4 (Visibility of (porous) lines)**.** *For all $N \ge 1$, $T \subset T_N$ and $x \in \mathbb{Z}^3$ such that $d_\infty(x,T) < N/100$, the following holds. If, for some $\gamma > 0$,*

$$|B_r(x) \cap T| \ge \gamma r, \text{ for all } r \text{ satisfying } d_\infty(x,T) \le r < N \tag{2.16}$$

then

$$P_x[H_T = \infty] \le C(\gamma)\,\frac{\log(1 + d_\infty(x,T))}{\log N}. \tag{2.17}$$

Proof. Throughout the proof, constants may depend implicitly on γ. Let k_0, k_1 be two integers with k_0 smallest so that $d_\infty(x,T) \le 10^{k_0}$ and k_1 largest such that $10^{k_1} \le N$. Notice that $k_1 > k_0$

when $N \geq C$, which we may assume and that $k_0 = k_0(x)$. Consider the boxes $U_k := B_{10^k}(x)$, for $k_0 \leq k \leq k_1$. By (2.16), one knows that

$$\big|U_k \cap T\big| \geq c(\gamma)10^k, \text{ for all } k_0 \leq k \leq k_1 \tag{2.18}$$

Since $\sum_{z' \in U_k \cap T} g(z, z') \leq Ck$ uniformly in $z \in U_k \cap T$ by (2.2), it follows from (2.6) and (2.18) that $\text{cap}(U_k \cap T) \geq c' 10^k/k$ for all $k_0 \leq k \leq k_1$. Therefore, fixing $L \geq 1$ such that

$$2^{-1} \leq c_2^{-1}(g(x)|x|) \leq 2, \text{ if } |x|_\infty \geq L \tag{2.19}$$

it follows that for all x such that $d_\infty(x, T) \geq L$, all $k_0(= k_0(x)) \leq k < k_1$ and $y \in \partial_{\text{out}} U_k$,

$$P_y\big[H_{U_k \cap T} < T_{U_{k+1}}\big] \overset{(2.5)}{\geq} \inf_{z,z' \in U_k} g_{U_{k+1}}(z, z')\text{cap}(U_k \cap T) \geq c10^{-k}\frac{c'10^k}{k} \geq \frac{c}{k}. \tag{2.20}$$

In obtaining (2.20), we also used the fact that $g_{U_{k+1}}(z, z') \geq cg(z, z')$ for $z, z' \in U_k$. Indeed, by the Markov property we have $g(z, z') = g_{U_{k+1}}(z, z') + E_z[g(X_{T_{U_{k+1}}}, z')]$, and on the other hand, the definition of $(U_k)_{k_0 \leq k \leq k_1}$ readily implies that $|y - z'| \geq 5(|z - z'| \vee L)$, for all $y \in \partial U_{k+1}$, which together with (2.19) gives $E_z[g(X_{T_{U_{k+1}}}, z')] \leq \frac{4}{5}g(z, z')$.

Now consider the process $\{Z_k : k \geq 0\}$ on $\mathbb{N} \cup \{\Delta\}$ defined by $Z_0 = 0$ and for $k \geq 1$, conditionally on $Z_0, \ldots Z_{k-1}$,

$$Z_k := \begin{cases} k, & \text{if } Z_{k-1} \neq \Delta \text{ and } H_{U_{k-1} \cap T} \circ \theta_{T_{U_{k-1}}} > T_{U_k}, \\ \Delta, & \text{otherwise.} \end{cases} \tag{2.21}$$

Using the strong Markov property we get that $Z.$ is a Markov chain under P_x and (2.20) implies that $P_x[Z_k \neq \Delta | Z_{k-1} \neq \Delta] \leq 1 - \frac{c}{k}$ for all $k_0 < k \leq k_1$. It follows that for all x with $d_\infty(x, T) \geq L$,

$$P_x[H_T = \infty] \overset{(2.21)}{\leq} P_x[Z_k \neq \Delta, k_0 \leq k \leq k_1] \leq \prod_{k_0 < k \leq k_1} \Big(1 - \frac{c}{k}\Big) \leq C\frac{k_0}{k_1}, \tag{2.22}$$

which yields (2.17) for such x, as $k_0 \leq C \log d_\infty(x, T)$ and $k_1 \geq c \log N$ (see above (2.18)). To handle the case $d_\infty(x, T) \leq L$, the strong Markov property at the time of first exit from $T_N(L)$, see (2.11), with L given by (2.19), implies that $P_x[H_T = \infty] \leq \sup_y P_y[H_T = \infty]$, with the supremum ranging over $y \in \partial_{\text{out}} T_N(L)$ and (2.18) still follows from (2.22) as $d_\infty(y, T) \geq L$. This completes the proof. □

We now move on to capacities of tubes whose width is a fractional power of their length. In the sequel, let $T^\delta_{N,k} = T_N(kN^\delta)$ for $\delta > 0$, $k > 0$ (cf. (2.11)), and abbreviate $T^\delta_N = T^\delta_{N,1}$.

Lemma 2.5 (Capacity of tubes)**.** *The following bounds hold for $d = 3$. There exists $C_1 \in (0, \infty)$ such that for every $\delta \in (0, 1)$ and $N \geq C(\delta)$,*

$$\text{cap}(T^\delta_N) \leq (1 + C_1\delta)\,\text{cap}(T_N), \tag{2.23}$$

$$\text{cap}_{T^\delta_{N,2}}(T^\delta_N) \leq C_1\delta^{-1}\,\text{cap}(T^\delta_N). \tag{2.24}$$

Proof. We claim that for every $x \in \partial T_N^\delta$ and $y \in \partial T_{N,2}^\delta$, one has

$$P_x[H_{T_N} = \infty] \le C\delta \text{ and } P_y[H_{T_N^\delta} = \infty] \ge c\delta, \text{ for all } N \ge C(\delta). \tag{2.25}$$

Before proving (2.25) let us deduce the lemma from it. By the sweeping identity (2.7),

$$\mathrm{cap}(T_N) = P_{e_{T_N^\delta}}[H_{T_N} < \infty] = \mathrm{cap}(T_N^\delta)\, P_{\overline{e}_{T_N^\delta}}[H_{T_N} < \infty],$$

and (2.23) follows directly from the upper bound in (2.25). Also, by decomposing on the first exit time of $T_{N,2}^\delta$, one finds that

$$P_x[\widetilde{H}_{T_N^\delta} = \infty] = \sum_y P_x[\widetilde{H}_{T_N^\delta} > H_{(T_{N,2}^\delta)^c},\ X_{H_{(T_{N,2}^\delta)^c}} = y]\, P_y[H_{T_N^\delta} = \infty],$$

for all $x \in T_N^\delta$, which combined with the lower bound in (2.25) implies that $e_{T_N^\delta}(x) \ge c\,\delta\, e_{T_N^\delta, T_{N,2}^\delta}(x)$. Summing over $x \in T_N^\delta$ yields (2.24). We proceed to the proof of the bounds in (2.25).

The upper bound in (2.25). The bound is obviously true for $\delta \ge 1/2$ by letting $C \ge 2$, whereas for $\delta \in (0, 1/2)$ the hypotheses of Lemma 2.4 hold with $T = T_N$ and any $x \in \partial T_N^\delta$ for all $N \ge C(\delta)$, whence the upper bound in (2.25) follows from (2.17).

The lower bound in (2.25). Below we will use B'_L to denote the two-dimensional box $[-L, L]^2$. First of all, notice that for any $y = (y_1, y_2, y_3) \in \partial T_{N,2}^\delta$, either $(y_2, y_3) \in \partial B'_{2N^\delta}$ or $y \in \{-\lfloor 2N^\delta \rfloor, N + \lfloor 2N^\delta \rfloor\} \times [-2N^\delta, 2N^\delta]^2$. We deal with the former case first. To this end let us consider the projection $X' = (X'_n)_{n\ge 0}$ of $(X_n)_{n\ge0}$ onto its last two coordinates, which has the law of a (lazy) simple random walk in $\mathbb{Z}^2$. Let H'_U denote the entrance time in U for X' and abbreviate $H'_{\mathrm{in}} = H'_{B'_{N^\delta}}$ and $H'_{\mathrm{out}} = H'_{\partial B'_{10N}}$. Applying Exercise 1.6.8 in [14], we get

$$P_{y'}[H'_{\mathrm{out}} < H'_{\mathrm{in}}] \ge c\delta, \text{ if } |y'| \ge 2N^\delta, \tag{2.26}$$

whenever $N \ge C(\delta)$. Now since $\{(x_2, x_3) : x \in T_{N^\delta}\} \subset B'_{N^\delta}$, the inclusion

$$\{H'_{\mathrm{out}} < H'_{\mathrm{in}}, H_{T_N^\delta} \circ \theta_{H'_{\mathrm{out}}} = \infty\} \subset \{H_{T_N^\delta} = \infty\}$$

holds and consequently, by the strong Markov property, we have

$$P_y[H_{T_N^\delta} = \infty] \ge P_{y'}[H'_{\mathrm{out}} < H'_{\mathrm{in}}] \inf_{z:\, d_\infty(z, T_N) \ge 10N} P_z[H_{T_N^\delta} = \infty]. \tag{2.27}$$

However, by the last-exit decomposition (2.5) and (2.2),

$$P_z[H_{T_N^\delta} < \infty] \le \mathrm{cap}(T_N^\delta) \max_{z' \in T_N^\delta} g(z, z') \le C \frac{\mathrm{cap}(T_N^\delta)}{N}, \text{ when } d_\infty(z, T_N) \ge 10N.$$

In view of Lemma 2.2 and (2.23) which, let us recall, requires only the upper bound in (2.25), the right-hand side in the previous display is bounded by $\frac{C}{\log N}$. Plugging this and the bound (2.26) into (2.27), we deduce the lower bound in (2.25) in the case $(y_2, y_3) \in \partial B'_{2N^\delta}$.

To deal with the case when $y \in \{-\lfloor 2N^\delta \rfloor, N + \lfloor 2N^\delta \rfloor\} \times [-2N^\delta, 2N^\delta]^2$ let us assume without loss of generality that $y_1 = -\lfloor 2N^\delta \rfloor$ and note that, by means of the strong Markov property and the previous case, it suffices to show that

$$P_y[H' < H_{T_N^\delta}] \ge c, \text{ where } H' := H'_{\partial B'_{2N^\delta}}. \tag{2.28}$$

To this end, let $(X_n^1)_{n\geq 0}$ denote the projection of $(X_n)_{n\geq 0}$ onto its first coordinate and consider the event

$$G \coloneqq \{H_{-\lfloor 10N^\delta \rfloor}(X^1) < H_{-\lfloor N^\delta \rfloor}(X^1)\}$$

which has a constant positive probability under P_y by the standard gambler's ruin estimate. It follows from the definition of T_N^δ that $H_{T_N^\delta} > H_{-\lfloor 10N^\delta \rfloor}(X^1) =: H^1$ on G and hence

$$G \cap \{H^1 \geq H'\} \subset \{H' < H_{T_N^\delta}\}. \tag{2.29}$$

On the other hand, we have

$$G \cap \{H^1 < H'\} \subset \{X_{H^1}^1 = -\lfloor 10N^\delta \rfloor, X'_{H^1} \in B'_{2N^\delta}\}.$$

An implication of the condition on the right hand side above is that $B_{8N^\delta}(X_{H^1}) \cap T_N^\delta = \emptyset$, whereas $B'_{2N^\delta} \subset B'_{8N^\delta}(X'_{H^1})$. Denoting $H \coloneqq H'_{\partial B'_{8N^\delta}(X'_0)} \circ \theta_{H^1}$, we therefore have

$$(G \cap \{H^1 < H'\}) \cap \{X'_H \subset \partial B'_{8N^\delta}(X'_{H^1})\} \subset \{H' < H_{T_N^\delta}\}. \tag{2.30}$$

However, since all two-dimensional axial projections of $(X_n)_{n\geq 0}$ have the same law, we can deduce via a union bound that $P_x[X'_{H_{\partial B_{8N^\delta(x)}}} \in \partial B'_{8N^\delta}(x)] \geq c$ uniformly for all $x \in \mathbb{Z}^d$. Hence, in view of (2.30), we get, applying the strong Markov property at time H^1,

$$P_y[H' < H_{T_N^\delta} \,|\, G, H^1 < H'] \geq c,$$

with y as in (2.28). Combined with (2.29) and the fact that $P_y[G] \geq c$, this yields (2.28) and consequently the lower bound in (2.25) in this case. □

We conclude this section by reviewing some important features of Gaussian free fields. For $U \subset \mathbb{Z}^d$, we write $\mathbb{P}_U$ for the law of the centered Gaussian process with covariance $g_U(\cdot,\cdot)$, with g_U as given by (2.1) (in particular $\mathbb{P} = \mathbb{P}_{\mathbb{Z}^d}$ following our above convention). Notice that under $\mathbb{P}_U$, the field φ is almost surely 0 on $\mathbb{Z}^d \setminus U$. For $U \subset \mathbb{Z}^d$, we further introduce the Gaussian fields (functions of φ)

$$\xi_x^U \coloneqq E_x[\varphi_{X_{T_U}}] = \sum_y P_x[X_{T_U} = y]\varphi_y, \quad \psi_x^U \coloneqq \varphi_x - \xi_x^U, \quad \text{for } x \in \mathbb{Z}^d. \tag{2.31}$$

The field ξ^U will be referred to as the harmonic average of φ in U and ψ^U as the local field in U. Plainly, $\xi_x^U = \varphi_x$ (and therefore $\psi_x^U = 0$) for all $x \in \mathbb{Z}^d \setminus U$. As in [19, Lemma 1.2], one observes that ξ^U is independent of ψ^U and that $(\psi_x^U)_{x\in\mathbb{Z}^d}$ has law $\mathbb{P}_U$ under $\mathbb{P}$.

It will be convenient at times to consider a certain extension of the above setup. Let $\mathbb{M}^d$ denote the set of mid-points of the edges of $\mathbb{Z}^d$. We regard $\tilde{\mathbb{Z}}^d \coloneqq \mathbb{Z}^d \cup \mathbb{M}^d$ as the graph obtained from $\mathbb{Z}^d$ by splitting every edge of $\mathbb{Z}^d$ into two (and adding the corresponding mid-point to the vertex set). Let $\tilde{X} = (\tilde{X}_n)_{n\geq 0}$ be the discrete-time random walk on $\tilde{\mathbb{Z}}^d$, which at each step jumps with uniform probability to one of its neighboring vertices in $\tilde{\mathbb{Z}}^d$. Let $\tilde{P}_{\tilde{x}}$ denote the canonical law of $\tilde{X}$ with starting point $\tilde{X}_0 = \tilde{x} \in \tilde{\mathbb{Z}}^d$. By suitable extension of $\mathbb{P}$, one defines a centered Gaussian field $\tilde{\varphi} = (\tilde{\varphi}_{\tilde{x}})_{\tilde{x}\in\tilde{\mathbb{Z}}^d}$ such that

$$\mathbb{E}[\tilde{\varphi}_{\tilde{x}}\tilde{\varphi}_{\tilde{y}}] \coloneqq \frac{1}{2}\sum_{n\geq 0} \tilde{P}_{\tilde{x}}[\tilde{X}_n = \tilde{y}], \text{ for } \tilde{x}, \tilde{y} \in \tilde{\mathbb{Z}}^d. \tag{2.32}$$

Indeed, it follows from (2.32) and (2.1) that $\mathbb{E}[\tilde{\varphi}_x \tilde{\varphi}_y] = g(x,y)$ whenever $x, y \in \mathbb{Z}^d$, whence

$$\tilde{\varphi}|_{\mathbb{Z}^d} = \varphi. \tag{2.33}$$

The decomposition (2.31) also extends and one obtains that

$$\tilde{\varphi} = \tilde{\xi}^V + \tilde{\psi}^V, \text{ for } V \subset \tilde{\mathbb{Z}}^d, \text{ where } \tilde{\xi}^V_{\tilde{x}} := \tilde{E}_{\tilde{x}}\big[\tilde{\varphi}_{\tilde{X}_{T_V}}\big]; \tag{2.34}$$

here $\tilde{\xi}^V$ and $\tilde{\psi}^V$ are independent Gaussian fields and

$$\mathbb{E}\big[\tilde{\psi}^V_{\tilde{x}} \tilde{\psi}^V_{\tilde{y}}\big] = \frac{1}{2} \sum_{n \geq 0} \tilde{P}_{\tilde{x}}[\tilde{X}_n = \tilde{y}, n < T_V], \text{ for } \tilde{x}, \tilde{y} \in \tilde{\mathbb{Z}}^d, \tag{2.35}$$

where with hopefully obvious notation, $T_V = T_V(\tilde{X})$ denotes the exit time of $\tilde{X}$ from V. The analogue of the restriction property (2.33) for the harmonic extension is then the following. For $U \subset \mathbb{Z}^d$, defining $\tilde{U} := U \cup \{\tilde{x} \in \tilde{\mathbb{Z}}^d : \exists x \in U \text{ s.t. } |x - \tilde{x}| = \frac{1}{2}\}$, noting that X_{T_U} under P_x has the same law as $\tilde{X}_{T_{\tilde{U}}}$ under $\tilde{P}_x$ for any $x \in \mathbb{Z}^d$ and using (2.33), one sees that

$$\tilde{\xi}^{\tilde{U}}|_{\mathbb{Z}^d} = \xi^U. \tag{2.36}$$

We conclude with a particular instance of (2.34), which will prove useful on several occasions. Let

$$\tilde{\varphi} = \hat{\xi} + \hat{\psi} \tag{2.37}$$

be the decomposition (2.34) corresponding to the choice $V := \mathbb{Z}^d (\subset \tilde{\mathbb{Z}}^d)$, i.e. $\hat{\xi}_x = \hat{\xi}^{\mathbb{Z}^d}_x = \frac{1}{2d} \sum_{m \in \mathbb{M}^d : m \sim x} \tilde{\varphi}_m$ if $x \in \mathbb{Z}^d$ (where $\sim$ refers to neighbors in $\tilde{\mathbb{Z}}^d$), $\hat{\xi}_m = \hat{\xi}^{\mathbb{Z}^d}_m = \tilde{\varphi}_m$ if $m \in \mathbb{M}^d$, and by (2.35),

(2.38) $(\hat{\psi}_x)_{x \in \mathbb{Z}^d}$ is a field of i.i.d. centered Gaussian variables with variance 1/2 each.

3 Lower bounds

The main result of this section is the following

Theorem 3.1 (Lower bounds)**.** *The following holds:*

i) If $d = 3$, then

$$\text{for all } h > h_*, \ \liminf_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}[0 \stackrel{\varphi \geq h}{\longleftrightarrow} \partial B_N] \geq -\frac{\pi}{6}(h - h_*)^2, \tag{3.1}$$

$$\text{for all } h < h_*, \ \liminf_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}[0 \stackrel{\varphi \geq h}{\longleftrightarrow} \partial B_N, \, 0 \stackrel{\varphi \geq h}{\not\longleftrightarrow} \infty] \geq -\frac{\pi}{6}(h - h_*)^2. \tag{3.2}$$

ii) If $d \geq 4$, then for all $h \in \mathbb{R}$,

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}[0 \stackrel{\varphi \geq h}{\longleftrightarrow} \partial B_N, \, 0 \stackrel{\varphi \geq h}{\not\longleftrightarrow} \infty] > -\infty. \tag{3.3}$$

Moreover, the bounds (3.2) *and* (3.3) *also hold for the event* $\text{LocUniq}(N,h)^c$ *(see* (1.5)*) in place of* $\{0 \stackrel{\varphi \geq h}{\longleftrightarrow} \partial B_N, \, 0 \stackrel{\varphi \geq h}{\not\longleftrightarrow} \infty\}$.

3.1. General entropic lower bound. The following lemma will be used in the course of proving Theorem 3.1, but is of independent interest. The lower bound it asserts in (3.6) will follow by a change of measure argument, see e.g. the proof of Theorem 2.1 in [20], or Lemma 2.3 in [1], for results of a similar flavor. Given an event $A \in \mathcal{B}(\mathbb{R}^K)$, $K \subset \mathbb{Z}^d$ and a height parameter $h \in \mathbb{R}$, we define

$$A^h = A^h(\varphi) = \{\varphi_{|K} - h \in A\}, \tag{3.4}$$

where, with hopefully obvious notation $\varphi_{|K} - h$ refers to the field (restricted to K) shifted by $-h$ coordinatewise.

Lemma 3.2 (Entropic lower bound). *Let $K_N \subset\subset U_N \subset \mathbb{Z}^d$ be subsets with $\text{cap}_{U_N}(K_N) \to \infty$. Let $A_N \in \mathcal{B}(\mathbb{R}^{K_N})$ and $I \subset \mathbb{R}$ be an interval such that, for every $h' \in I$,*

$$\mathbb{P}_{U_N}[A_N^{h'}] \to 1. \tag{3.5}$$

Then for every $h \notin I$,

$$\liminf_{N\to\infty} \frac{1}{\text{cap}_{U_N}(K_N)} \log \mathbb{P}_{U_N}\big[A_N^h\big] \geq -\frac{1}{2} d(h, I)^2. \tag{3.6}$$

Proof. Recall the following fact, which is a consequence of Jensen's inequality, see e.g. the discussion following (2.7) in [1] for a proof. Given two probability measures $\tilde{\mathbb{P}}$ and $\mathbb{P}$ such that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$, and an event A with positive $\tilde{\mathbb{P}}$-probability, one has

$$\mathbb{P}[A] \geq \tilde{\mathbb{P}}[A] e^{-(1/\tilde{\mathbb{P}}[A])(H(\tilde{\mathbb{P}}|\mathbb{P})+1/e)}. \tag{3.7}$$

where $H(\tilde{\mathbb{P}}|\mathbb{P}) := \tilde{\mathbb{E}}\left[\log \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right]$ is the relative entropy of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. Abbreviate $K = K_N$ and $U = U_N$ in the sequel. Pick $h \notin I$ and $h' \in I$. Using (2.5), it follows by an application of the Cameron-Martin theorem, see e.g. [13, Corollary 14.1] with the choice (in the notation of [13]) $\xi := (h - h')\langle e_{K,U}, \varphi\rangle \in L^2(\mathbb{P})$, that $\tilde{\mathbb{P}}_U$ defined by

$$\frac{d\tilde{\mathbb{P}}_U}{d\mathbb{P}_U} = \exp\Big\{(h - h')\langle e_{K,U}, \varphi\rangle - \frac{(h - h')^2}{2} \text{cap}_U(K)\Big\} \tag{3.8}$$

is a probability such that φ has the same law under $\tilde{\mathbb{P}}_U$ as $\varphi + f$ under $\mathbb{P}_U$, where

$$f(x) = (h - h')P_x[H_K < T_U], \ x \in \mathbb{Z}^d. \tag{3.9}$$

(indeed observe to this effect that in (14.3) of [13], one obtains $\rho_\xi(\varphi_\cdot) = \varphi_\cdot + \mathbb{E}_U[\xi\varphi_\cdot] = \varphi_\cdot + f_\cdot$ by (2.5)). In particular, $f = h - h'$ on K, whence $\tilde{\mathbb{P}}_U[A^h(\varphi)] = \mathbb{P}_U[A^h(\varphi + f)] = \mathbb{P}_U[A^{h'}(\varphi)]$ which tends to 1 as $N \to \infty$ by (3.5). Moreover, by (3.8) and (3.9), noting that $\tilde{\mathbb{E}}_U\big[\langle e_{K,U}, \varphi\rangle\big] = \langle e_{K,U}, f\rangle$, one sees that

$$H\big(\tilde{\mathbb{P}}_U\big|\mathbb{P}_U\big) = (h - h')\tilde{\mathbb{E}}_U\big[\langle e_{K,U}, \varphi\rangle\big] - \frac{(h - h')^2}{2}\text{cap}_U(K) = \frac{(h - h')^2}{2}\text{cap}_U(K).$$

Applying (3.7), taking logarithms and letting $N \to \infty$ now readily yields (3.6), since $h' \in I$ was arbitrary. □

3.2. Lower bounds for $d = 3$. In this section, we show the lower bounds (3.1) and (3.2), which will both follow from an application of Lemma 3.2, with carefully chosen events A^h in (3.4) as to implement sufficiently cost-effective strategies for connection. The asserted bound (3.2) bears the additional difficulty that the event in question is not monotone, which makes its proof more involved than (3.1).

Proof of (3.1) *and* (3.2). We begin with the proof of (3.1). Recalling the notation from (2.11), define the thin cylinder $K_N \coloneqq T_N(L)$, with

$$L \coloneqq 8\lceil (\log N)^{2/c_1} \rceil \tag{3.10}$$

(cf. (1.3), (1.4) regarding c_1) and let $F_N^- = \{0\} \times [-L, L]^{d-1}$, $F_N^+ = \{N\} \times [-L, L]^{d-1}$. Note that $F_N^\pm \subset K_N$ and that $0 \in F_N^-$, while $F_N^+ \subset \partial B_N$. For $h \in \mathbb{R}$, consider the event

$$A_N^h \coloneqq A_{N,L}^h \coloneqq \{F_N^- \text{ and } F_N^+ \text{ are connected by a path in } \{\varphi \geq h\} \cap K_N\}, \tag{3.11}$$

which is of the form (3.4).

Let us first check that $\mathbb{P}[A_N^{h'}] \to 1$ for all $h' \in I := (-\infty, h_*)$. In view of (1.5), one readily sees that

$$\bigcap_{k \in \{-L, -L+1, \dots, N+L\}} \big(\{B_{L/8} \xleftrightarrow{\varphi \geq h'} \partial B_L\} \cap \text{LocUniq}(L/4, h')\big) \circ \tau_{ke_1} \subset A_N^{h'},$$

where τ_x, $x \in \mathbb{Z}^d$, denote the canonical space shifts for $\mathbb{P}$. Now, combining the bounds of (1.6) and in view of (3.10) one deduces that $\mathbb{P}[B_{L/8} \xleftrightarrow{\varphi \geq h'} \partial B_L] \geq 1 - e^{-c\,(\log N)^2}$. Together with (1.4) and a union bound, this is easily seen to imply $\lim_N \mathbb{P}[A_N^{h'}] = 1$, as desired. Lemma 3.2 thus applies with $U_N = \mathbb{Z}^d$ and yields for any $h > h_*$,

$$\liminf_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}\big[A_N^h\big] \geq -\frac{\pi}{6}(h - h_*)^2, \tag{3.12}$$

using that $\text{cap}(K_N) \sim \frac{\pi}{3} \frac{N}{\log N}$ as $N \to \infty$, which follows directly from (2.23), monotonicity $\text{cap}(\cdot)$ and (2.12).

Next, by the FKG-inequality for φ, one knows that $\mathbb{P}[F_N^- \subset \{\varphi \geq h\}] \geq \exp\{-C(h)L^{d-1}\}$. In particular, in view of (3.10), it follows that

$$\liminf_{N \to \infty} \frac{\log N}{N} \log \mathbb{P}\big[F_N^- \subset \{\varphi \geq h\}\big] = 0, \text{ for all } h \in \mathbb{R}. \tag{3.13}$$

Recalling A_N^h from (3.11) and the fact that $0 \in F_N^-$, one observes that $A_N^h \cap \{F_N^- \subset \{\varphi \geq h\}\}$ implies $\{0 \xleftrightarrow{\varphi \geq h} \partial B_N\}$. Hence, (3.1) follows directly from (3.12), (3.13) and the FKG-inequality.

We now show (3.2). For arbitrary $\delta > 0$, let $D_N \subset U_N \subset K_N$ be defined as $D_N \coloneqq T_N(N^\delta)$ $U_N \coloneqq T_N(2N^\delta)$, $K_N \coloneqq T_N(3N^\delta)$ (see (2.11) for notation) and abbreviate $\xi^N = \xi^{U_N}$, $\psi^N = \psi^{U_N}$, cf. (2.31). For an additonal parameter $\varepsilon > 0$ and $h < h_* + \varepsilon$, consider the event

$$C_N^h \coloneqq \big\{\partial_{\text{out}} U_N \overset{\varphi \geq h}{\not\longleftrightarrow} \partial K_N\big\} \cap \big\{\inf_{D_N} \xi^N \geq -(h_* + \varepsilon - h)\big\}, \tag{3.14}$$

and note that C_N^h is of the form (3.4) with $K = K_N$; observe to this effect that due to (2.31), the condition $\xi_\cdot^N \geq -(h_* + \varepsilon - h)$ can be recast as $\sum_{y\in U_N} P_\cdot[X_{H_{U_N}} = y](\varphi_y - h) \geq -(h_* + \varepsilon)$, i.e. as a condition on the field $\varphi_\cdot - h$ restricted to $U_N(\subset K_N)$.

We now argue that C_N^h is typical as $N \to \infty$ for every $h \in (h_*, h_* + \varepsilon)$. For such h, the probability of $\{\partial_{\text{out}} U_N \stackrel{\varphi\geq h}{\longleftrightarrow} \partial K_N\}$ vanishes as $N \to \infty$ follows from (1.3) and a union bound. Next, for $x \in D_N$ one has

$$\begin{aligned}\mathbb{E}[(\xi_x^N)^2] &\stackrel{(2.31)}{=} \sum_{y,z} P_x[X_{T_{U_N}} = y]P_x[X_{T_{U_N}} = z]g(y,z) = \sum_y P_x[X_{T_{U_N}} = y]E_x[g(X_{T_{U_N}}, y)]\\ &= \sum_y P_x[X_{T_{U_N}} = y]g(x,y) \stackrel{(2.2)}{\leq} C\,\mathrm{dist}(x, U_N^{\mathrm{c}})^{-(d-2)} \leq CN^{-\delta(d-2)},\end{aligned}$$

where the equality in the second line is obtained by applying the strong Markov property at time T_{U_N}, noting that $x \in U_N$ whereas $y \notin U_N$. It then follows from a union bound over D_N and a standard Gaussian tail estimate that $\mathbb{P}[\inf_{D_N} \xi^N \geq -(h_* + \varepsilon - h)] \to 1$ as $N \to \infty$ for all $h \in (h_*, h_* + \varepsilon)$ (alternatively one could also use Lemma 4.1 below to deduce this). All in all, in view of (3.14), one obtains

$$\mathbb{P}[C_N^h] \to 1 \text{ as } N \to \infty \text{ for every } h \in (h_*, h_* + \varepsilon). \tag{3.15}$$

Using (3.16) and applying Lemma 3.2, one infers that for every $h < h_*$ (and all $\delta, \varepsilon > 0$, implicit in the definition of C_N^h),

$$\liminf_{N\to\infty} \frac{1}{\mathrm{cap}(K_N)} \log \mathbb{P}\big[C_N^h\big] \geq -\frac{1}{2}(h_* - h)^2. \tag{3.16}$$

In the notation of (3.11), let $\tilde{A}_N^h := \tilde{A}_{N,\lfloor N^\delta\rfloor}^h$. In words, $\tilde{A}_N^h$ is the event that the cylinder D_N contains a crossing in $\{\varphi \geq h\}$ intersecting both 'slices' $F_N^\pm$ as defined below (3.10), but with $L = \lfloor N^\delta \rfloor$. Combining the occurrence of $\tilde{A}_N^h$ and the insulating property of the disconnection event in (3.14), one deduces that

$$\tilde{A}_N^h \cap C_N^h \subset \bigcup_{x\in F_N^-} \{x \stackrel{\varphi\geq h}{\longleftrightarrow} \partial B_N(x),\, x \stackrel{\varphi\geq h}{\not\longleftrightarrow} \infty\}$$

for all $h \in \mathbb{R}$, and hence by a union bound and translation invariance, that

$$\mathbb{P}[0 \stackrel{\varphi\geq h}{\longleftrightarrow} \partial B_N,\, 0 \stackrel{\varphi\geq h}{\not\longleftrightarrow} \infty] \geq \frac{1}{|F_N^-|}\mathbb{P}[A_N^h \cap C_N]. \tag{3.17}$$

Since the event C_N^h in (3.14) is $\mathcal{F}_{U_N^{\mathrm{c}}} = \sigma(\varphi_x; x \in U_N^{\mathrm{c}})$-measurable, one has, applying the decomposition (2.31),

$$\mathbb{P}[\tilde{A}_N^h \cap C_N^h] = \mathbb{E}\big[\mathbb{P}[\tilde{A}_N^h|\mathcal{F}_{U_N^{\mathrm{c}}}]1_{C_N^h}\big] = \mathbb{E}\big[\mathbb{P}_{U_N}[\tilde{A}_N^h(\cdot + \xi^N)]1_{C_N^h}\big] \geq \mathbb{P}_{U_N}[\tilde{A}_N^{h_*+\varepsilon}]\mathbb{P}[C_N^h], \tag{3.18}$$

where in the last step, we used monotonicity of the event $\tilde{A}_N^h(\cdot)$ along with the control on ξ^N supplied by (3.14). A useful lower bound on $\mathbb{P}_{U_N}[\tilde{A}_N^{h_*+\varepsilon}]$ is obtained as follows. First, mimicking the argument following (3.11) (the fact that the value of L has increased is only beneficial), one deduces that

$$\lim_N \mathbb{P}[\tilde{A}_N^h] = 1 \text{ for all } h < h_*. \tag{3.19}$$

Then one writes, for all $h < h_*$, with $\varepsilon_0 = \frac{1}{2}(h_* - h)$,

$$\mathbb{P}_{U_N}[\tilde{A}_N^h] = \mathbb{P}[\tilde{A}_N^h(\psi^N)] \geq \mathbb{P}\big[\tilde{A}_N^h(\psi^N), \sup_{D_N} \xi^N \leq \varepsilon_0\big] \geq \mathbb{P}[\tilde{A}_N^{h+\varepsilon_0}(\varphi)] - \mathbb{P}\big[\sup_{D_N} \xi^N > \varepsilon_0\big].$$

By (3.19) and since $h + \varepsilon_0 = \frac{1}{2}(h + h_*) < h_*$, the first term on the right-hand side tends to 1 as $N \to \infty$. By similar considerations as above (3.16), one sees that $\mathbb{P}\big[\sup_{D_N} \xi^N > \varepsilon_0\big] \to 0$, yielding that $\mathbb{P}_{U_N}[\tilde{A}_N^h] \to 1$ as $N \to \infty$ for every $h < h_*$. Thus, Lemma 3.2 applies and gives, for all $\delta, \varepsilon > 0$

$$\liminf_N \frac{1}{\mathrm{cap}_{U_N}(D_N)} \log \mathbb{P}_{U_N}\big[\tilde{A}_N^{h_*+\varepsilon}\big] \geq -\frac{1}{2}\varepsilon^2 \tag{3.20}$$

(note also that $\mathrm{cap}_{U_N}(D_N) \geq \mathrm{cap}(D_N) \to \infty$, as required for Lemma 3.2 to apply). Finally, combining (3.17), (3.18) with the bounds (3.16), (3.20), and using (2.23), (2.24), noting that $|F_N^-| \leq CN^{\delta(d-1)}$, one obtains, for all $h < h_*$ and $\varepsilon, \delta > 0$,

$$\liminf_{N\to\infty} \frac{1}{\mathrm{cap}(T_N)} \log \mathbb{P}\big[0 \overset{\varphi\geq h}{\longleftrightarrow} \partial B_N,\, 0 \overset{\varphi\geq h}{\not\longleftrightarrow} \infty\big] \geq -\frac{1}{2}\Big((h_* - h)^2(1 + C_1\delta) + C_1\delta^{-1}\varepsilon^2\Big). \tag{3.21}$$

The result (3.2) now follows from (3.21) by first letting $\varepsilon \to 0$ and then $\delta \to 0$ (recall (2.12)).

It remains to argue that (3.2) continues to hold for the event $\mathrm{LocUniq}(N, h)^c$ defined in (1.5). To this end, using invariance of $\mathbb{P}$ under translations and rotations by multiples of $\pi/2$ along coordinate axes, denoting by $H_N(x) = x + \{y = (y_1, \dots, y_d) \in \mathbb{Z}^d : y_1 = N\}$, one deduces that for all $h < h_*$ and $x \in \mathbb{Z}^d$,

$$\liminf_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[x \overset{\varphi\geq h}{\longleftrightarrow} (\partial B_N(x) \cap H_N^1(x)),\, x \overset{\varphi\geq h}{\not\longleftrightarrow} \infty] \geq -\frac{\pi}{6}(h - h_*)^2. \tag{3.22}$$

Now, fix a point $x_0 \in \partial B_N \cap H_N^1(0)$ (for instance $x_0 = (N, 0, \dots, 0)$ say). Then it follows that

$$\{B_N \overset{\varphi\geq h}{\longleftrightarrow} \infty\} \cap \Big\{x_0 \overset{\varphi\geq h}{\longleftrightarrow} (\partial B_N(x_0) \cap H_N^1(x_0)),\, x_0 \overset{\varphi\geq h}{\not\longleftrightarrow} \infty\Big\} \subset \mathrm{LocUniq}(N, h)^c. \tag{3.23}$$

Indeed, the first event on the left-hand side of (3.23) implies that $\mathscr{C}$, the infinite cluster in $\{\varphi \geq h\}$ is a crossing cluster of $B_{2N} \setminus B_N$, whereas the second event yields that $\mathscr{C}_{x_0}$, the cluster of x_0 in $\{\varphi \geq h\}$, also crosses $B_{2N} \setminus B_N$ and that $\mathscr{C}_{x_0} \cap \mathscr{C} = \emptyset$, whence (3.23). From (5.74) in Section 5.2, one knows that for all $h < h_*$,

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[B_N \overset{\varphi\geq h}{\not\longleftrightarrow} \infty] = -\infty \tag{3.24}$$

Note that (5.74) makes use of the upper bounds derived in Theorem 5.1 below. However, the proof of Theorem 5.1 does not rely the lower bound (3.2) which is being presently derived (nor on any of the bounds stated in Theorem 3.1 for that matter). Combining (3.24) with (3.23) and (3.22), the claim readily follows. □

3.3. Lower bound for $d \geq 4$. We now supply the proof of (3.3). A straightforward strategy consists of opening all the vertices in the line segment $T_N = T_N(0)$ (recall (2.11) for notation) and closing all vertices in its outer boundary $\partial_{\text{out}} T_N$. However, the Gaussian free field does not satisfy a uniform finite-energy property, which would make this easy to implement. Moreover, unlike in the subcritical case, one cannot immediately apply the FKG-inequality as the event in question is not monotonic. In order to deal with these issues, we make usage of the midpoint extension $\tilde{\varphi}$ introduced at the end of Section 2, see in particular (2.32) and (2.33).

Proof of (3.3). Recall the decomposition (2.37) of $\tilde{\varphi}$. Now let $\overline{T}_N := T_N \cup \partial_{\text{out}} T_N$ (where T_N is viewed as a subset of $\mathbb{Z}^d$, whence $\overline{T}_N \subset \mathbb{Z}^d$) and let $M_N := \{m \in \mathbb{M}^d : \ m \sim x, \text{ for some } x \in \overline{T}_N\}$. Using the fact that the absolute value of the Gaussian free field on any transient graph (and thus in particular of $\tilde{\varphi}$ on $\tilde{\mathbb{Z}}^d$) satisfies the FKG-inequality, see e.g. (1.3) and Corollary 1.3 in [12], one obtains that

$$\mathbb{P}[|\hat{\xi}_x| \leq 1, \ \forall x \in \overline{T}_N] \geq \mathbb{P}[|\tilde{\varphi}_m| \leq 1, \ \forall m \in M_N] \geq \mathbb{P}[|\tilde{\varphi}_{m_0}| \leq 1]^{|M_N|} \geq e^{-CN}, \tag{3.25}$$

where the first inequality follows because $\hat{\xi}_x$ is the mean of $\tilde{\varphi}$ evaluated at its neighbors and m_0 refers to an arbitrary reference point in $\mathbb{M}^d$. As a consequence, one has, for all $h \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N, \, 0 \overset{\varphi \geq h}{\not\leftrightarrow} \infty] \geq \mathbb{P}[\varphi_x \geq h, \ \forall x \in T_N, \ \varphi_y < h, \ \forall \partial_{\text{out}} T_N] \\
&\quad \overset{(2.37)}{\geq} \mathbb{P}[|\hat{\xi}_x| \leq 1, \ \forall x \in \overline{T}_N, \ \hat{\psi}_x \geq h+1, \ \forall x \in T_N, \ \hat{\psi}_y < h-1, \ \forall y \in \partial_{\text{out}} T_N] \\
&\quad \overset{(2.38)}{=} \mathbb{P}[|\hat{\xi}_x| \leq 1, \ \forall x \in \overline{T}_N] \, P[X \geq h+1]^{|T_N|} \, P[X < h-1]^{|\partial_{\text{out}} T_N|} \overset{(3.25)}{\geq} e^{-C(h)N},
\end{aligned}$$

where in the penultimate step, we also used independence of $\hat{\psi}$ and $\hat{\xi}$ and X is a $N(0, 1/2)$-distributed random variable. One easily adapts the above argument in order to create two "insulated" paths in $\{\varphi \geq h\}$ joining B_N to ∂B_{2N} at an exponential cost in N, thus obtaining the lower bound (3.3) for the event $\text{LocUniq}(N, h)^c$ instead. This completes the proof of (3.3) and with it that of Theorem 3.1. □

4 Coarse-graining

We now prepare the ground for the upper bounds that will be derived in the next section. The main result of this section is a certain coarse-graining scheme for paths, see Proposition 4.3 below. Its proof is split over Sections 4.1 and 4.2, which deal with the case $d = 3$ and $d \geq 4$, respectively. The key effect of the scheme is to keep the capacity of the 'coarse-grained path' above a certain threshold, see (4.15), (4.16), while maintaining good control on the entropy factor for its possible choices. The notion of 'coarse-grained path' is formalized in Definition 4.2. It depends on a function $\Gamma(\cdot)$ parametrizing this entropy, see (4.13) and (4.14).

We now describe the precise setup. We consider positive integers $L \geq 1$ and $K \geq 100$ and introduce the lattice

$$\mathbb{L} = \mathbb{L}(L) := L\mathbb{Z}^d \tag{4.1}$$

along with the boxes

$$C_z := z + [0, L)^d, \quad D_z := z + [-3L, 4L)^d, \quad U_z := z + [-KL+1, L+KL-1)^d \tag{4.2}$$

as well as

$$\tilde{C}_z := z + [-L, 2L)^d, \quad \tilde{D}_z := z + [-2L, 3L)^d \tag{4.3}$$

attached to $z \in \mathbb{L}$. Notice that $C_z \subset \tilde{C}_z \subset \tilde{D}_z \subset D_z \subset U_z$. When considering more than one scale in a given context, we will sometimes explicitly refer to the associated length scale L by writing $C_{z,L} = C_z, \tilde{C}_{z,L} = \tilde{C}_z$ etc. Using the notation of [20, Section 4], for any $z \in \mathbb{L}$, we introduce the decomposition

$$\varphi = \xi^z + \psi^z \tag{4.4}$$

where $\xi^z_x := \xi^{U_z}_x = E_x[\varphi_{X_{T_{U_z}}}]$ for all $x \in \mathbb{Z}^d$, cf. (2.31), and $\psi^z = \psi^{U_z}$, with U_z as in (4.2). Recall that ψ^z has law $\mathbb{P}_{U_z}$. Letting $\tilde{U}_z := (U_z \cap \mathbb{Z}^d) \cup \{\tilde{x} \in \tilde{\mathbb{Z}}^d : \exists x \in (U_z \cap \mathbb{Z}^d) \text{ s.t. } |x - \tilde{x}| = \frac{1}{2}\}$, we further define for $z \in \mathbb{L}$ the extended harmonic average

$$\tilde{\xi}^z := \tilde{\xi}^{\tilde{U}_z} \ \big(\overset{(2.34)}{=} \tilde{E}_{\cdot}\big[\tilde{\varphi}_{\tilde{X}_{T_{\tilde{U}_z}}}\big]\big), \quad \tilde{\psi}^z := \tilde{\varphi} - \tilde{\xi}^z, \tag{4.5}$$

where $\tilde{\varphi}$ refers to the extension of φ to the graph $\tilde{\mathbb{Z}}^d$, see the discussion leading to (2.32). On account of (2.33) and (2.36), one thus has $\tilde{\xi}^z|_{\mathbb{Z}^d} = \xi^z$ and $\tilde{\psi}^z|_{\mathbb{Z}^d} = \psi^z$. Moreover, if

(4.6) $\quad \mathcal{C} \subset \mathbb{L}$ is a non-empty collection of sites with mutual $|\cdot|_\infty$-distance at least $2KL + L$,

then denoting by $\tilde{\psi}^z = (\tilde{\psi}^z_{\tilde{x}})_{\tilde{x} \in \tilde{\mathbb{Z}}^d}$, for $z \in \mathcal{C}$, one has that

(4.7) $\quad$ the Gaussian fields $\{\tilde{\xi}, \tilde{\psi}^z, z \in \mathcal{C}\}$, where $\tilde{\xi} = (\tilde{\xi}^z_{\tilde{x}})_{z \in \mathcal{C}, \tilde{x} \in \tilde{U}_z}$, are independent.

Given $\mathcal{C}$ as in (4.6), we write

$$\Sigma := \Sigma(\mathcal{C}) := \bigcup_{z \in \mathcal{C}} C_z. \tag{4.8}$$

The following precise result will be useful.

Lemma 4.1 (Control of harmonic average)**.** *For all $K \geq 100$, there exists $\alpha(K) > 1$ with $\lim_{K \to \infty} \alpha(K) = 1$ such that, for every $a > 0$, one has*

$$\limsup_L \sup_{\mathcal{C}} \left\{ \log \mathbb{P}\Big[\bigcap_{z \in \mathcal{C}} \{\sup_{D_z} \tilde{\xi}^z \geq a\}\Big] + \frac{1}{2}\left(a - \frac{c_4}{K}\sqrt{\frac{|\mathcal{C}|}{\text{cap}(\Sigma)}}\right)_+^2 \frac{\text{cap}(\Sigma)}{\alpha(K)} \right\} \leq 0, \tag{4.9}$$

where the supremum over $\mathcal{C}$ runs over the sets satisfying (4.6)*, $\sup_{D_z}$ refers to the supremum over all points in $D_z \cap \tilde{\mathbb{Z}}^d$ and $r_+ = r \vee 0$ for any $r \in \mathbb{R}$.*

Proof. With ξ^z in place of $\tilde{\xi}^z$, (4.9) is proved in [20, Corollary 4.4]. To extend the result to $\tilde{\xi}^z$, observe that, for any mid-point $\tilde{x} \in (D_z \cap \tilde{\mathbb{Z}}^d) \setminus \mathbb{Z}^d$, the neighbors x_1 and x_2 of $\tilde{x}$ (in $\tilde{\mathbb{Z}}^d$) both belong to $D_z \cap \mathbb{Z}^d$. Furthermore, by harmonicity of $\tilde{\xi}^z_{\cdot}$ in $D_z \cap \tilde{\mathbb{Z}}^d$, cf. (4.2) and (4.5), one has $\tilde{\xi}^z_{\tilde{x}} = \frac{1}{2}(\tilde{\xi}^z_{x_1} + \tilde{\xi}^z_{x_2}) = \frac{1}{2}(\xi^z_{x_1} + \xi^z_{x_2})$, whence $\tilde{\xi}^z_{\tilde{x}} \leq a$ whenever $\xi^z_{x_i} \leq a$ for $i = 1, 2$. Together, these observations yield that

$$\{\sup_{D_z \cap \tilde{\mathbb{Z}}^d} \tilde{\xi}^z \geq a\} \subset \{\sup_{D_z \cap \mathbb{Z}^d} \xi^z \geq a\},$$

and the claim follows. □

We will be interested in families of collections $\mathcal{C}$ satisfying (4.6) with certain finer properties. In what follows, let Λ_N be any of the elements in

$$\mathcal{S}_N := \{B_N,\, B_{2N} \setminus B_N,\, \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}\}, \text{ for } N \geq 1 \text{ (see below (4.3) for notation).} \tag{4.10}$$

In line with the wording below (1.5), for $U \subset V \subset\subset \mathbb{Z}^d$, we say that a $*$-path γ in $\mathbb{Z}^d$ (see Section 2 for its definition) *crosses* $V \setminus U$ if γ intersects both U and ∂V. If $U = \{0\}$ we omit the reference to U; e.g. when γ crosses B_N we mean that γ intersects both 0 and ∂B_N.

Definition 4.2. ($K \geq 100$, $L \geq 1$, $N \geq 10KL$). Let $\Gamma : [1, \infty) \mapsto [0, \infty)$. A family $\mathcal{A} = \mathcal{A}^K_{N,L}(\Lambda_N)$ of collections $\mathcal{C} \subset \mathbb{L}$ is called Γ-admissible if

(4.11) all $\mathcal{C} \in \mathcal{A}$ have equal cardinality $n := |\mathcal{C}|$ satisfying $n \in [\frac{c_5 N}{u(KL)}, \frac{N}{u(KL)}]$, where $u(x) = x$ if $d = 3$ and $u(x) = x(\log x)^2$ if $d \geq 4$, (4.6) holds for all $\mathcal{C} \in \mathcal{A}$ and $\tilde{D}_z = \tilde{D}_{z,L} \subset \Lambda_N$ for all $z \in \mathcal{C}$,

(4.12) for any $*$-path γ crossing Λ_N, there exists $\mathcal{C} \in \mathcal{A}$ such that γ crosses $\tilde{D}_z \setminus C_z$ (with $C_z = C_{z,L}$, $\tilde{D}_z = \tilde{D}_{z,L}$ as in (4.2), (4.3)) for all $z \in \mathcal{C}$,

$$\log |\mathcal{A}| \leq \Gamma(N/L). \tag{4.13}$$

The main result of this section is the following:

Proposition 4.3 (Coarsening of crossing paths)**.** *For all $K \geq 100$, $L \geq 1$, $N \geq 10KL$ and $\Lambda_N \in \mathcal{S}_N$, there exists a Γ-admissible collection $\mathcal{A} = \mathcal{A}^K_{N,L}(\Lambda_N)$ with the following properties.*

i) If $d = 3$,

$$\Gamma(r) = C_2 K^{-1} r \log(r \vee 2), \tag{4.14}$$

and there exist positive numbers $\lambda(K) \in (0, 1]$ satisfying

$$\lim_{K \to \infty} \lambda(K) = 1, \tag{4.15}$$

such that, for all $\rho \in (0, 1)$,

$$\liminf_{N \to \infty} \inf_{L_0(N) \leq L \leq L_1(N)} \inf_{\mathcal{C} \in \mathcal{A}} \inf_{\substack{\tilde{\mathcal{C}} \subset \mathcal{C} \\ |\tilde{\mathcal{C}}| \geq (1-\rho)|\mathcal{C}|}} \frac{\mathrm{cap}(\tilde{\Sigma})}{\mathrm{cap}(T_N)} \geq \lambda(K)\,(1 - \rho) \tag{4.16}$$

with $\tilde{\Sigma} = \Sigma(\tilde{\mathcal{C}})$ (see (4.8) for $\Sigma(\cdot)$) and $(L_0(N), L_1(N))$ any sequence with $\lim_N L_0(N) = \infty$ and $L_1(N) \leq N/10K$ satisfying $\lim_N \frac{\log L_1(N)}{\log N} = 0$.

ii) If $d \geq 4$,

$$\Gamma(r) = C_2 r \tag{4.14'}$$

and (4.16) remains valid with $L_0(N) = 1$, $L_1(N) = N/10K$ and for some $\lambda(K) \in (0, 1]$.

Moreover, the above conclusions continue to hold with $\Lambda_N = B_N \setminus B_{\varepsilon N}$, for any $\varepsilon \in (0, \frac{1}{3})$, upon replacing T_N by $T_{(1-\varepsilon)N}$ in (4.16).

Remark 4.4. 1) The statement of Proposition 4.3 could even be generalized to the case $\Lambda_N = B_N \setminus B_{\varepsilon N}$, for any $\varepsilon \in (0,1)$, with suitable modifications. Specifically, the condition in (4.16) relating L and N would involve K and ε, and c_5 in (4.11) would depend on ε. We refrain from such extensions since we will only be interested in the limit $\varepsilon \downarrow 0$ in the sequel.

2) The differing complexities $\Gamma(r)$ in (4.14), (4.14'), reflect different procedures in implementing the coarse-graining depending on whether $d = 3$ or $d \geq 4$. An interesting question this brings about is the following: can one devise a coarsening scheme in dimension 3 (for instance the strategy employed for $d \geq 4$, or a variation thereof), which would be more cost-effective, i.e. reduce $\Gamma(r)$, ideally getting rid of the logarithmic factor in (4.14), while retaining the controls (4.15), (4.16) on the capacity?

3) The coarse-graining scheme used in case $d = 3$ can also be employed in dimensions $d \geq 4$ in such a way that (4.16) continues to hold. The resulting higher combinatorial complexity, see (4.14), already yields near-optimal asymptotic upper bounds for the quantities of interest. We refer to Remarks 4.7 and 5.10 below for further details.

The proof of Proposition 4.3 is split over Sections 4.1 and 4.2, which deal separately with the cases $d = 3$ and $d \geq 4$.

4.1. Proof of Proposition 4.3 for $d = 3$**.** Let $L \geq 1$ and $N \geq 10KL$ be integers. We consider the case $\Lambda_N = B_N$. The small adaptations needed to account for the remaining cases in $\mathcal{S}_N$, see (4.10), as well as for $\Lambda_N = B_N \setminus B_{\varepsilon N}$, are indicated at the end of the proof. For each $i = 1, \dots, n$ with $n := \lfloor N/3KL \rfloor - 1$ (note that $n \geq 1$), define the concentric shells $S_i := \partial B_{3KLi}$.

By paving S_i with boxes of the form $C_z = z + [0,L)^d$ for $z \in \mathbb{L}$, and considering the successive first exits of the path γ from each of the sets *enclosed* by S_i, one finds for each $i = 1, \dots, n$ a point $z_i \in \mathbb{L}$ such that

$$C_{z_i} \cap S_i \neq \emptyset \text{ and } \gamma \text{ crosses } \tilde{D}_{z_i} \setminus C_{z_i}. \tag{4.17}$$

Note that $\tilde{D}_{z_i} \subset B_N$ for all $i = 1, \dots, n$. We then define $\mathcal{A} = \mathcal{A}^K_{N,L}(B_N)$ as the family consisting of all collections $\mathcal{C} := \{z_i : 1 \leq i \leq n\}$ that can be obtained in this way.

We proceed to verify the conditions of Definition 4.2 for $\Gamma(\cdot)$ as in (4.14) (with $d = 3$). Properties (4.11) and (4.12) are immediate by construction. Regarding the cardinality of $\mathcal{A}$, one notes that the number of choices for z_i is bounded by $(C\frac{N}{L})^{d-1}$, whence $|\mathcal{A}| \leq (C\frac{N}{L})^{n(d-1)}$, from which (4.13) follows with $\Gamma(\cdot)$ given by (4.14), for suitable choice of C_2.

It remains to argue that (4.16) holds, with $\lambda(K)$ satisfying (4.15). This will be done in two steps, corresponding to Lemmas 4.5 and 4.6 below. We begin with a reduction step (Lemma 4.5), consisting of replacing the set $\tilde{\Sigma}$ appearing in (4.16) (recall (4.8)) by the 'porous line'

$$\widetilde{T}_N := \bigcup_{i=1}^{\lceil (1-\rho)n \rceil} \big(\tilde{z}_i + [0,L) \times \{0\}^2\big), \quad \text{with} \quad \tilde{z}_i := S_i \cap ([0,\infty) \times \{0\}^2) \, \big(= (3LKi, 0, 0)\big). \tag{4.18}$$

Lemma 4.6 then compares the capacity of porous and non-porous lines.

Lemma 4.5. *For all* $\rho \in (0,1)$,

$$\liminf_{N \to \infty} \inf_{L_0(N) \leq L \leq N/10K} \inf_{\mathcal{C} \in \mathcal{A}} \inf_{\substack{\tilde{\mathcal{C}} \subset \mathcal{C} \\ |\tilde{\mathcal{C}}| \geq (1-\rho)|\mathcal{C}|}} \frac{\mathrm{cap}(\tilde{\Sigma})}{\mathrm{cap}(\widetilde{T}_N)} \geq \lambda(K), \tag{4.19}$$

for suitable $\lambda(K) > 0$ *satisfying* (4.15) *and any* $L_0(N)$ *with* $\lim_N L_0(N) = \infty$.

We prove (4.19) using a projection argument, first by rigidly displacing the boxes in $\tilde{\Sigma}$ onto $\mathbb{R}\times\{0\}^2$, then by "packing them together" along this axis to obtain the "homogenous porosity" of $\widetilde{T}_N$. Since these operations essentially reduce the (ℓ^2-)distances between pairs of points in $\tilde{\Sigma}$, the capacity expectedly decreases. This intuition is formalized below.

Proof of Lemma 4.5. In view of (2.2), for any $\varepsilon \in (0,1)$ we can choose $\widetilde{L}_0(\varepsilon) \geq 100$ such that

$$c_2^{-1}|x|g(0,x) \in \big[1-\varepsilon, 1/(1-\varepsilon)\big], \text{ whenever } |x|_\infty \geq \widetilde{L}_0. \tag{4.20}$$

Let $L \geq \widetilde{L}_0(\varepsilon)$. Notice that $\tilde{z}_i$ in (4.18) is such that $C_{\tilde{z}_i}$, cf. (4.2), is the (unique) box among those paving S_i intersecting the positive half-line $\mathbb{Z}_+ \times \{0\}^2$. Now for $\mathcal{C} \in \mathcal{A}$ and $z_i \in \mathcal{C}$ as in (4.17), let $\tau(z_i) \coloneqq \tilde{z}_i$. Importantly, τ is an (ℓ^2-)projection, in that

$$|\tau(z) - \tau(z')| \leq |z - z'|, \text{ for all } z, z' \in \mathcal{C} \text{ and } \mathcal{C} \in \mathcal{A}. \tag{4.21}$$

Indeed, by construction, (4.21) holds with $|\cdot|_\infty$ in place of $|\cdot|$ and (4.21) follows because $|\tau(z) - \tau(z')| = |\tau(z) - \tau(z')|_\infty$ and $|\cdot|_\infty \leq |\cdot|$. The map τ extends naturally to a bijection defined on the set Σ (cf. (4.8)) by setting $\tau(y) = \tau(z) + y - z$, if $y \in C_z$ for some $z \in \mathcal{C}$. In words, τ sends any point in C_z to the corresponding point in $C_{\tau(z)}$.

Recalling the notation from (2.9), for any probability measure μ supported on $\tilde{\Sigma}$, with $\mu' = \tau \circ \mu$ and $\tilde{\Sigma}_\tau \coloneqq \{\tau(z) : z \in \tilde{\Sigma}\}$, it follows that

$$E(\mu) = \sum_{x,y\in\tilde{\Sigma}_\tau} \mu'(x) g\left(\tau^{-1}(x), \tau^{-1}(y)\right) \mu'(y) \leq \kappa \cdot E(\tau\circ\mu) \tag{4.22}$$

with

$$\kappa \coloneqq \sup_{x,y\in\tilde{\Sigma}_\tau} \frac{g\left(\tau^{-1}(x),\tau^{-1}(y)\right)}{g(x,y)}. \tag{4.23}$$

The quantity κ is suitably bounded as follows. By translation invariance of the Green's function, if x, y belong to the same box $C_{\tau(z)}$ for some $z \in \tilde{\mathcal{C}}$, then $g(\tau^{-1}(x), \tau^{-1}(y)) = g(x-y)$. Otherwise, i.e. if $x \in C_{\tau(z)}$ and $y \in C_{\tau(z')}$ for $z, z' \in \mathcal{C}$ with $z \neq z'$, using the triangle inequality and (4.21), one readily infers that $|x-y| \leq |\tau^{-1}(x) - \tau^{-1}(y)| + 4\sqrt{3}L$, for all $x, y \in \tilde{\Sigma}_\tau$. With this observation, returning to (4.23) and using (4.20), which is in force as $|x-y| \wedge |\tau^{-1}(x) - \tau^{-1}(y)| \geq KL \geq \widetilde{L}_0$ whenever x and y belong to different boxes, one finds that

$$\kappa \leq 1 \vee \left(\frac{1}{(1-\varepsilon)^2} \sup_{\substack{x\in C_{\tau(z)}, y\in C_{\tau(z')} \\ z,z'\in\mathcal{C},\, z\neq z'}} \frac{|x-y|}{|\tau^{-1}(x) - \tau^{-1}(y)|} \right) \leq \frac{1+C/K}{(1-\varepsilon)^2},$$

using in the last step that $|\tau^{-1}(x) - \tau^{-1}(y)| \geq KL$. Substituting this bound into (4.22), taking an infimum over μ, noting that $\mu \mapsto \tau\circ\mu$ is a bijection between probability measures with support on $\tilde{\Sigma}$ and $\tilde{\Sigma}_\tau$, and applying the variational formula (2.9), one obtains the lower bound

$$\mathrm{cap}(\tilde{\Sigma}) \geq (1 - C/K)(1-\varepsilon)^2 \mathrm{cap}(\tilde{\Sigma}_\tau). \tag{4.24}$$

In view of (4.19), in order to produce the set $\widetilde{T}_N$, cf. (4.18), one trims $\tilde{\Sigma}_\tau$ as follows. First, let $\widetilde{T}'_N \coloneqq \bigcup_{i=1}^{\lceil(1-\rho)n\rceil} C_{\tilde{z}_i}$ and note that $\widetilde{T}_N \subset \widetilde{T}'_N$, whence $\mathrm{cap}(\widetilde{T}_N) \leq \mathrm{cap}(\widetilde{T}'_N)$. Observe that the elements of $\mathcal{C}$, and hence of $\tilde{\mathcal{C}}$, are naturally ordered according to increasing index i, cf. below

(4.17). Now one only retains the boxes in $\tilde{\Sigma}_\tau$ corresponding to the first $\lceil(1-\rho)n\rceil$ elements of $\tilde{\mathcal{C}}$ in this ordering (recall that $\tilde{\mathcal{C}}$ has at least this many elements), and only keeps the intersection of the resulting set of boxes with the line $\mathbb{Z}\times\{0\}^2$, thus obtaining overall a smaller set $\tilde{\Sigma}'_\tau \subset \tilde{\Sigma}_\tau$, whence $\mathrm{cap}(\tilde{\Sigma}'_\tau) \leq \mathrm{cap}(\tilde{\Sigma}_\tau)$. The resulting set $\tilde{\Sigma}'_\tau$ is in natural bijection with $\widetilde{T}'_N$, essentially by removing the gaps, one by one by rigidly shifting all the intervals to the (say) right of the gap by a suitable constant amount to the left. This operation either leaves the relative position between two points $x, y \in \tilde{\Sigma}'_\tau$ unchanged or *reduces* their Euclidean norm, but in the latter case never as to fall below $KL \geq \widetilde{L}_0$ (a lower bound on the gap size in $\widetilde{T}_N$). With this observation, a similar argument as above, using (2.9) and (4.20) yields that

$$\mathrm{cap}(\tilde{\Sigma}_\tau) \geq \mathrm{cap}(\tilde{\Sigma}'_\tau) \geq (1-\varepsilon)^2\mathrm{cap}(\widetilde{T}'_N) \geq (1-\varepsilon)^2\mathrm{cap}(\widetilde{T}_N). \tag{4.25}$$

By letting $\varepsilon \to 0$ (and therefore $L \geq \widetilde{L}_0 \to \infty$ as well as $N \geq 10KL \to \infty$), (4.24) and (4.25) imply (4.19) with $\lambda(K) = 1 - C/K$. □

We proceed with

Lemma 4.6. *For all $K \geq 100$, $\rho \in (0,1)$,*

$$\liminf_{\frac{\log N}{\log L}\to\infty} \frac{\mathrm{cap}(\widetilde{T}_N)}{\mathrm{cap}(T_N)} \geq 1-\rho, \tag{4.26}$$

where the $\liminf$ *regards any sequence $(N_k, L_k)_{k\geq 0}$ satisfying* $\log N_k/\log L_k \to \infty$ *as $k \to \infty$.*

Proof. Let $\widehat{T}_N := T_{\lceil(1-\rho)n\rceil 3KL+L}$ (recall that $n = \lfloor N/3LK\rfloor - 1$ and (2.11) for notation). On account of (2.12), it suffices to show that $\liminf_{N/L\to\infty} \frac{\mathrm{cap}(\widetilde{T}_N)}{\mathrm{cap}(\widehat{T}_N)} \geq 1$ and (4.26) directly follows. By definition, see (4.18), $\widetilde{T}_N \subset \widehat{T}_N$ and

$$\text{for all } x \in \widehat{T}_N,\ d(x, \widetilde{T}_N) \leq 3KL. \tag{4.27}$$

Hence, by (2.8), (4.26) follows at once if we argue that

$$\liminf_{N/L\to\infty} \inf_{x\in\widehat{T}_N} P_x[H_{\widetilde{T}_N} < \infty] = 1. \tag{4.28}$$

In view of (4.27), Lemma 2.4 applied to $T = \widehat{T}_N \subset \widetilde{T}_N$ and $x \in \widehat{T}_N$ yields that

$$P_x[H_{\widehat{T}_N} = \infty] \leq C(K)\frac{\log L}{\log N} \to 0 \text{ as } \log N/\log L \to \infty.$$

Thus, (4.28) follows, which completes the proof. □

With Lemmas 4.5 and 4.6 at hand, we can conclude the proof of (4.16) (and (4.15)) for $\Lambda_N = B_N$. The remaining cases in $\mathcal{S}_N$, see (4.10), i.e. $\Lambda_N = B_{2N} \setminus B_N$, resp. $\Lambda_N = \tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$ are dealt with by considering instead $S_i = \partial B_{N+3KLi}$, resp. $S_i = \partial([-3KLi - N, 2N + 3KLi)^d)$ (cf. (4.3)), for $1 \leq i \leq n$, and adapting the subsequent arguments accordingly. In particular, the bound for $|\mathcal{A}|$ remains valid for these choices. The set $\widetilde{T}_N$ in (4.18) changes accordingly whence $\tilde{z}_i = (N + 3LKi, 0, 0)$, resp. $\tilde{z}_i = (2N + 3LKi - 1, 0, 0)$. The statements and proofs of Lemmas 4.5 and 4.6 then remain valid.

In the case $\Lambda_N = B_N \setminus B_{\varepsilon N}$ for a given $\varepsilon \in (0, \frac{1}{3})$, one considers the shells $S_i \coloneqq \partial B_{\lceil \varepsilon N \rceil + 3KLi}$, for $1 \le i \le n \coloneqq \lfloor (1-\varepsilon)N/3KL \rfloor - 1$ (note that $n \ge 1$ when $N \ge 10KL$). Then, defining $\widetilde{T}_N$ as in (4.18) (whence $\tilde{z}_i = (\lceil \varepsilon N \rceil + 3LKi, 0, 0)$), Lemma 4.5 remains valid and Lemma 4.6 as well upon replacing T_N by $T_{(1-\varepsilon)N}$ in (4.26). The lower bound (4.16) with $T_{(1-\varepsilon)N}$ instead of T_N then follows as above. This concludes the proof of Proposition 4.3 in the case $d = 3$. □

Remark 4.7. The above coarse-graining scheme also applies when $d \ge 4$. As a result, one obtains a Γ-admissible collection $\mathcal{A}' = \mathcal{A}^K_{N,L}(\Lambda_N)'$, with $\Gamma(\cdot)$ as defined in (4.14) for $d = 3$, i.e. $\Gamma(r) = \frac{C_2}{K} r \log r$, and with $u(x) = x$ in (4.11), so that the statement (4.16) for $d \ge 4$ holds with $\mathcal{A}'$ in place of $\mathcal{A}$. Only (4.16) requires an explanation. Repeating the steps leading up to (4.18), one shows an analogue of Lemma 4.5 without the condition (4.15). The proof essentially remains the same except that one simply uses $c \le |x|^{d-2} g(0,x) \le C$ for all $x \in \mathbb{Z}^d$ instead of (4.20), which is sufficient since one only aims at a $\lambda(K) > 0$ in (4.19). Then Lemma 4.6 gets replaced by the statement that $\frac{\mathrm{cap}(\widetilde{T}_N)}{\mathrm{cap}(T_N)} \ge cK^{-1}(1-\rho)$ for all $L \ge 1$ and $N \ge 10KL$, which follows by covering T_N with at most CK shifted copies of the set $\widetilde{T}_N$ and using monotonicity and subadditivity of $\mathrm{cap}(\cdot)$ (see e.g. Proposition 2.2.1(b) in [14] regarding the latter). The slightly higher combinatorial complexity of the collection $\mathcal{A}'$, reflected by the logarithmic factor in $\Gamma(\cdot)$, cf. (4.14), only yields near-optimal upper bounds for $d \ge 4$, see Remark 5.10 below. The presence of the additional logarithm is remedied by the approach of Section 4.2.

4.2. Proof of Proposition 4.3 for $d \ge 4$. We introduce the length scales

$$L_0 \coloneqq 1, \quad L_{m+1} \coloneqq \lceil 2(1+\varepsilon_m) L_m \rceil, \text{ with } \varepsilon_m \coloneqq (m+1)^{-2},\ m \ge 0. \tag{4.29}$$

Note that $2^m \le L_m \le C2^m$ for all $m \ge 0$, which will be used frequently below. Throughout the proof we use $C_{z,k}, \tilde{C}_{z,k}, D_{z,k}$ and $U_{z,k}$ to denote the boxes $C_z, \tilde{C}_z, D_z$ and U_z, see (4.2), corresponding to the length $L = L_k$ for any $z \in \mathbb{Z}^d$. Also let $\hat{C}_{z,k} \coloneqq z + [-L_k + L_{k-1}, 2L_k - L_{k-1})^d$ and observe that $C_{z,k} \subset \hat{C}_{z,k} \subset \tilde{C}_{z,k}$.

We first define a certain coarse graining of paths crossing generic shapes, see Lemma 4.8 below, which will later be applied inductively to define an admissible collection with the desired properties. Roughly speaking, the idea is to implement a cascading scheme on the path γ of diameter N, thus only retaining its trace in a system of well-separated boxes at scale L (naturally indexed by the leaves of a binary tree of depth $\approx \log_2(N/L)$), see for instance [10], Section 8.1 for a gentle introduction to this circle of ideas in a related model. The precise recursive scheme underlying the proof of Lemma 4.8 refines ideas from [19, 18], thereby improving the amount of information kept when iterating the construction from a scale $L_{n_0} \approx N$ to $L_{k_0} \approx L$ for suitable $0 \le k_0 \le n_0$. In particular, retaining merely the information that *full* annuli are crossed at smaller scales (as done in [18]) makes it difficult to derive the lower bound (4.16). This leads to the notion of *shapes* which we introduce now.

A *shape* at level k anchored at $z \in \mathbb{Z}^d$ is any $*$-connected subset S of $\tilde{C}_{z,k} \setminus C^-_{z,k}$, where $C^-_{z,k} = C_{z,k} \setminus \partial C_{z,k}$, intersecting both $\partial C_{z,k}$ and $\partial \tilde{C}_{z,k}$. The collection of all shapes at level k anchored at z will be denoted by $\mathcal{S}_{z,k}$ and $\mathcal{S}_k \coloneqq \bigcup_{z \in \mathbb{Z}^d} \mathcal{S}_{z,k}$ is the set of all shapes at level k. Any face F of $C_{z,k}$ (i.e. any set of the form $F = (\partial C_{z,k}) \cap \{x^i = a\}$ for some $1 \le i \le d$, $a \in \mathbb{Z}$) intersecting S will be called an *exposed face* (with respect to S).

In what follows, let $\mathbb{T}_n$, $n \ge 0$, denote the (rooted) complete binary tree of depth n (with $|\mathbb{T}_n| = 2^n$) and let $\mathcal{L}(\mathbb{T}_n) = \{0,1\}^n$ (with $\mathcal{L}(\mathbb{T}_0) = \{\varnothing\}$, the root) be its set of leaves. The leaves of $\mathbb{T}_n$ provide a natural indexing set due to the recursive dyadic manner in which the coarsening scheme operates, cf. (4.32).

Lemma 4.8 (Coarse-graining of shapes)**.** *For any integers $n \geq k > 0$, $z \in \mathbb{Z}^d$ and all shapes $S \in \mathcal{S}_{z,n}$, there exists a family $\mathcal{A}_S = \mathcal{A}_{S,k}$ with*

$$\log |\mathcal{A}_S| \leq C L_{n-k} \tag{4.30}$$

of collections $\mathcal{D} = \{(z(\ell), S(\ell)) : \ell \in \mathcal{L}(\mathbb{T}_{n-k})\} \subset \mathbb{Z}^d \times \mathcal{S}_k$ satisfying the following three properties:

(4.31) $S(\ell) \subset S$ *and* $S(\ell) \in \mathcal{S}_{z(\ell),k}$ *for all* $\ell \in \mathcal{L}(\mathbb{T}_{n-k})$,

(4.32) *if $n > k$, the sub-collections $\mathcal{D}_i \coloneqq \{(z(\ell), S(\ell)) : \ell \in \mathcal{L}(\mathbb{T}_{n-k,i})\}$, $i = 1, 2$, of $\mathcal{D}$, where $\mathbb{T}_{n-k,i}$, $i = 1, 2$, denote the two binary sub-trees of $\mathbb{T}_{n-k}$ with depth $n - k - 1$, belong to $\mathcal{A}_{S_1}$ and $\mathcal{A}_{S_2}$ for some shapes $S_1, S_2 \in \mathcal{S}_{n-1}$ such that $d_\infty(S_1, S_2) \geq 2\varepsilon_{n-1} L_{n-1}$,*

(4.33) *for any $*$-path $\gamma \subset S$ crossing $\tilde{C}_{z,n} \setminus C_{z,n}$, there exists $\mathcal{D} \in \mathcal{A}_S$ such that for all $\ell \in \mathcal{L}(\mathbb{T}_{n-k})$, γ induces a $*$-path $\gamma' \subset S(\ell)$ crossing $\tilde{C}_{z(\ell),k} \setminus C_{z(\ell),k}$.*

Proof. Fix $k > 0$. We proceed by induction over $n \geq k$. When $n = k$, we simply choose $\mathcal{A}_S$ to be the singleton set consisting of $\mathcal{D} \coloneqq \{(z, S)\}$ whence (4.30), (4.31) and (4.33) hold.

Suppose now that for some $n \geq 0$ and each $S \in \mathcal{S}_n$, there exists a family $\mathcal{A}_S$ satisfying (4.31)-(4.33), and such that, for some $b_{n-k} \in (0, \infty)$,

$$\sup_{S \in \mathcal{S}_n} \log |\mathcal{A}_S| \leq b_{n-k} L_{n-k} \tag{4.34}$$

(note that (4.34) holds for $n = k$ with $b_0 = 0$). Consider a shape $S \in \mathcal{S}_{z,n+1}$ for some $z \in \mathbb{Z}^d$. Choose a fixed set of vertices $T_1 \subset C_{z,n+1}$ of cardinality $|T_1| \leq C_3$ such that $C_{w,n} \subset C_{z,n+1}$ for all $w \in T_1$ and the faces of the boxes $\{C_{w,n}; w \in T_1\}$ form a cover of the exposed faces of $C_{z,n+1}$. Now, since $C_{w,n} \subset C_{z,n+1}$ for all $w \in T_1$, given any $*$-path γ crossing $\tilde{C}_{z,n+1} \setminus C_{z,n+1}$, one finds $z_1 \in T_1$ such that γ induces a path γ_1 crossing $\tilde{C}_{z_1,n} \setminus C_{z_1,n}$. Furthermore if $\gamma \subset S$, it follows that $\gamma_1 \subset S_{z_1}$ where $S_{z_1} \subset S$ is the $*$-connected component of $S \cap (\tilde{C}_{z_1,n} \setminus C^-_{z_1,n})$ containing γ_1. It is clear from this definition that $S_{z_1} \in \mathcal{S}_{z_1,n}$.

By a similar reasoning, one finds a set $T_2 \subset \hat{C}_{z,n+1}$, see below (4.29) for notation, with $|T_2| \leq C_4$ such that for some $z_2 \in T_2$, γ exits $\hat{C}_{z,n+1}$ for the last time through $C_{z_2,n} \subset \hat{C}_{z,n+1}$. Since $z_2 \in \hat{C}_{z,n+1}$ and consequently $\tilde{C}_{z_2,n} \subset \tilde{C}_{z,n+1}$, we deduce from these definitions that γ induces a path γ_2 crossing $\tilde{C}_{z_2,n} \setminus C_{z_2,n}$. Now, let $S_{z_2} \in \mathcal{S}_{z_2,n}$ be defined as the $*$-connected component of $S \cap (\tilde{C}_{z_2,n} \setminus C^-_{z_2,n})$ containing γ_2. Note that $\gamma_2 \subset S_{z_2} \subset S \setminus \hat{C}_{z,n+1}$. Now recalling that $S_{z_1} \subset \tilde{C}_{z_1,n} \subset z + [-L_n, L_{n+1} + L_n)^d$, it follows that

$$d_\infty(S_{z_1}, S_{z_2}) \geq d_\infty([-L_n, L_{n+1} + L_n)^d, \partial_{\text{out}} \hat{C}_{0,n+1}) \geq L_{n+1} - 2L_n \overset{(4.29)}{\geq} 2\varepsilon_n L_n. \tag{4.35}$$

Therefore, upon defining $\mathcal{A}_S$ to be the collection of all $\mathcal{D}$ such that the restriction of $\mathcal{D}$ to the leaves of the left and right sub-trees of $\mathbb{T}_{n+1-k}$ of depth $n - k$ correspond to some $\mathcal{D}_1 \in \mathcal{A}_{S_{z_1}}$ and $\mathcal{D}_2 \in \mathcal{A}_{S_{z_2}}$, for some $z_1 \in T_1$ and $z_2 \in T_2$ respectively, then the properties (4.31)-(4.33) follow as an immediate consequence of the above construction and the induction hypothesis. In particular, the distance constraint in (4.32) (at level $n + 1$) is exactly (4.35).

Finally, observe that

$$|\mathcal{A}_S| \leq |T_1| \cdot |T_2| \sup_{S' \in \mathcal{S}_n} |\mathcal{A}_{S'}|^2 \leq C_3 C_4 \sup_{S' \in \mathcal{S}_n} |\mathcal{A}_{S'}|^2. \tag{4.36}$$

Together, (4.34), (4.36) and (4.29) readily imply that a bound similar to (4.34) holds with $n + 1$ in place of n and $b_{n+1-k} \coloneqq b_{n-k} + \frac{\log(C_3 C_4)}{L^{n-k}}$, from which (4.30) follows with the choice $C = \lim_{n\to\infty} b_n (< \infty)$. □

The next result entails a capacity estimate which will be key in deducing (4.16). For a given shape $S \in \mathcal{S}_n$ and a collection $\mathcal{D} = \{(z(\ell), S(\ell)) : \ell \in \mathcal{L}(\mathbb{T}_{n-k})\} \in \mathcal{A}_{S,k}$ (with $\mathcal{A}_{S,k}$ as given by Lemma 4.8), consider a collection $\mathcal{B} = \{B(\ell) : \ell \in \mathcal{L}(\mathbb{T}_{n-k})\}$ of boxes such that, for all $\ell \in \mathcal{L}(\mathbb{T}_{n-k})$,

$$B(\ell) = z + [0,r)^d, \text{ for some } z \in \mathbb{Z}^d \text{ s.t. } B(\ell) \cap S(\ell) \neq \emptyset \text{ and } 1 \leq r \leq \frac{1}{2}\varepsilon_k L_k. \tag{4.37}$$

Now, define the set

$$S_{\mathcal{B}}(\tilde{\mathcal{D}}) := \bigcup_{\substack{\ell \in \mathcal{L}(\mathbb{T}_{n-k}): \\ (z(\ell), S(\ell)) \in \tilde{\mathcal{D}}}} B(\ell), \quad \text{ for } \tilde{\mathcal{D}} \subset \mathcal{D} \tag{4.38}$$

and

$$\kappa_{n,k} := \inf_{\rho \in (0,1]} \inf_{S \in \mathcal{S}_n} \inf_{\mathcal{D} \in \mathcal{A}_{S,k}} \inf_{\mathcal{B}} \inf_{\substack{\tilde{\mathcal{D}} \subset \mathcal{D}: \\ |\tilde{\mathcal{D}}| \geq \rho |\mathcal{D}|}} \rho^{-1} \text{cap}\,(S_{\mathcal{B}}(\tilde{\mathcal{D}})), \tag{4.39}$$

with the infimum over $\mathcal{B}$ ranging over all collections of boxes satisfying (4.37). The following lemma supplies suitable lower bounds on the quantity $\kappa_{n,k}$.

Lemma 4.9. ($d \geq 4$) *For any $n \geq k > 0$, one has*

$$\kappa_{n+1,k} \geq 2^{n+1} \wedge \frac{2\kappa_{n,k}}{1 + C\frac{2^{n+1}}{(\varepsilon_n L_n)^{d-2}}}. \tag{4.40}$$

As a consequence, for all for all $n \geq k > 0$, one has

$$\kappa_{n,k} \geq 2^n \wedge c_6 2^{n-k} \kappa_{k,k}. \tag{4.41}$$

Proof. Consider $S \in \mathcal{S}_{n+1}$, $\mathcal{D} \in \mathcal{A}_S = \mathcal{A}_{S,k}$, a collection $\mathcal{B}$ satisfying (4.37) and $\tilde{\mathcal{D}} \subset \mathcal{D}$ with $|\tilde{\mathcal{D}}| \geq \rho \mathcal{D}$ for some $\rho \in (0,1]$. Consider the sub-collections $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\mathcal{D}$ given by (4.32), define $\tilde{\mathcal{D}}_i := \tilde{\mathcal{D}} \cap \mathcal{D}_i$ as well as the sets $S_{\mathcal{B}}(\tilde{\mathcal{D}}_i)$ for $i = 1, 2$ in similar fashion as (4.38), so that $S_{\mathcal{B}}(\tilde{\mathcal{D}}) = S_{\mathcal{B}}(\tilde{\mathcal{D}}_1) \cup S_{\mathcal{B}}(\tilde{\mathcal{D}}_2)$. By (4.32) and (4.37) and since $n \geq k$, one has

$$d_2(S_{\mathcal{B}}(\tilde{\mathcal{D}}_1), S_{\mathcal{B}}(\tilde{\mathcal{D}}_2)) \geq \varepsilon_n L_n. \tag{4.42}$$

Now, since $S_{\mathcal{B}}(\tilde{\mathcal{D}}) \supset S_{\mathcal{B}}(\tilde{\mathcal{D}}_i)$ for $i = 1, 2$, using the sweeping identity (2.7), one bounds

$$\text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_i)) \leq \sum_{z \in S_{\mathcal{B}}(\tilde{\mathcal{D}}_i)} e_{S_{\mathcal{B}}(\tilde{\mathcal{D}})}(z) + \max_{z \in S_{\mathcal{B}}(\tilde{\mathcal{D}}_{3-i})} P_z[H_{S_{\mathcal{B}}(\tilde{\mathcal{D}}_i)} < \infty] \sum_{z \in S_{\mathcal{B}}(\tilde{\mathcal{D}}_{3-i})} e_{S_{\mathcal{B}}(\tilde{\mathcal{D}})}(z). \tag{4.43}$$

Using (2.5), one finds, for $i = 1, 2$,

$$\max_{z \in S_{\mathcal{B}}(\tilde{\mathcal{D}}_{3-i})} P_z[H_{S_{\mathcal{B}}(\tilde{\mathcal{D}}_i)} < \infty] \leq \text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_i)) \max_{z \in S_{\mathcal{B}}(\tilde{\mathcal{D}}_{3-i}), y \in S_{\mathcal{B}}(\tilde{\mathcal{D}}_i)} g(z, y),$$

The maximum of Green's functions on the right-hand side is bounded by $C(\varepsilon_n L_n)^{2-d}$ in view of (4.42) and (2.2). Now substituting these bounds into (4.43) and adding the resulting estimates for $i = 1, 2$, one deduces in view of (4.39) that

$$\text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}})) \geq \frac{\text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_1)) + \text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_2))}{1 + C\frac{\text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_1)) \vee \text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_2))}{(\varepsilon_n L_n)^{d-2}}} \geq \frac{|\tilde{\mathcal{D}}|}{|\mathcal{D}|} \frac{2\kappa_{n,k}}{1 + C\frac{\text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_1)) \vee \text{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_2))}{(\varepsilon_n L_n)^{d-2}}}. \tag{4.44}$$

However, since $\mathrm{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}})) \geq \mathrm{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_1)) \vee \mathrm{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}_2)) =: \Xi$ due to the monotonicity of $\mathrm{cap}(\cdot)$, see (2.8), distinguishing whether $\Xi \geq 2^{n+1}$, in which case $\mathrm{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}))$ inherits this lower bound, or $\Xi < 2^{n+1}$, in which case one applies (4.44), it follows that

$$\mathrm{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}})) \geq \frac{|\tilde{\mathcal{D}}|}{|\mathcal{D}|}\Big(2^{n+1} \wedge \frac{2\kappa_{n,k}}{1 + C\,\frac{2^{n+1}}{(\varepsilon_n L_n)^{d-2}}}\Big),$$

yielding (4.40). The lower bound (4.41) follows from (4.40) and a straightforward induction argument, with $c_6 = \prod_{n=0}^{\infty}(1 + C\frac{2^{n+1}}{(2\varepsilon_n L_n)^{d-2}})^{-1} > 0$ (see (4.29)). □

We now complete the proof of Proposition 4.3 for $d \geq 4$. Let $L \geq 1$ $K \geq 100$ and $N \geq 10KL$. We first introduce the collection $\mathcal{A} = \mathcal{A}^K_{N,L}(\Lambda_N)$ for any $\Lambda_N \in \mathcal{S}_N$ and verify that it is Γ-admissible. Let n_0 be maximal such that $L_{n_0} \leq N/10$, which by (4.29) implies that

$$N/30 \leq L_{n_0} \leq N/10, \text{ and let } L_{k_0-1} \leq 5L \leq L_{k_0}. \tag{4.45}$$

Note that $k_0 \leq n_0$. The set $\mathcal{A}$ will be defined in terms of the coarse grainings $\mathcal{A}_{S,k_0}$ of a fixed number of shapes $S \in \mathcal{S}_{n_0}$, which we now introduce. Let $V^0 \subset \mathbb{Z}^d$ consist of 0 and any point $z \in L_{n_0}\mathbb{Z}^d$ such that $b_{z,n_0} = z + [0, L_{n_0})^d$ intersects $\partial B_{3N/2}$ or $\partial[-\frac{N}{2}, \frac{3N}{2})^d$. Define $\mathcal{S}^0 = \{\tilde{C}_{z,n_0} \setminus b_{z,n_0} : z \in V^0\} \subset \mathcal{S}_{n_0}$. In view of (4.10), observe that any $*$-path γ crossing Λ_N induces a $*$-path crossing S for some $S \in \mathcal{S}^0$.

Now, let $k \in [k_0, n_0]$ be such that

$$2\varepsilon_{k-2}L_{k-2} \leq 2KL + 3L \leq 2\varepsilon_{k-1}L_{k-1} \tag{4.46}$$

and consider the subset of leaves $\ell' \in \mathcal{L}(\mathbb{T}_{n_0-k_0})$ of the form $\ell' = \ell \times (0, \dots, 0) =: \ell_0$, where $\ell \in \mathcal{L}(\mathbb{T}_{n_0-k})$ is arbitrary and $(0, \dots, 0) \in \mathcal{L}(\mathbb{T}_{k-k_0})$ is fixed. For a given shape $S \in \mathcal{S}^0$ and a collection $\mathcal{D}_0 \in \mathcal{A}_{S,k_0}$, "prune" $\mathcal{D}_0 = \{(z(\ell'), S(\ell')) : \ell' \in \mathcal{L}(\mathbb{T}_{n_0-k_0})\}$ (and forget the anchor point $z(\ell')$) to obtain the collection $\mathcal{D} = \{S(\ell_0) : \ell \in \mathcal{L}(\mathbb{T}_{n_0-k})\}$. Let $\mathcal{A}'$ denote the collections $\mathcal{D}$ thereby obtained as $\mathcal{D}$ ranges over $\mathcal{A}_{S,k_0}$ and $S \in \mathcal{S}^0$. The collection $\mathcal{A} = \mathcal{A}^K_{N,L}(\Lambda_N)$ is then defined as (recall $\mathbb{L}$ from (4.1))

$$\begin{aligned}\mathcal{A} = \big\{\mathcal{C} = (y(\ell))_{\ell \in \mathcal{L}(\mathbb{T}_{n_0-k})} \ &: y(\ell) \in \mathbb{L},\ \exists\, \mathcal{D} = \{S(\ell_0) : \ell \in \mathcal{L}(\mathbb{T}_{n_0-k})\} \in \mathcal{A}' \\ &\text{s.t. } C_{y(\ell),L} \cap S(\ell_0) \neq \emptyset \text{ for all } \ell \in \mathcal{L}(\mathbb{T}_{n_0-k})\big\}.\end{aligned} \tag{4.47}$$

We now verify that $\mathcal{A}$ defined in (4.47) is Γ-admissible. The fact that $|\mathcal{A}|$ satisfies (4.13) with $\Gamma(\cdot)$ as in (4.14) follows from (4.30), the fact that $|\mathcal{S}^0| \leq C$, whence $\log|\mathcal{A}'| \leq CL_{n_0-k_0}$, and since the choice of points in $\mathcal{C}$ for a given $\mathcal{D} \in \mathcal{A}'$ is bounded by $C^{|\mathbb{T}_{n_0-k}|}$. Overall, this gives $\log|\mathcal{A}| \leq CL_{n_0-k_0}$, as desired (note that $L_{n_0-k_0} \leq CN/L$ by (4.45)).

The crossing property (4.12) can be seen as follows. Let γ be a $*$-path crossing Λ_N. As noted above, γ induces a crossing for one of the annuli shapes $S \in \mathcal{S}^0$. By (4.33) there exists $\mathcal{D} \in \mathcal{A}_{S,k_0}$ such that the following holds for every $\ell \in \mathcal{L}(\mathbb{T}_{n_0-k})$. The $*$-path γ induces a $*$-path γ' crossing $\tilde{C}_{z(\ell),k_0} \setminus C_{z(\ell),k_0}$ with $\gamma' \subset S(\ell_0)$. In particular, by paving the part of $\partial S(\ell_0)$ adjacent to $C_{z(\ell),k_0}$ by boxes $C_{y,L}$, for $y \in \mathbb{L}$ and by choice of L_{k_0} in (4.45), one finds a point $y(\ell)$ such that $\tilde{D}_{y(\ell),L} \setminus C_{y(\ell),L}$ is crossed by γ'. The resulting collection $\mathcal{C}$ belongs to $\mathcal{A}$ and (4.12) follows.

Regarding (4.11), observe that $n = |\mathcal{C}| = |\mathcal{L}(\mathbb{T}_{n_0-k})|$, whose logarithm is comparable to L_{n_0}/L_k, hence to $N/(KL(\log KL)^2)$ using (4.45), (4.46) and the fact that $c(\log KL)^{-2} \leq \varepsilon_k \leq$

$C(\log KL)^{-2}$. The required separation property (4.6) then follows from (4.31) and (4.32). Indeed, the latter (applied inductively) implies that any two shapes $S(\ell_0)$, $S(\ell'_0)$ with $\ell \neq \ell' \in \mathcal{L}(\mathbb{T}_{n_0-k})$ are each subsets of two shapes $S(\ell), S(\ell') \in \mathcal{S}_k$ separated by $2\varepsilon_{k-1}L_{k-1} \geq 2KL + 3L$, see (4.46). Hence $S(\ell_0)$, $S(\ell'_0)$ inherit this separation. On account of (4.47), the resulting points $y(\ell)$, $y(\ell')$ are then at ℓ^∞-distance at least $2KL + L$.

Thus $\mathcal{A}$ is Γ-admissible. To see that the capacity lower bound (4.16) holds, first observe that $L \geq \frac{c_7}{K}\varepsilon_k L_k$ by (4.46) upon choosing c_7 small enough and choose $r(\geq 1)$ satisfying

$$\frac{c_7}{2K} \leq \frac{r}{\varepsilon_k L_k} \leq \Big(\frac{1}{2} \wedge \frac{c_7}{K}\Big). \tag{4.48}$$

In particular $r \leq L$. Now consider an arbitrary collection $\mathcal{C} \in \mathcal{A}$ and note that

(4.49) each box $C_{y(\ell),L}$, $y(\ell) \in \mathcal{C}$, contains a box $B(\ell)$ satisfying (4.37) (with $n = n_0$);

indeed this follows immediately by construction of $C_{y(\ell),L}$, which intersects $S(\ell)$ by definition, see (4.47), and the fact that $r \leq L$.

Together, (4.49), (4.8) and (4.38) imply that for any $\mathcal{C} \in \mathcal{A}$ and any sub-collection $\tilde{\mathcal{C}} \subset \mathcal{C}$ with $|\tilde{\mathcal{C}}| \geq (1-\rho)|\mathcal{C}|$ for some $\rho \in (0,1)$, $\mathrm{cap}(\Sigma(\tilde{\mathcal{C}})) \geq \mathrm{cap}(S_{\mathcal{B}}(\tilde{\mathcal{D}}))$, for some family $\mathcal{B}$ satisfying (4.37), where $\mathcal{D}$ refers to the collection generating $\mathcal{C}$, see (4.47), and $\tilde{\mathcal{D}} \subset \mathcal{D}$ is the sub-collection of $\mathcal{D}$ corresponding to the indices $\ell \in \mathcal{L}(\mathbb{T}_{n_0-k})$ appearing in $\tilde{\mathcal{C}} \subset \mathcal{C}$. It follows that

$$(1-\rho)^{-1}\mathrm{cap}(\Sigma(\tilde{\mathcal{C}})) \overset{(4.39)}{\geq} \kappa_{n_0,k} \overset{(4.41)}{\geq} cK^{-(d-2)}N, \tag{4.50}$$

where the last inequality is obtained by combining the fact that $2^{n_0} \geq cN$ due to (4.45) (see also the note following (4.29)) and observing that

$$\begin{aligned} 2^{-k}\kappa_{k,k} &\overset{(4.39)}{\geq} cL_k^{-1}\mathrm{cap}(B_r) \overset{(4.48)}{\geq} cK^{-(d-2)} \inf_{m\geq 0} L_m^{-1}\mathrm{cap}(B_{\varepsilon_m L_m}) \\ &\overset{(2.10)}{\geq} cK^{-(d-2)} \inf_{m\geq 0} L_m^{d-3}\varepsilon_m^{d-2} \overset{(4.29)}{\geq} c'K^{-(d-2)}. \end{aligned} \tag{4.51}$$

The bound (4.16) follows immediately from (4.50). If $\Lambda_N = B_N \setminus B_{\varepsilon N}$ for some $\varepsilon \in (0, \frac{1}{3})$, one simply sets $\mathcal{A}^K_{N,L}(\Lambda_N) := \mathcal{A}^K_{N,L}(B_N \setminus B_{N/2})$, which has the desired properties. This completes the proof of Proposition 4.3 for $d \geq 4$. □

5 Upper bounds

Using the coarse-graining scheme developed in the last section, see in particular Proposition 4.3, we now derive companion upper bounds to the lower bounds obtained in Theorem 3.1. The main result of this section is:

Theorem 5.1 (Upper bounds)**.**

i) If $d = 3$, then

$$\textit{for all } h > h_*, \ \limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \overset{\varphi \geq h}{\longleftrightarrow} \partial B_N] \leq -\frac{\pi}{6}(h - h_*)^2, \tag{5.1}$$

$$\textit{for all } h < h_*, \ \limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \overset{\varphi \geq h}{\longleftrightarrow} \partial B_N, \, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty] \leq -\frac{\pi}{6}(h_* - h)^2. \tag{5.2}$$

ii) If $d \geq 4$, then

$$\text{(5.3)} \qquad \textit{for all } h > h_*, \ \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N] < 0,$$

$$\text{(5.4)} \qquad \textit{for all } h < h_*, \ \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N, \, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty] < 0.$$

Moreover, the bounds (5.2) *and* (5.4) *also hold for the event* $\mathrm{LocUniq}(N,h)^c$ *(see* (1.5)*) in place of* $\{0 \xleftrightarrow{\varphi \geq h} \partial B_N, \, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty\}$.

In spite of a common thread, the treatment of the subcritical ($h > h_*$) and supercritical ($h < h_*$) regimes involve significantly different ideas. The supercritical case is more involved, mostly due to the additional disconnection constraint present in the events. Correspondingly, the upper bounds of Theorem 5.1 are furnished separately in two subsections. Section 5.1 contains the proof of (5.1) and (5.3), Section 5.2 that of (5.2) and (5.4).

5.1. Upper bounds for the subcritical phase. We start by giving an overview of the proof strategy leading to (5.1) and (5.3). To any path connecting 0 to ∂B_N in $\{\varphi \geq h\}$, one associates, in view of Proposition 4.3, a collection of well-separated boxes of carefully chosen size $L \ll N$, each containing a box-to-box crossing at scale L. By the decomposition (4.4) of φ into the sum of ψ and ξ within each such box, it follows that when $h > h_*$, either ξ is atypical for all but a small proportion of the boxes, or the localized version of the event, involving crossings in $\{\psi \geq h_* + \varepsilon\}$, behaves atypically for the remaining boxes. The corresponding events $E_{N,L}$ and $F_{N,L}$ (for ψ and ξ, respectively), are defined in (5.7) below. Together, they yield the central estimate (5.8), which drives the subsequent upper bounds. The key control on the event $F_{N,L}$ involving the harmonic average, derived in Lemma 5.5 for $d = 3$ and Lemma 5.9 for $d \geq 4$, is obtained by combining Lemma 4.1 and the capacity estimates of (4.16). The resulting bound ends up carrying the leading order in (5.1). The localized event $E_{N,L}$ is dealt with in Lemmas 5.4 and 5.7, and essentially inherits a given a-priori estimate (for instance (1.3)). Pitting the resulting bounds against the entropy factor (4.14) coming from the choice of coarse-grainings in Proposition 4.3 leads to an improved bound on the one-arm probability, for suitably chosen box sizes L. This scheme can be applied as a bootstrapping mechanism, see Proposition 5.2 below, thus yielding the desired bound (5.1) starting from (1.3) in several steps (in contrast, a single step suffices when $d \geq 4$).

We now render the above precise. Let $h > h'$ and $\varepsilon \in (0, h - h')$. Referring to the notations from (4.1)–(4.4), given $L \geq 1$, $K \geq 100$ and a vertex $z \in \mathbb{L} = \mathbb{L}(L)$, we introduce the events

$$\text{(5.5)} \qquad \{z \text{ is } \psi\text{-bad}\} := \{C_z \xleftrightarrow{\psi^z \geq h' + \frac{\varepsilon}{4}} \partial \tilde{C}_z\}, \text{ and}$$

$$\text{(5.6)} \qquad \{z \text{ is } \xi\text{-bad}\} := \big\{ \sup_{D_z} \xi^z \geq h - h' - \tfrac{\varepsilon}{4} \big\}.$$

We also refer to the box C_z as ψ/ξ-bad whenever z is ψ/ξ-bad. Next, for any $N \geq 4KL$ and and $\rho \in (0,1)$, consider the events

$$\text{(5.7)} \qquad \begin{aligned} E_{N,L} = E^K_{N,L}(\rho, h, h', \varepsilon) &:= \left\{ \begin{array}{c} \exists \mathcal{C} \in \mathcal{A}_{N,L} \text{ and } \tilde{\mathcal{C}} \subset \mathcal{C} \text{ with } |\tilde{\mathcal{C}}| = \lceil \rho |\mathcal{C}| \rceil \\ \text{such that all the sites } z \in \tilde{\mathcal{C}} \text{ are } \psi\text{-bad} \end{array} \right\}, \\ F_{N,L} = F^K_{N,L}(\rho, h, h', \varepsilon) &:= \left\{ \begin{array}{c} \exists \mathcal{C} \in \mathcal{A}_{N,L} \text{ and } \tilde{\mathcal{C}} \subset \mathcal{C} \text{ with } |\tilde{\mathcal{C}}| = |\mathcal{C}| - \lceil \rho |\mathcal{C}| \rceil \\ \text{such that all the sites } z \in \tilde{\mathcal{C}} \text{ are } \xi\text{-bad,} \end{array} \right\} \end{aligned}$$

where $\mathcal{A}_{N,L} = \mathcal{A}^K_{N,L}(B_N)$ is the admissible collection given by Proposition 4.3. Since $\varphi = \xi^z + \psi^z$ on $U_z \supset \tilde{C}_z$ and $D_z \supset \tilde{C}_z$, it is then a consequence of the property (4.12) of $\mathcal{A}_{N,L}$ and (5.5), (5.6), (5.7) that

$$\mathbb{P}[0 \overset{\varphi\geq h}{\longleftrightarrow} \partial B_N] \leq \mathbb{P}[E_{N,L}] + \mathbb{P}[F_{N,L}]. \tag{5.8}$$

The following (a-priori) bound will be useful in dealing with (5.5) and the event $E_{N,L}$ in (5.7). It will also apply to a different notion of ψ-badness in the next subsection, hence the general formulation. Consider an arbitrary increasing set $A \in \mathcal{B}(\mathbb{R}^{\tilde{C}_z})$. Then, in the notation of (3.4), if for some $h' \in \mathbb{R}$, $\varepsilon > 0$ and $L \geq 1$,

$$\mathbb{P}[A^{h'}(\varphi)] \leq \mathrm{e}^{-2f(L)} \text{ with } \log 2 \leq f(L) \leq c_8(\varepsilon)L, \tag{5.9}$$

then

$$\mathbb{P}\big[A^{h'+\frac{\varepsilon}{4}}(\psi^z)\big] \leq \mathrm{e}^{-f(L)}. \tag{5.10}$$

Indeed, (5.10) follows immediately from the decomposition $\varphi = \xi^z + \psi^z$ valid on U_z, whence

$$\mathbb{P}\big[A^{h'+\frac{\varepsilon}{4}}(\psi^z)\big] \leq \mathbb{P}[A^{h'}(\varphi)] + \mathbb{P}\big[\inf_{D_z} \xi^z \leq -\varepsilon/4\big] \leq \mathbb{P}[A^{h'}(\varphi)] + \mathrm{e}^{-2c_8(\varepsilon)L} \overset{(5.9)}{\leq} \mathrm{e}^{-f(L)},$$

where in the penultimate step we used Lemma 4.1 for the singleton $\mathcal{C} := \{z\}$ along with the lower bound $\mathrm{cap}(C_z) \geq cL$ from (2.10) (valid for all $d \geq 3$).

At this point we consider the cases $d = 3$ and $d \geq 4$ separately.

Upper bound for $d = 3$. Recall from (1.3) that the quantity $\mathbb{P}[0 \overset{\varphi\geq h}{\longleftrightarrow} \partial B_N]$ decays stretched exponentially in N for every $h > h_*$ with some exponent $\beta = c_1(h) \in (0,1)$. In what follows, we will bootstrap this decay to the one asserted by (5.1) in – as will soon turn out to be necessary – two steps. This is encapsulated in the following proposition, from which the upper bound (5.1) will quickly follow.

Proposition 5.2 (Bootstrap)**.** *Let $h' \in \mathbb{R}$ and $\beta' \in (0,1)$ be such that*

$$\limsup_{N\to\infty} \frac{1}{N^{\beta'}} \log \mathbb{P}[0 \overset{\varphi\geq h'}{\longleftrightarrow} \partial B_N] < 0. \tag{5.11}$$

Then for all $h > h'$, the following improved bounds hold, depending on the value of β'. If $\beta' \leq 1/2$, then

$$\limsup_{N\to\infty} \frac{1}{N^{\beta}} \log \mathbb{P}[0 \overset{\varphi\geq h}{\longleftrightarrow} \partial B_N] < 0 \textit{ for every } \beta < 1, \tag{5.12}$$

whereas if $\beta' > 1/2$, then

$$\limsup_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \overset{\varphi\geq h}{\longleftrightarrow} \partial B_N] \leq \frac{\pi}{6}(h-h')^2. \tag{5.13}$$

Assuming Proposition 5.2 to hold, we first give the short:

Proof of (5.1). Let $h > h_*$. By (1.3) we have (5.11) at any height $h' > h_*$ with exponent $\beta' = c_1(h') > 0$. Therefore, by (5.12), we obtain that (5.11) holds for every $h' > h_*$ and $\beta' < 1$ – in particular for $\beta' = 3/4$ (say). Consequently, we obtain the bound in (5.13) for any $h' \in (h_*, h)$. The result now follows by sending $h' \to h_*$. □

Remark 5.3. Proposition 5.2 highlights in a transparent form the paradigm underlying our strategy to obtain sharp upper bounds. Indeed, a similar (but considerably more involved) bootstrapping mechanism is at work in the supercritical regime; see Section 5.2. The choice (5.11) as a starting point for the bootstrap reflects the fact that stretched exponential estimates naturally come out of the static renormalization arguments leading to the existence of a non-trivial subcritical regime, see [19]. One could forego one step in deducing (5.1) as (5.12) is implied by the strongest available results [17, 18], but our findings do not rely on these. Moreover, we will face similar issues in the supercritical regime, where such results are not available *a-priori*. In fact one could even deduce the desired bound (5.13) from a much weaker a-priori estimate than a stretched-exponential bound by bootstrapping a few more times, see Remark 5.6 below.

We now aim at showing Proposition 5.2. Its proof combines individual estimates for $\mathbb{P}[E_{N,L}]$ and $\mathbb{P}[F_{N,L}]$, cf. (5.7) and (5.8), which are supplied in the following two lemmas.

Lemma 5.4. $(\rho \in (0,1),\ K \geq 100,\ h > h',\ \varepsilon \in (0, h-h'))$. *If* (5.9) *holds with* $A^{h'}(\varphi) = \{C_0 \stackrel{\varphi \geq h'}{\longleftrightarrow} \partial \tilde{C}_0\}$ *for some* $L \geq 1$, *then with* $E_{N,L} = E^K_{N,L}(\rho, h, h', \varepsilon)$, *for all* $N \geq 10KL$ *one has*

$$\log \mathbb{P}[E_{N,L}] \leq n(C\log(nK) - \rho f(L)), \tag{5.14}$$

where $n = |\mathcal{C}|$ *for any* $\mathcal{C} \in \mathcal{A}$ *(cf.* (4.11) *in Definition 4.2).*

Proof. On account of (5.7) and by a union bound, one obtains

$$\begin{aligned} \mathbb{P}[E_{N,L}] &\leq |\mathcal{A}_{N,L}| \binom{n}{\lceil \rho n \rceil} \sup_{\mathcal{C} \in \mathcal{A}_{N,L}} \sup_{\substack{\tilde{\mathcal{C}} \subset \mathcal{C} \\ |\tilde{\mathcal{C}}| = \lceil \rho n \rceil}} \mathbb{P}[z \text{ is } \psi\text{-bad}, \forall z \in \tilde{\mathcal{C}}] \\ &\leq e^{Cn\log(nK)} \sup_{\mathcal{C} \in \mathcal{A}_{N,L}} \sup_{\substack{\tilde{\mathcal{C}} \subset \mathcal{C} \\ |\tilde{\mathcal{C}}| = \lceil \rho n \rceil}} \mathbb{P}[z \text{ is } \psi\text{-bad}, \forall z \in \tilde{\mathcal{C}}] \end{aligned} \tag{5.15}$$

where the second line follows using (4.13), (4.14) and the lower bound on n from (4.11) in order to bound $|\mathcal{A}_{N,L}|$. Now, due to the independence property (4.7) and by translation invariance, one has, for any $\tilde{\mathcal{C}} \subset \mathcal{C} \in \mathcal{A}$ with $|\tilde{\mathcal{C}}| = \lceil \rho n \rceil$, since (5.9) holds,

$$\mathbb{P}[z \text{ is } \psi\text{-bad}, \forall z \in \tilde{\mathcal{C}}] = \mathbb{P}[0 \text{ is } \psi\text{-bad}]^{\lceil \rho n \rceil} \stackrel{(5.5),(5.10)}{\leq} e^{-\rho n f(L)},$$

which together with (5.15) gives (5.14). □

Next we present the relevant bound for $\mathbb{P}[F_{N,L}]$. Fix any function $L_1(N)$ such that $L_1(N) \geq (\log N)^\gamma$ for any $\gamma > 0$ and $N \geq C(\gamma)$ and such that $(\log L_1(N)/\log N) \to 0$ (cf. below (4.16) in Proposition 4.3).

Lemma 5.5. $(\rho \in (0,1),\ K \geq 100,\ h > h',\ \varepsilon \in (0, h-h'))$. *For any* $\theta > 0$, *one has*

$$\limsup_{N\to\infty} \sup_{L \in [(\log N)^{2+\theta}, L_1(N)]} \frac{\log N}{N} \log \mathbb{P}[F_{N,L}] \leq -\frac{\pi}{6} \frac{\lambda(K)(1-\rho)}{\alpha(K)} (h - h' - \varepsilon/4)^2, \tag{5.16}$$

with $\alpha(K)$ *and* $\lambda(K)$ *as appearing in Lemma 4.1 and Proposition 4.3, respectively.*

Proof. Recalling (5.6) and (5.7), and proceeding as in (5.15), one finds that

$$\log \mathbb{P}[F_{N,L}] \le Cn\log(nK) + \sup_{\mathcal{C}\in\mathcal{A}_{N,L}} \sup_{\substack{\tilde{\mathcal{C}}\subset\mathcal{C}\\ |\tilde{\mathcal{C}}|=n-\lceil\rho n\rceil}} \log \mathbb{P}\Big[\bigcap_{z\in\tilde{\mathcal{C}}}\big\{\sup_{x\in D_z}\xi_x^z \ge h-h'-\varepsilon/4\big\}\Big].$$

Denoting the event on the right hand side above by $F(\tilde{\mathcal{C}})$, we get, combining Lemma 4.1, the capacity lower bound (4.16) from Proposition 4.3 and the fact that $n \le N/LK$ by (4.11),

$$\begin{aligned}&\sup_{L\in[(\log N)^{2+\theta},L_1(N)]} \sup_{\mathcal{C}\in\mathcal{A}_{N,L}} \sup_{\substack{\tilde{\mathcal{C}}\subset\mathcal{C}\\ |\tilde{\mathcal{C}}|=n-\lceil\rho n\rceil}} \log \mathbb{P}[F(\tilde{\mathcal{C}})]\\ &\qquad \le -\frac{1}{2}(h-h'-\varepsilon/4-\delta)_+^2\frac{\lambda(K)(1-\rho)}{\alpha(K)}\mathrm{cap}(T_N)(1+o_N(1)),\end{aligned}$$

where

$$\delta=\delta(N,K,\rho,\theta) \coloneqq C\sqrt{\frac{N}{\lambda(K)(1-\rho)K^3(\log N)^{2+\theta}\mathrm{cap}(T_N)}} \longrightarrow 0, \text{ as } N\to\infty,$$

using the asymptotics for $\mathrm{cap}(T_N)$ from (2.12) in the last step. Finally notice that

$$n\log(nK) \le \frac{N}{K(\log N)^{1+\theta}} \text{ for } L \ge (\log N)^{2+\theta}.$$

The lemma follows by combining the previous displays along with the asymptotic of $\mathrm{cap}(T_N)$. □

With Lemmas 5.4 and 5.5 at hand, we proceed to the

Proof of Proposition 5.2. Let $h' \in \mathbb{R}$ be such that (5.11) holds and consider $h > h'$ and $\varepsilon \in (0, h-h')$. First choose $K \ge 100$ large enough and $\rho \in (0,1)$ close enough to 0, both depending on h, h' and ε, such that, applying Lemma 5.5, one obtains, for all $\theta > 0$, $N \ge C(\varepsilon, h, h', \theta)$ and $L \in [(\log N)^{2+\theta}, L_1(N)]$,

$$\log \mathbb{P}[F_{N,L}] \le -\frac{\pi}{6}(h-h'-\varepsilon/2)^2\frac{N}{\log N} \tag{5.17}$$

(recall to that effect that both $\alpha(K)$ and $\lambda(K)(1-\rho)$ converge to 1 in the limit $\rho\to 0$ and $K\to\infty$ by Lemma 4.1 and (4.15) in Proposition 4.3, respectively). Now for any $N \ge 10^3$, let

$$L = L(N,\theta) \coloneqq (\log N)^{2+\theta}, \tag{5.18}$$

for some $\theta > 0$ to be chosen. Notice that with (5.18) and by (5.11), the condition (5.9) holds with $A^{h'}(\varphi) = \{C_0 \stackrel{\varphi\ge h'}{\longleftrightarrow} \partial\tilde{C}_0\}$ and $f(L) = c(h',\beta')L^{\beta'}$ whenever $N \ge C(\varepsilon, h, h', \theta, \beta')$. With this choice of $f(\cdot)$ and since $nK \le N/L$ by (4.11), it follows that

$$C\log(nK) - \rho f(L) \le -\rho f(L)/2, \text{ for all } N \ge C(\varepsilon, h, h', \theta, \beta')$$

as soon as

$$(2+\theta)\beta' > 1. \tag{5.19}$$

Hence applying Lemma 5.4 and using the lower bound on n from (4.11), one gets (with L as in (5.18))

$$\log \mathbb{P}[E_{N,L}] \leq -\frac{c(h',\beta')\rho N}{K(\log N)^{(2+\theta)(1-\beta')}}, \tag{5.20}$$

provided (5.19) is satisfied and N is sufficiently large. Plugging the bounds from (5.17) and (5.20) into (5.8) we immediately deduce, letting $N \to \infty$ and then $\varepsilon \to 0$, that

$$\lim_{N\to\infty} \frac{(\log N)^{\beta}}{N} \log \mathbb{P}[0 \overset{\varphi \geq h}{\longleftrightarrow} \partial B_N] \leq -\frac{\pi}{6}(h-h')^2 \tag{5.21}$$

for any value of β satisfying

$$\begin{array}{ll} \beta > (2+\theta)(1-\beta'), & \text{if } (2+\theta)(1-\beta') \geq 1 \\ \beta = 1, & \text{otherwise} \end{array} \tag{5.22}$$

and any choice of $\theta > 0$ such that (5.19) is satisfied. If $\beta' \leq 1/2$, the conditions $\theta > 0$ and $(2+\theta)(1-\beta') < 1$ cannot simultaneously hold. Hence, in this case, choosing for example $\theta = \theta(\beta')$ so that $(2+\theta)\beta' = 2$, whence (5.19) is satisfied, (5.21) yields the bound (5.12). On the other hand when $\beta' > 1/2$, the conditions (5.19) and $(2+\theta)(1-\beta') < 1$ can be recast as

$$(0 <)\ \frac{1}{\beta'} - 2 < \theta < \frac{1}{1-\beta'} - 2$$

(note that the interval of admissible values for θ is non-degenerate because $\beta' > 1/2$). So choosing for instance $\theta = \frac{1}{2\beta'(1-\beta')} - 2$, we obtain (5.21) with $\beta = 1$ (since $(2+\theta)(1-\beta') < 1$ holds), which is (5.13). □

Remark 5.6. A careful examination of the proof of Proposition 5.2 reveals that a stretched exponential a-priori bound such as (5.11) is not required to arrive at (5.1). Indeed one could for instance obtain the same result by means of a few additional bootstrapping steps starting from a much weaker estimate of the type $\limsup_L \frac{\mathbb{P}[C_{0,L} \overset{\varphi\geq h}{\longleftrightarrow} \partial \tilde{C}_{0,L}]}{(\log L)^{\beta'(h)}} < 0$ for some $\beta'(h) > 0$ and all $h > h_*$ (or even a k-fold composition of log, for some fixed integer $k \geq 1$). Combining with other existing methods, see e.g. [17], one would further obtain that (5.1) holds as soon as $\mathbb{P}[C_{0,L} \overset{\varphi\geq h}{\longleftrightarrow} \partial \tilde{C}_{0,L}]$ is bounded from above by a suitable $c(d) \in (0,1)$ uniformly along a diverging subsequence of scales L. Similar conclusions could be drawn in the supercritical regime, cf. Remark 5.17,1).

Upper bound for $d \geq 4$. We now supply the proof of (5.3). Throughout the remainder of Section 5.1, for an arbitrary level $h > h_*$ (as appearing in (5.3)), we simply fix $h' = (h_*+h)/2$, $\varepsilon = (h-h')/8$, $\rho = 1/2$ and $K = 100$ in (5.5)–(5.7). The events $E_{N,L}$ and $F_{N,L}$ thus effectively depend on the sole parameter h. The following two results replace Lemmas 5.4 and 5.5, respectively.

Lemma 5.7. *($d \geq 4$, $h > h_*$) For all $L \geq C(h)$ and $N \geq 10^3 L$ one has*

$$\mathbb{P}[E_{N,L}] \leq \mathrm{e}^{-c(h)N/L}. \tag{5.23}$$

Proof. The proof mimics that of Lemma 5.4, with small modifications. Proceeding as in (5.15), using (4.13), (4.14) and the bound $\frac{cN}{L(\log L)^2} \leq n$ from (4.11) to bound $|\mathcal{A}^{100}_{N,L}|$, one finds,

$$\mathbb{P}[E_{N,L}] \leq e^{Cn(\log L)^2}\mathbb{P}[0 \text{ is } \psi\text{-bad}]^{\lceil n/2\rceil} \leq \mathrm{e}^{n(C(\log L)^2 - c(h)L^{c_1})}, \tag{5.24}$$

for all $L \geq 1$, $N \geq 10^3 L$, where the first inequality also relies on the independence property (4.7) and the second one on the fact that (1.3) and (5.9)–(5.10) combine to give a suitable bound on $\mathbb{P}[0 \text{ is } \psi\text{-bad}]$. Using the lower bound on n yet again, (5.23) readily follows from (5.24). □

Remark 5.8. The conclusions of Lemma 5.7 would remain unaltered if one replaced (1.3) by the (weaker) assumption that $\mathbb{P}[C_{0,L} \overset{\varphi\geq h}{\longleftrightarrow} \partial\tilde{C}_{0,L}] \leq \mathrm{e}^{-c(h)(\log L)^{2+\varepsilon}}$, for some $\varepsilon = \varepsilon(h) > 0$ and all $L \geq 1$, $h > h_*$. This is related to the power in the definition of ε_m in (4.29), and could be relaxed to a "$1+\varepsilon$"-condition by suitable modification of (4.29) and the subsequent arguments of Section 4.2, which would lead to a corresponding improvement of the lower bound on n in (4.11).

The analogue of Lemma 5.5 is

Lemma 5.9. ($d \geq 4$, $h > h_*$) *For some* $C(h) > 0$,

$$\limsup_{N\to\infty} \sup_{L\in[C(h),N/10K]} \frac{1}{N}\log\mathbb{P}[F_{N,L}] < 0. \tag{5.25}$$

Proof. For arbitrary $L \geq 1$, $N \geq 10^3 L$ and any given collection $\mathcal{C} \in \mathcal{A} = \mathcal{A}^{100}_{N,L}$ and $\tilde{\mathcal{C}} \subset \mathcal{C}$ with $|\tilde{\mathcal{C}}| = n - \lceil n/2\rceil$, one obtains by virtue of (4.16) and (2.13) that $\mathrm{cap}(\tilde{\Sigma}) \geq cN$. Together with Lemma 4.1, this is seen to imply that,

$$\mathbb{P}\Big[\bigcap_{z\in\tilde{\mathcal{C}}}\big\{\sup_{x\in D_z}\xi^z_x \geq h - h' - \tfrac{\varepsilon}{4}\big\}\Big] \leq \mathrm{e}^{-c(h)N} \tag{5.26}$$

(with h', ε as defined above Lemma 5.7) whenever $L \geq C(h)$ and $N \geq 10^3 L$, noting that $\frac{|\tilde{\mathcal{C}}|}{\mathrm{cap}(\tilde{\Sigma})} \leq cL^{-1}$ becomes suitably small for such L, cf. (4.9). In view of (5.7), applying a union bound over the choices of $\mathcal{C} \in \mathcal{A}_{N,L}$ and $\tilde{\mathcal{C}} \subset \mathcal{C}$, (5.25) is easily seen to follow since $\log|\mathcal{A}| \leq Cn(\log L)^2 \leq C\frac{N}{L}$, see (5.24) and the upper bound on n from (4.11). Thus, the resulting combinatorial complexity doesn't spoil the upper bound in (5.26) whenever $L \geq C'(h)$. □

Proof of (5.3)*.* The upper bound (5.3) follows immediately by combining (5.8), (5.23) and (5.25) upon choosing $L = C(h)$ large enough to ensure both Lemmas 5.7 and 5.9 are in force. □

Remark 5.10. Following up on Remark 4.7, we describe which upper bounds can be derived for $d \geq 4$ using the collection $\mathcal{A}'$ (obtained by following the coarse-graining scheme used for $d = 3$). For $h > h_*$, fix h', ε, K and ρ as above (5.7) and define $E'_{N,L} = E^K_{N,L}(\rho, h, h', \varepsilon)'$, $F'_{N,L}$ as in (5.7), but with $\mathcal{A}'$ in place of $\mathcal{A}$. Similarly as in (5.8), by admissibility of $\mathcal{A}'$, one sees that $\mathbb{P}[0 \overset{\varphi\geq h}{\longleftrightarrow} \partial B_N] \leq \mathbb{P}[E'_{N,L}] + \mathbb{P}[F'_{N,L}]$. Using (4.16), (5.26) and recalling the larger combinatorial complexity of $|\mathcal{A}'|$ (as in (5.15) for instance), one finds that

$$\mathbb{P}[F'_{N,L}] \leq \mathrm{e}^{Cn\log(nK)}\mathrm{e}^{-c(h)N} \overset{(4.11)}{\leq} \mathrm{e}^{CN\frac{\log N}{L} - cN} \leq \mathrm{e}^{-c'(h)N}, \text{ if } C(h)\log N \leq L \leq cN. \tag{5.27}$$

Regarding $\mathbb{P}[E'_{N,L}]$, observe that the bound (5.14) from Lemma 5.4 holds with $E'_{N,L}$ instead of $E_{N,L}$. This crucially uses the fact that $n = |\mathcal{C}|$ for $\mathcal{C} \in \mathcal{A}'$ satisfies the lower bound (4.11) with $u(x) = x$, which is used in the proof of Lemma 5.4. As a consequence, one obtains

$$\mathbb{P}[E'_{N,L}] \leq \mathrm{e}^{-c\frac{f(L)N}{L}}, \text{ if } f(L) \geq C \log N,\ L \leq cN, \tag{5.28}$$

assuming (5.9) holds with $A^{h'}(\varphi) = \{C_0 \xleftrightarrow{\varphi \geq h'} \partial \tilde{C}_0\}$.

As we now explain, one can deduce from (5.27) and (5.28) that for any integer $k \geq 1$

$$\mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N] \leq \mathrm{e}^{-c(h)N/(\log^{(k)} N)^{C_5}}, \text{ for all } N \geq 1,\ h > h_*, \tag{5.29}$$

where $\log^{(k)}(\cdot)$ denotes the k-fold composition of $\log(\cdot)$. To obtain (5.29) one proceeds as follows: in a separate (first) step, starting from (1.3), one chooses $f(L) = c(h)L^{c_1}$ and $L = C(h)(\log N)^{1+c_1^{-1}}$, whence (5.27) and (5.28) apply and yield (5.29) for $k = 1$ with $C_5 = c_1^{-1} - c_1$. Now, assuming (5.29) to hold for some $k \geq 1$, one chooses $f(L) = c(h)L/(\log^{(k)} L)^{C_5}$ and $L = C(h)\log N(\log^{(k+1)} N)^{C_5}$, whence (5.27) and (5.28) apply (in particular (5.9) holds with this choice of $f(\cdot)$ due to (5.29), and the conditions on L and $f(L)$ in (5.27), (5.28) are met) and readily yield (5.29) with $k + 1$ instead of k.

Note that (5.29) is nearly (5.3) and that the obstruction to obtaining the desired result comes from competition between the entropy of $\mathcal{A}'$ and the local fields leading to (5.28), not the contribution (5.27) from the harmonic average, which exhibits the desired exponential decay.

5.2. Supercritical phase. We now proceed to the proofs of (5.2) and (5.4) in Theorem 5.1, along with the corresponding statements for $\mathrm{LocUniq}(N, h)^c$, $h < h_*$, which we will actually prove first. In all cases, our argument revolves around a notion of good event G_N, see (5.32), which will allow us to construct ambient clusters with certain desirable properties. The bottom line is that it will be costly for any large connected set to avoid connecting to any such ambient cluster. This is quantified in Lemma 5.11. The desired estimate for $\mathbb{P}[G_N]$ is then arrived to by means of a renormalization scheme, starting from a certain localized good event $\mathcal{G}_z$, see Definition 5.43, which satisfies a suitable a-priori estimate, see Lemma 5.16. Importantly, the scheme, whose essential features are captured by Proposition 5.14, improves not only probabilistic estimates but also the number a of contact points inherent to the definition of $\mathcal{G}_z$, see (5.44), in each step of the iteration. The proofs of (5.2) and (5.4) for all events of interest follow by combining Lemma 5.11, Proposition 5.14 and Lemma 5.16 and are presented at the end of this section.

For $\tilde{f} : \tilde{\mathbb{Z}}^d \to \mathbb{R}$ (cf. above (2.32) regarding $\tilde{\mathbb{Z}}^d \supset \mathbb{Z}^d$), we define the local average $(A\tilde{f})_x = (2d)^{-1} \sum_{m \in \mathbb{M}^d : m \sim x} \tilde{f}_m$, for $x \in \mathbb{Z}^d$. For integers $L_0 \geq 1$ and $M > 1$, we then introduce the set

$$\mathcal{M}(\tilde{f}) := \big\{x \in \mathbb{Z}^d : (A\tilde{f})_y \geq -M \text{ for all } y \in B_{2L_0}(x)\big\}, \qquad \mathcal{M} := \mathcal{M}(\tilde{\varphi}) \tag{5.30}$$

(note that $B_{2L_0}(x) \subset \mathbb{Z}^d$ by definition and recall the extension $\tilde{\varphi}$ from (2.33)). Due to the decomposition $\tilde{\varphi} = \hat{\xi} + \hat{\psi}$ from (2.37), this means that $\mathcal{M} = \{x \in \mathbb{Z}^d : \hat{\xi}_y \geq -M,\ y \in B_{2L_0}(x)\}$. The condition used in the definition (5.30) provides us with a uniform insertion tolerance bound *on* the set $\mathcal{M}$ which will be used in the proof of Lemma 5.11 below (see around the display (5.37)) and which — as already noted in the beginning of Section 3.3 — is not otherwise true.

We now introduce a key (good) event, involving various parameters, which roughly ensures the existence of ambient clusters with desirable properties. Throughout this section, let

$$N, L, L_0 \text{ be integers with } N \geq L > 2L_0 \geq 1,\ M > 1 \text{ and } a, b \in \mathbb{N}_{>0}, \tag{5.31}$$

and assume that $\Lambda_N \in \{B_{2N} \setminus B_N,\ B_N \setminus B_{\sigma N}, \sigma \in (0, \frac{1}{3})\}$ is arbitrary, unless specified otherwise. To avoid clumsy notation, we will keep dependence on the quantities appearing in (5.31) and Λ_N implicit in the sequel, except for the ones that are subject to change in any given context. The intermediate scale L will first appear in Definition 5.12 below. For $h' \in \mathbb{R}$, the event $G_N = G_N(L_0, a, b, h', M)$ is defined as

$$G_N = \left\{ \begin{array}{c} \exists \text{ disjoint connected sets } \mathscr{C}_i \subset (\Lambda_N \cap \{\varphi \geq h'\}), \text{ for } 1 \leq i \leq b, \text{ such} \\ \text{that any } *\text{-path } \gamma \text{ crossing } \Lambda_N \text{ contains points } x_{i,j} = x_{i,j}(\gamma) \text{ satisfying} \\ B_{L_0}(x_{i,j}) \cap \mathscr{C}_i \cap \mathcal{M} \neq \emptyset, \text{ for all } 1 \leq i \leq b \text{ and } 1 \leq j \leq a. \end{array} \right\}. \tag{5.32}$$

Notice that the sets $\mathscr{C}_i$ in (5.32) may very well be connected in $\Lambda_N \cap \{\varphi \geq h'\}$. For $h' \in \mathbb{R}$, consider the sigma-algebra $\mathcal{F}_{h'} := \sigma(\tilde{\varphi}_{\tilde{y}}, 1\{\varphi_x \geq h'\};\ x \in \mathbb{Z}^d, \tilde{y} \in \mathbb{M}^d)$ and notice for later reference that $\mathcal{M}$, the event G_N as well as the sets $\mathscr{C}_1, \dots, \mathscr{C}_b$ are all $\mathcal{F}_{h'}$-measurable. Our interest in G_N stems from the following

Lemma 5.11. ($h < h'$, (5.31)). *There exists $c = c(h, h', L_0, M) > 0$ such that, with $G_N = G_N(L_0, a, b, h')$ as defined in* (5.32)*, the following holds if $a \geq C(L_0, h, h', M)$ and $b \geq C(L_0)$.*

$$\text{For } \Lambda_N = B_N \setminus B_{\sigma N},\ \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty] \leq \mathbb{P}[G_N^c] + \mathbb{P}[B_{\sigma N} \overset{\varphi \geq h'}{\not\longleftrightarrow} \infty] + \mathrm{e}^{-cba}. \tag{5.33}$$

$$\text{For } \Lambda_N = B_{2N} \setminus B_N,\ \mathbb{P}[\mathrm{LocUniq}(N, h)^c] \leq \mathbb{P}[G_N^c] + \mathbb{P}[B_N \overset{\varphi \geq h'}{\not\longleftrightarrow} \partial B_{2N}] + \mathrm{e}^{-cba}. \tag{5.34}$$

Proof. We only give the proof of (5.33). The proof of (5.34) follows by straightforward modifications of the argument. We begin by introducing an auxiliary event G'_N as follows. With $h'' := (h + h')/2$ and for $\lambda > 0$, the event $G'_N(\lambda)$ occurs if all of the following hold:

i) the set $\{\varphi \geq h''\}$ contains an infinite cluster $\mathscr{C}$ which intersects $B_{\sigma N}$.

ii) Letting $\mathscr{S} = B_N \cap \mathscr{C} \cap \mathcal{M}$, there exists — for any $*$-path γ crossing $B_N \setminus B_{\sigma N}$ — a set $S \subset \gamma$ with $|S| \geq \lambda ba$ such that $B_{L_0}(x) \cap \mathscr{S} \neq \emptyset$ for every $x \in S$.

The parameter λ, chosen below in (5.38), will ensure a certain well-separatedness property. Clearly the event $G'_N(\lambda)$ is measurable relative to $\mathcal{F}_{h''}$ (cf. below (5.32)). We will argue that for suitable $c, \lambda > 0$ each depending on h, h', L_0 only,

$$\mathbb{P}\big[\, 0 \xleftrightarrow{\varphi \geq h} \partial B_N, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty \,\big|\, \mathcal{F}_{h''} \big] 1_{G'_N(\lambda)} \leq \mathrm{e}^{-cba}, \text{ and} \tag{5.35}$$

$$\mathbb{P}\big[\, G'_N(\lambda)^c \cap G_N \cap \{B_{\sigma N} \xleftrightarrow{\varphi \geq h'} \infty\} \big] \leq \mathrm{e}^{-cba}, \tag{5.36}$$

from which (5.33) readily follows.

The proofs of (5.35) and (5.36) both hinge on the following representation of the conditional distribution of $\{\varphi \geq h\}$ under $\mathbb{P}[\,\cdot\,|\mathcal{F}_{h''}]$. In what follows, given $\mathbf{p} = (\mathbf{p}_x)_{x \in \mathbb{Z}^d}$ with $\mathbf{p}_x \in [0, 1]$, let $P_{\mathbf{p}}$ denote the corresponding product measure on $\{0, 1\}^{\mathbb{Z}^d}$, with canonical coordinates $Y = (Y_x)_{x \in \mathbb{Z}^d}$, so that $P_{\mathbf{p}}[Y_x = 1] = \mathbf{p}_x$ for all $x \in \mathbb{Z}^d$. By means of (2.37) and (2.38), one infers that for all $\tilde{h} > h$,

$$\begin{array}{l} \text{the law of } (1\{\varphi_x \geq h\})_{x \in \mathbb{Z}^d} \text{ under } \mathbb{P}[\,\cdot\,|\mathcal{F}_{\tilde{h}}] \text{ is } P_{\mathbf{p}} \text{ with } \mathbf{p}_x = \mathbb{P}[\varphi_x \geq h|\mathcal{F}_{\tilde{h}}],\ x \in \mathbb{Z}^d, \\ \mathbf{p}_x = 1 \text{ if } x \in \{\varphi \geq \tilde{h}\} \text{ and } \mathbf{p}_x \geq c_9 = c_9(h, \tilde{h}, M) \text{ if } x \in B_{L_0}(\mathcal{M}). \end{array} \tag{5.37}$$

Indeed, regarding the last part, on the event $\{\varphi_x < \tilde{h}\}$, one has by (2.37) and (2.38) that $\mathbb{P}[\varphi_x \geq h|\mathcal{F}_{\tilde{h}}] = 1 - \frac{\Phi(\sqrt{2}(h-\hat{\xi}_x))}{\Phi(\sqrt{2}(\tilde{h}-\hat{\xi}_x))}$, where $\Phi(\cdot)$ denotes the distribution function of a standard Gaussian random variable. In case $x \in B_{L_0}(\mathcal{M})$, one obtains the desired lower bound using that $\hat{\xi}_x \geq -M$ by (5.30) and $\tilde{h} > h$ whilst noting that $\lim_{\xi\to\infty} \frac{\Phi(s-\xi)}{\Phi(t-\xi)} = 0$ for any $s, t \in \mathbb{R}$ with $s < t$.

With (5.37) at hand, we proceed to show (5.35), (5.36), starting with (5.35) and choosing

$$\lambda := 1/(3|B_{3L_0}|)^2 \text{ (so that } \lceil 2\sqrt{\lambda}b\rceil \leq b/|B_{3L_0}| \text{ when } b \geq C(L_0)). \tag{5.38}$$

The choice of λ in (5.38) will soon become clear. Let $C(0)$ denotes the cluster of 0 in $B_L \cap \{\varphi \geq h\}$, under the measure $\mathbb{P}[\,\cdot\,|\mathcal{F}_{h'}] \overset{\text{law}}{=} P_{\mathbf{p}}$. We explore $C(0)$ vertex by vertex starting from 0 in a canonical way, i.e. checking at each step the state of the smallest (in a fixed deterministic ordering of the points in $\mathbb{Z}^d$) unexplored vertex in the exterior neighborhood of the currently explored piece of $C(0)$. We do so until the first time we discover a vertex $x_1 \in C(0)$ (there may not be any) which is in the exterior neighborhood of some $B_{L_0}(y_1)$ with $y_1 \in \mathscr{S}$ (note that $\mathscr{S}$ as defined in ii) above is $\mathcal{F}_{h''}$-measurable). At this point, we explore the state of every vertex in $B_{L_0}(y_1)$. By definition, $\mathscr{S} \subset \mathscr{C}$ so $\mathscr{C}$ intersects $B_{L_0}(y_1)$. We stop the exploration if some vertex of $\mathscr{C} \cap B_{L_0}(y_1)$ lies in $C(0)$, which occurs for instance if all the vertices of $B_{L_0}(y_1)$ belong to $C(0)$. Otherwise we continue exploring $C(0)$ until we discover a vertex $x_2 \in C(0)$ in the exterior neighborhood of $B_{L_0}(y_2)$ for some $y_2 \in \mathscr{S} \setminus \{y_1\}$ which was not visited by the exploration yet. As before, we then explore the state of every vertex in $B_{L_0}(y_2)$, stopping the exploration if $C(0)$ intersects $\mathscr{C}$ in that box and continuing otherwise. We proceed like this until either stopping or discovering the whole cluster $C(0)$. By construction, if the algorithm stops prior to discovering $C(0)$, the event $\{0 \overset{\varphi\geq h}{\longleftrightarrow} \infty\}$ occurs.

Using the above algorithm, one deduces the following. Let $\tau := |\{y_1, y_2, \dots, \}|$ denote the number of points y_i discovered until the algorithm stops (possibly $\tau = 0$). Then in view of property ii) above, with $\lambda' = \lambda/|B_{L_0+1}|$,

$$G'_N(\lambda) \cap \{0 \overset{\varphi\geq h}{\longleftrightarrow} \partial B_N, 0 \overset{\varphi\geq h}{\not\longleftrightarrow} \infty\} \subset \{\tau \geq \lambda' ba\} \tag{5.39}$$

(in particular, the right-hand side implies $\{\tau \geq 1\}$ whenever $a \geq C(h, h', L_0)$, as stipulated above (5.33)). Moreover, by means of (5.37) and (5.38), one sees that under $\mathbb{P}[\,\cdot\,|\mathcal{F}_{h'}]$ and on the event $G'_N(\lambda)$, conditionally on $\{\tau > n\}$ for some $0 \leq n < \lambda' ba$, the event $\{\tau = n + 1\}$ occurs with probability at least c'_9, where $c'_9(h, h'', L_0, M) = c_9(h, h'', M)^{|B_{L_0+1}|}$ (with c_9 as in (5.37)). Together with (5.39), this readily implies that the left-hand side of (5.35) is bounded by $(1 - c'_9)^{\lambda' ba}$, as claimed.

We now turn to the proof of (5.36). Recall that $G_N \cap \{B_{\sigma N} \overset{\varphi\geq h'}{\longleftrightarrow} \infty\}$ is $\mathcal{F}_{h'}$-measurable. We will in fact show that

$$\mathbb{P}\big[\, G'_N(\lambda)^c \,\big|\, \mathcal{F}_{h'}\big] 1\{G_N, B_{\sigma N} \overset{\varphi\geq h'}{\longleftrightarrow} \infty\} \leq \mathrm{e}^{-cba}, \tag{5.40}$$

which implies (5.36). Under $\mathbb{P}[\,\cdot\,|\mathcal{F}_{h'}]$ and on the event $G_N \cap \{B_{\sigma N} \overset{\varphi\geq h'}{\longleftrightarrow} \infty\}$, fix a path γ_0 in the infinite cluster of $\{\varphi \geq h'\}$ crossing Λ_N. Notice in particular that $\gamma_0 \subset \mathscr{C}$, cf. i) above. By definition of G_N, see (5.32), and by suitable labeling (using for instance the above ordering of $\mathbb{Z}^d$), one finds disjoint connected sets $\mathscr{C}_i \subset (\Lambda_N \cap \{\varphi \geq h'\})$, for $1 \leq i \leq \lceil 2\sqrt{\lambda}b\rceil$ (with λ as in (5.38)) and corresponding points $x_{i,j}(\gamma_0)$, $1 \leq j \leq \lceil\sqrt{\lambda}a\rceil$, satisfying $B_{L_0}(x_{i,j}(\gamma_0)) \cap \mathscr{C}_i \cap \mathcal{M} \neq \emptyset$ and

$|x_{i,j}(\gamma_0) - x_{i',j'}(\gamma_0)|_\infty > 2L_0$ for any $(i,j) \neq (i',j')$ and all $1 \leq i, i' \leq \lceil 2\sqrt{\lambda} b \rceil$, $1 \leq j, j' \leq \lceil \sqrt{\lambda} a \rceil$. Thus, letting

$$I := \big\{ i : x_{i,j}(\gamma_0) \xleftrightarrow{\varphi \geq h''} \mathscr{C}_i \text{ for some } 1 \leq j \leq \lceil \sqrt{\lambda} a \rceil \big\} \tag{5.41}$$

it follows that under $\mathbb{P}[\,\cdot\,|\mathcal{F}_{h'}]$ and on the event $\{B_{\sigma N} \xleftrightarrow{\varphi \geq h'} \infty, G_N\}$,

$$G'_N(\lambda)^c \subset \big\{ |I| < \sqrt{\lambda} b \big\}; \tag{5.42}$$

indeed for an arbitrary $*$-path γ crossing Λ_N, the occurrence of G_N guarantees the existence of vertices $x_{i,j}(\gamma)$ such that $B_{L_0}(x_{i,j}(\gamma)) \cap \mathscr{C}_i \cap \mathcal{M} \neq \emptyset$, for every $1 \leq i \leq \lceil 2\sqrt{\lambda} b \rceil$ and $1 \leq j \leq \lceil \sqrt{\lambda} a \rceil$. Moreover, if $i \in I$, then in fact $\mathscr{C}_i \subset \mathscr{C}$ on account of (5.41). Hence, if $|I| \geq \sqrt{\lambda} b$, the set $S := \{ x_{i,j}(\gamma) : i \in I, 1 \leq j \leq \lceil \sqrt{\lambda} a \rceil \}$ satisfies the properties listed in ii) above and (5.42) follows (the occurrence of i) is ensured by that of $\{B_{\sigma N} \xleftrightarrow{\varphi \geq h'} \infty\}$, which is conditioned on).

Finally, using (5.37), (5.38), (5.41) and the separation of the points $x_{i,j}(\gamma_0)$, $1 \leq i \leq \lceil 2\sqrt{\lambda} b \rceil$, $1 \leq j \leq \lceil \sqrt{\lambda} a \rceil$, one infers that $|I|$ stochastically dominates (under $\mathbb{P}[\,\cdot\,|\mathcal{F}_{h'}]$ and on G_N) a binomial random variable with $\lceil 2\sqrt{\lambda} b \rceil$ trials and success probability $1 - (1 - c_{10})^{\lceil \sqrt{\lambda} a \rceil}$, where $c_{10} := c'_9(h, h', L_0, M)$. Thus,

$$\mathbb{P}\big[|I| < \sqrt{\lambda} b \big| \mathcal{F}_{h'} \big] 1_{G_N} \leq 2^{\lceil 2\sqrt{\lambda} b \rceil} (1 - c_{10})^{\lceil \sqrt{\lambda} a \rceil \sqrt{\lambda} b} \leq e^{-cba}$$

for suitable $c = c(L_0, h, h', M)$, as soon as $\sqrt{\lambda} a \geq C(L_0, h, h', M)$, whence (5.40) readily follows from (5.42). This completes the proof of (5.36) and with it that of Lemma 5.11. □

The upper bounds (5.2) and (5.4), as well as the corresponding bounds for $\mathrm{LocUniq}(h,N)^c$, will eventually follow from Lemma 5.11. As will turn out, all estimates will ultimately be carried by the upper bound for $\mathbb{P}[G_N^c]$. Our next step is thus to derive suitable upper bounds on $\mathbb{P}[G_N^c]$, which we achieve by a renormalization scheme operating as follows. In a single step, the scheme goes from a base scale L to a larger scale N, cf. (5.31). Whereas b as appearing in (5.32) remains effectively fixed through the iteration (as will turn out b will grow roughly like $N/u(KL)$, cf. (4.11)), the scheme is designed to *simultaneously* improve on both the growth of $a := a_L$ in (5.32) and the strength of the estimate on $\mathbb{P}[G_N^c]$ in each step. Roughly speaking, this will boost a_L from being of order 1 to growing faster than $L/(\log L)^C$ when $d = 3$ and linearly in L when $d \geq 4$, whence the error terms in (5.33), (5.34) become sufficiently small.

The starting point of the argument is a certain good event $\mathcal{G}_z$, which we now introduce, and for which we will later supply a suitable a-priori estimate, see Lemma 5.16. In the sequel, $\tilde{\chi} = (\tilde{\chi}^z)_{z \in \mathbb{L}}$, where $\tilde{\chi}^z$ refers to either $\tilde{\varphi}$ or $\tilde{\psi}^z$, see (4.5). We write χ^z for the restriction of $\tilde{\chi}^z$ to $\mathbb{Z}^d$. Recall (4.1)-(4.4), (5.30) and (5.31) regarding notation.

Definition 5.12. (Good event). For $h_1 \leq h_2 \leq h_3 \in \mathbb{R}$ and $z \in \mathbb{L}$, the event $\mathcal{G}_z(\tilde{\chi}) = \mathcal{G}_z(\tilde{\chi}, L_0, L, a, h_1, h_2, h_3, M)$ occurs if all of the following hold:

(5.43) C_z is connected to ∂D_z in $\{\chi^z \geq h_3\}$ and for every $z' \in \mathbb{L}$ such that $|z - z'|_\infty \leq L$, all clusters of $\{\chi^{z'} \geq h_2\}$ crossing $\tilde{D}_z \setminus \tilde{C}_z$ are connected inside $D_z \cap \{\chi^z \geq h_1\}$.

(5.44) letting $\mathscr{S}_z = \tilde{D}_z \cap \mathscr{C}_z \cap \mathcal{M}(\tilde{\chi}^z)$ where $\mathscr{C}_z$ denotes the cluster of C_z in $\{\chi^z \geq h_1\}$, there exists – for any $*$-path γ crossing $\tilde{D}_z \setminus \tilde{C}_z$ – a set $S \subset \gamma$ with $|S| \geq a$ such that $B_{L_0}(x) \cap \mathscr{S}_z \neq \emptyset$ for every $x \in S$.

Henceforth we routinely suppress the dependence of $\mathcal{G}_z$ on parameters which stay fixed. Note that $\mathcal{G}_z$ indeed depends on $\tilde{\varphi}$, the extension of φ to $\tilde{\mathbb{Z}}^d$, through $\mathcal{M}(\tilde{\chi}^z)$ in (5.44), see (5.30). We now introduce, abbreviating $\tilde{\psi} = (\tilde{\psi}^z)_{z\in\mathbb{L}}$ and for $\varepsilon > 0$,

$$\mathcal{H}_z^1 := \mathcal{G}_z(\tilde{\psi}) \cap \big\{ \sup_{D_z} |\tilde{\xi}^z| \leq \varepsilon \big\}, \tag{5.45}$$

$$\mathcal{H}_z^2 := \mathcal{G}_z(\tilde{\psi}) \cap \big\{ \inf_{D_z} \tilde{\xi}^z \geq -\varepsilon \big\}, \tag{5.46}$$

each inheriting the dependence on parameters from $\mathcal{G}_z$ and where, as in (4.9), $\sup_{D_z}/\inf_{D_z}$ refer to the suprema/infima over all points in $D_z \cap \tilde{\mathbb{Z}}^d$. Let $\varepsilon_0 := \frac{1}{3}(1 \wedge (h_2 - h_1) \wedge (h_3 - h_2))$. The key features of (5.43)-(5.46) are summarized in the following

Lemma 5.13 (Gluing; $h_1 < h_2 < h_3$, $\varepsilon \in (0, \varepsilon_0)$, (5.31))**.**

For $z \in \mathbb{L}$, with $(\tilde{\chi}_1, \tilde{\chi}_2) \in \{(\tilde{\varphi}, \tilde{\psi}), (\tilde{\psi}, \tilde{\varphi})\}$ and $\mathcal{G}^1 := \mathcal{G}_z(\tilde{\chi}_1, h_1, h_2, h_3, M)$, $\mathcal{G}^2 := \mathcal{G}_z(\tilde{\chi}_2, h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon)$, one has

$$\mathcal{G}^1 \cap \{\sup_{z'} \sup_{D_{z'}} |\tilde{\xi}^{z'}_{\cdot}| \leq \varepsilon\} \subset \mathcal{G}^2, \tag{5.47}$$

with z' ranging over all points in $\mathbb{L}$ with $|z' - z|_\infty \leq L$. In particular,

$$\mathbb{P}[\mathcal{G}^2] \geq 1 - \mathrm{e}^{-f(L)} \text{ whenever } \mathbb{P}[\mathcal{G}^1] \geq 1 - \mathrm{e}^{-2f(L)} \text{ with } 4\log(d) \leq 4f(L) \leq c_8(4\varepsilon)L. \tag{5.48}$$

Moreover, for $z, z' \in \mathbb{L}$ with $|z' - z|_\infty = L$, if $\mathcal{H}_z^i \cap \mathcal{H}_{z'}^i$ occurs for some $i \in \{1, 2\}$,

(5.49) *all clusters of $\{\Xi_i^y \geq h_2 + \varepsilon_i\}$ crossing $\tilde{D}_y \setminus \tilde{C}_y$, for $y \in \{z, z'\}$, belong to a single conn. subsets $\mathscr{C}_{z,z'}$ of $\{\varphi \geq h_1 - \varepsilon\} \cap (D_z \cup D_{z'})$, which crosses both $D_z \setminus C_z$ and $D_{z'} \setminus C_{z'}$,*

where $\Xi_1^y = \varphi$, $\varepsilon_1 = \varepsilon$ and $\Xi_2^y = \psi^y$, $\varepsilon_2 = 0$, and

(5.50) *for any $*$-path γ crossing $\tilde{D}_y \setminus \tilde{C}_y$, for $y \in \{z, z'\}$, there exists a set $S \subset \gamma$ with $|S| \geq a$ such that $B_{L_0}(x) \cap \tilde{D}_y \cap \mathscr{C}_{z,z'} \cap \mathcal{M}_{M+\varepsilon}(\tilde{\varphi}) \neq \emptyset$, for all $x \in S$.*

Proof. We write $\mathcal{M} = \mathcal{M}_M$ throughout the proof to make the dependence on M explicit, cf. (5.30). The two-sided control on the harmonic average in (5.47) and (4.4) imply that for all $h \in \mathbb{R}$ and $i \in \{1, 2\}$, any connected subset of $\{\chi_i \geq h\} \cap D_{z'}$ is contained in a connected subset of $\{\chi_{3-i} \geq h - \varepsilon\} \cap D_{z'}$, for any $z' \in \mathbb{L}$ with $|z' - z|_\infty \leq L$. Applying this repeatedly in the context of (5.43) readily yields the corresponding property for the event on the right-hand side of (5.47). Regarding (5.44), recalling the local averaging operator A from above (5.30), on $B_{2L_0}(\tilde{D}_z)(\subset \mathbb{Z}^d)$ one has that $A\tilde{\chi}_2 = A(\tilde{\chi}_1 \pm \tilde{\xi}^z) \geq A\tilde{\chi}_1 - |A\tilde{\xi}^z| \geq A\tilde{\chi}_1 - \varepsilon$ by linearity and using that $|A\tilde{\xi}^z| \leq A|\tilde{\xi}^z| \leq \varepsilon$ on $B_{2L_0}(\tilde{D}_z)$, whence $\mathcal{M}_M(\tilde{\chi}_1^z) \cap \tilde{D}_z$ is contained in $\mathcal{M}_{M+\varepsilon}(\tilde{\chi}_2^z) \cap \tilde{D}_z$.

The implication (5.48) follows readily from (5.47) upon applying e.g. (4.9) for a single box (along with the bound on the capacity of a box as given by (2.10)) to control the tails of the harmonic average.

To obtain (5.49), first observe that regardless of the choice of $i \in \{1, 2\}$, by (5.43), (5.45), (5.46) and the fact that any path crossing $D_{z'} \setminus C_{z'}$ also crosses $\tilde{D}_z \setminus \tilde{C}_z$ (see (4.2) and (4.3)), the following holds: all clusters of $\{\psi^y \geq h_2\}$ crossing $\tilde{D}_y \setminus \tilde{C}_y$, for $y \in \{z, z'\}$, belong to a single connected component $\mathscr{C}_{z,z'}$ of $(\{\psi^z \geq h_1\} \cap D_z) \cup (\{\psi^{z'} \geq h_1\} \cap D_{z'})$, which crosses both $D_z \setminus C_z$ and $D_{z'} \setminus C_{z'}$ (the last part is due to the first item in (5.43), which guarantees the existence of such a crossing for ψ^y above level $h_3 > h_2$). The control on the lower tail of ξ^z, $\xi^{z'}$ present in (5.46) (and (5.45)) implies that $\mathscr{C}_{z,z'}$ belongs to a connected component of $\{\varphi \geq h_1 - \varepsilon\}$, thus yielding (5.49) in case $i = 2$. The upper bound on ξ^y, $y \in \{z, z'\}$, in (5.45) further implies that any cluster in $\{\varphi \geq h_2 + \varepsilon\}$ crossing $\tilde{D}_y \setminus \tilde{C}_y$ is part of a crossing cluster of $\{\psi^y \geq h_2\}$, whence (5.49) follows for $i = 1$.

Finally, the property (5.50) is inherited from (5.44) when $\mathcal{H}_z^i \cap \mathcal{H}_{z'}^i$ occurs, regardless of $i \in \{1, 2\}$. For, by construction, the cluster of C_y in $\mathscr{C}_{z,z'}$ contains the cluster of C_y in $\{\psi^y \geq h_1\}$ for $y \in \{z, z'\}$, and $(\mathcal{M}_M(\tilde{\psi}^y) \cap \tilde{D}_y) \subset (\mathcal{M}_{M+\varepsilon}(\tilde{\varphi}) \cap \tilde{D}_y)$, since $A\tilde{\varphi} = A(\tilde{\psi}^y + \tilde{\xi}^y) \geq A\tilde{\psi}^y - \varepsilon$ in $B_{2L_0}(\tilde{D}_y)$ if the event $\mathcal{H}_y^i$ occurs (this only uses the lower bound on $\tilde{\xi}^y$). □

We now set up the bootstrapping scheme that will lead to a suitable control of $\mathbb{P}[G_N^c]$. The index $i \in \{1, 2\}$ in (5.45), (5.46) and below reflects the fact that intermediate steps (corresponding to $i = 1$) and the final step (corresponding to $i = 2$) of the argument need to be dealt with in distinct ways. The coarse-graining scheme developed in Section 4 now enters the picture. Referring to Proposition 4.3, for integers $K \geq 100$, $N \geq 10KL$, we let $\mathcal{A}^1 = \mathcal{A}_{N,L}^K(\tilde{D}_{0,N} \setminus \tilde{C}_{0,N})$ and $\mathcal{A}^2 = \mathcal{A}_{N,L}^K(\Lambda_N)$ (with $\Lambda_N \in \{B_N \setminus B_{\sigma N}, B_{2N} \setminus B_N\}$, $\sigma \in (0, 1/3)$, as below (5.31)), and define, for $\rho \in (0, 1)$,

$$H_{N,L}^i := \bigcap_{\mathcal{C} \in \mathcal{A}^i} \bigcup_{\substack{\tilde{\mathcal{C}} \subset \mathcal{C} \\ |\tilde{\mathcal{C}}| \geq \rho |\mathcal{C}|}} \bigcap_{z \in \tilde{\mathcal{C}}} \mathcal{H}_z^i. \tag{5.51}$$

We elaborate a bit more on the central role of the event $H_{N,L}^i$ in Remark 5.15,2) below. The next result is at the heart of our argument. It shows that $H_{N,L}^i$ typically reproduces the event from Definition 5.12 (implicit in $\mathcal{H}_z^i$) at a higher scale N and with an *improved* choice of a (for $i = 1$), as well as the target event G_N from (5.32) (for $i = 2$).

Proposition 5.14 (Bootstrap; $h_1 < h_2 < h_3$, (5.31), $\rho \in (0, 1), K \geq 100, i = 1, 2$)**.** *With $H_{N,L}^i = H_{N,L}^i(a, h_1, h_2, h_3, \varepsilon, M, \rho)$, whenever $\frac{\rho N}{u(KL)} \geq C$, one has the inclusion*

$$\big(H_{N,L}^i \cap \Omega_N^i\big) \subset \begin{cases} \mathcal{G}_0(\tilde{\varphi}, N, a', h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon), & \text{if } i = 1, \text{ for } \varepsilon \in (0, \varepsilon_0), \\ G_N(a, b, h_1 - \varepsilon, M + \varepsilon), & \text{if } i = 2, \text{ for all } \varepsilon > 0, \end{cases} \tag{5.52}$$

where $\Omega_N^1 = \{C_{0,N} \xleftrightarrow{\varphi \geq h_3 - \varepsilon} \partial D_{0,N}\}$, $\Omega_N^2 = \mathbb{R}^{\tilde{\mathbb{Z}}^d}$ and

$$b := \lfloor \tfrac{c_5}{10} \tfrac{\rho N}{u(KL)} \rfloor, \quad a' := ba. \tag{5.53}$$

Remark 5.15. 1) In (5.52), one also has the inclusion $H_{N,L}^1 \subset G_N(a, b, h_1 - \varepsilon)$ since $H_{N,L}^1 \subset H_{N,L}^2$. The weaker condition on the harmonic field entering the definition of $\mathcal{H}_z^2$, cf. (5.45) and (5.46), will play a role in obtaining the sharp bound (5.2) for $d = 3$. The asserted result (5.4) for $d \geq 4$ could be obtained by means of $\mathcal{H}_z^1$ (and $H_{N,L}^1$) alone.

2) (The role of ρ). In view of (5.33), (5.34) and (5.52), our goal in bounding $\mathbb{P}[G_N^c]$ becomes to control the probability of $(H_{N,L}^i)^c$, which by definition, see (5.51), entails the existence of at least one collection $\mathcal{C} \in \mathcal{A}^i$ containing a large fraction $(1 - \rho)$ of (bad) points z for which $\mathcal{H}_z^i$ does not occur. The sacrifice of a small fraction $\rho > 0$ of (good) points inherent in (5.51) is utilized in the proof below in order to create b interfaces (growing linearly with ρ) with certain good properties, implied locally by the occurrence of $\mathcal{H}_z^i$. This eventually leads to the desired improvement (5.53) over a.

Proof. The proof uses Lemma 2.1 as a crucial ingredient, applied to the renormalized lattice $\mathbb{L} = \mathbb{L}(L)$ rather than $\mathbb{Z}^d$. For any $A \subset \mathbb{Z}^d$, we use $\mathbb{L}_A$ to denote the set of all $z \in \mathbb{L} = \mathbb{L}(L)$ such that $A \cap C_{z,L} \neq \emptyset$ (see (4.1) and (4.2) for notation). In what follows, let $U = \mathbb{L}_{\tilde{C}_{0,N}}$ if $i = 1$ and $U = \mathbb{L}_{B_{\sigma N}}$ (resp. $U = \mathbb{L}_{B_N}$) if $i = 2$ and $\Lambda_N = B_N \setminus B_{\sigma N}$ (resp. $\Lambda_N = B_{2N} \setminus B_N$). In a similar vein, let $V = \mathbb{L}_{\tilde{D}_{0,N}}$ if $i = 1$ and $V = \mathbb{L}_{B_N}$ or $\mathbb{L}_{B_{2N}}$ if $i = 2$.

Now let $\Sigma \subset \mathbb{L}$ denote the collection of all points $z \in \mathbb{L} \setminus U$ such that $\mathcal{H}_z^i$ occurs and $\tilde{D}_{z,L} \subset \tilde{D}_{0,N}$ if $i = 1$, resp. $\tilde{D}_{z,L} \subset B_N$ or B_{2N} if $i = 2$. As we now explain, on $H_{N,L}^i$,

(5.54) $\Sigma \subset \mathbb{L}$ satisfies the hypotheses of Lemma 2.1 with U, V as above and $k = \lfloor \frac{c_5 \rho N}{u(KL)} \rfloor - 1$.

To see (5.54), consider any $*$-path $\gamma_{\mathbb{L}}$ in $\mathbb{L}$ crossing $V \setminus U$. By suitably interpolating between successive vertices of $\gamma_{\mathbb{L}}$, one creates a $*$-path γ in $\mathbb{Z}^d$ such that $\gamma|_{\mathbb{L}} = \gamma_{\mathbb{L}}$. Let $\mathcal{C} \in \mathcal{A}_i$ be the admissible collection corresponding to γ, i.e. such that (4.12) holds. Then by definition, see (5.51), if $H_{N,L}^i$ occurs, there exists $\tilde{\mathcal{C}} \subset \mathcal{C}$ such that $\mathcal{H}_z^i$ occurs for all $z \in \tilde{\mathcal{C}}$, hence $\tilde{\mathcal{C}} \subset \Sigma$, and (5.54) follows since $|\tilde{\mathcal{C}}| \geq k$ on account of (5.51) and (4.11).

With (5.54) in force, applying Lemma 2.1 we deduce the existence of disjoint $*$-connected subsets $O_1 \preceq \ldots \preceq O_k$ (each part of $\mathbb{L}$) of Σ all of which surround U. By definition of $\preceq$, it follows that $(O_i + D_{0,L}) \cap (O_j + D_{0,L}) = \emptyset$ as soon as $|i - j| \geq 7$. Consequently we can extract from $\{O_i : 1 \leq i \leq k\}$ a subcollection $\{O'_i : 1 \leq i \leq k'\}$ with $k' \geq k/8$ such that $(O'_i + D_{0,L})$ are pairwise disjoint subsets of A with $V = L_A$. Now for each $j \in \{1, \ldots, k'\}$, by $*$-connectedness of O'_j, the fact that $\mathcal{H}_z^i$ occurs for every $z \in O'_j (\subset \Sigma)$ and using property (5.49), one finds that the connected sets $\mathscr{C}_{z,z'}$, for $z, z' \in O'_j$ with $|z - z'|_\infty = L$, are contained in a single connected subset $\mathscr{C}_j$ of $\{\varphi \geq h_1 - \varepsilon\} \cap (O'_j + D_{0,L})$. The sets $\mathscr{C}_j$, $1 \leq j \leq k'$, are disjoint by construction. Moreover by (5.50),

(5.55) for any $*$-path γ crossing $\tilde{D}_{z,L} \setminus \tilde{C}_{z,L}$, for some $z \in O'_j$, there exists a set $S \subset \gamma$ with $|S| \geq a$ such that $B_{L_0}(x) \cap \tilde{D}_{z,L} \cap \mathscr{C}_j \cap \mathcal{M}_{M+\varepsilon} \neq \emptyset$, for all $x \in S$.

In addition, if $i = 1$, (5.49) yields that

(5.56) all conn. subsets of $\{\varphi \geq h_2 + \varepsilon\}$ crossing $\tilde{D}_{z,L} \setminus \tilde{C}_{z,L}$ for some $z \in O'_j$ intersect $\mathscr{C}_j$.

We now explain how the inclusions (5.52) follow from this, and first consider the case $i = 1$. In view of Definition 5.12, this amounts to verifying (5.43) for $\chi^z, \chi^{z'} = \varphi$, with N in place of L and at the heights given by (5.52), as well as (5.44) with a' in place of a. First, the connection required in (5.43) is ensured by Ω_N^1.

To proceed further, we need the following observation. For a $*$-path γ on $\mathbb{Z}^d$, define the trace $\gamma^{\mathbb{L}}$ of γ on $\mathbb{L}$ as follows: $\gamma^{\mathbb{L}}(0)$ is the unique point in $\mathbb{L}$ such that $\gamma(0) \in C_{\gamma^{\mathbb{L}}(0),L}$ (recall that these boxes partition $\mathbb{Z}^d$, see (4.2)). Set $n_0 = 0$. Given $\gamma^{\mathbb{L}}(0), \ldots, \gamma^{\mathbb{L}}(k-1)$ and $n_0, \ldots n_{k-1}$ for some $k \geq 1$, set $n_k = \inf\{n > n_{k-1} : \gamma(n) \notin C_{\gamma^{\mathbb{L}}(k-1),L}\}$ and $\gamma^{\mathbb{L}}(k) \in \mathbb{L}$ is such that $\gamma(n_k) \in C_{\gamma^{\mathbb{L}}(k),L}$. By construction $\gamma^{\mathbb{L}}$ is a $*$-path on $\mathbb{L}$. Moreover, if γ crosses $\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$, then $\gamma^{\mathbb{L}}$ crosses $V \setminus U$. As the sets $\mathscr{C}_j$, $1 \leq j \leq k'$, each surround U, it follows that for each $*$-path γ crossing $\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$,

(5.57) there exists $\{z_j : 1 \leq j \leq k'\} \subset \gamma^{\mathbb{L}}$ such that $\mathrm{dist}_{\ell^\infty(\mathbb{L})}(z_j, O'_j) \leq 1$.

Indeed, (5.57) follows for instance by extending $\gamma^{\mathbb{L}}$ to a nearest-neighbor path $\bar{\gamma}^{\mathbb{L}}$ on $\mathbb{L}$, which only requires adding vertices at unit $\ell^\infty(\mathbb{L})$-distance from $\gamma^{\mathbb{L}}$. The path $\bar{\gamma}^{\mathbb{L}}$ crossing $V \setminus U$ in turn intersects O'_j for all $1 \leq j \leq k'$ by [6, Lemma 2.1] and the surrounding property of each O'_j.

Now, returning to the verification of (5.43), consider a cluster of $\{\varphi \geq h_2 + \varepsilon\}$ crossing $\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$. Extracting a crossing path γ from this cluster, it follows by definition of $\gamma^{\mathbb{L}}$ that γ induces a crossing in $D_{z_1,L} \setminus C_{z_1,L}$, with z_1 as in (5.57) (recall that $O_1' + D_{0,L} \subset \tilde{D}_{0,N}$). Hence, γ induces a crossing in $\tilde{D}_{y_1,L} \setminus \tilde{C}_{y_1,L}$ for some $y_1 \in O_1'$ with $|z_1 - y_1|_{\ell^\infty(\mathbb{L})} \leq 1$. Thus, γ intersects $\mathscr{C}_1$ by (5.56). All in all, each cluster of $\{\varphi \geq h_2 + \varepsilon\}$ crossing $\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$ intersects $\mathscr{C}_1 \subset \{\varphi \geq h_1 - \varepsilon\}$. Since $\mathscr{C}_1$ is connected, the second part of (5.43) follows.

To deduce (5.44) relative to the event $\mathcal{G}_0(\tilde{\varphi}, N, a', h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon)$ in (5.52), one proceeds as follows. Repeating the above argument for all $j \in \{1, \dots, k'\}$ using (5.56) and (5.57), one first observes that the cluster of $\{\varphi \geq h_3 - \varepsilon\}$ crossing $D_{0,N} \setminus C_{0,N}$ stipulated by Ω_N^1 intersects each $\mathscr{C}_j$, and therefore

$$\text{(5.58)} \qquad \mathscr{C}_0 \text{, the cluster of } C_{0,N} \text{ in } \{\varphi \geq h_1 - \varepsilon\}\text{, contains } \mathscr{C}_j \text{ for all } 1 \leq j \leq k'.$$

Now, still by the same argument, every $*$-path γ crossing $\tilde{D}_{0,N} \setminus \tilde{C}_{0,N}$ induces a crossing in $\tilde{D}_{y_j,L} \setminus \tilde{C}_{y_j,L}$ for some $y_j \in O_j'$ and all $1 \leq j \leq k'$ (in fact γ is also connected to every $\mathscr{C}_j$ but we won't use this). By (5.55), there exist sets $S_j \subset \gamma$ for all $1 \leq j \leq k'$, each of cardinality at least a, such that $B_{L_0}(x) \cap \tilde{D}_{y_j} \cap \mathscr{C}_j \cap \mathcal{M}_{M+\varepsilon} \neq \emptyset$ for all $x \in S_j$ and one can replace $\mathscr{C}_j$ by $\mathscr{C}_0$ in the previous intersection due to (5.58). By construction the sets S_j are disjoint, Thus letting $S \coloneqq \bigcup_j S_j$, one obtains by (5.55) that $|S| \geq k'a \geq (k/8)a$, whence $|S| \geq a'$ on account of (5.54) and (5.53) whenever $\frac{\rho N}{u(KL)} \geq C$. All in all, S has all the properties required by (5.44).

We now verify (5.52) in case $i = 2$. Consider a $*$-path γ crossing Λ_N. By (5.57), γ induces crossings in $\tilde{D}_{y_j,L} \setminus \tilde{C}_{y_j,L}$ for suitable $y_j = y_j(\gamma) \in O_j'$ and all $1 \leq j \leq k'$. Applying (5.55), we obtain for every $j \in \{1, \dots, k'\}$ a set of points $\{x_{j,k} = x_{j,k}(\gamma) : 1 \leq k \leq a\}$ such that $B_{L_0}(x_{j,k}) \cap \tilde{D}_{y_j} \cap \mathscr{C}_j \cap \mathcal{M}_{M+\varepsilon} \neq \emptyset$. In view of (5.32), the inclusion (5.52) for $i = 2$ follows since the sets $\mathscr{C}_j$ are connected subsets of $\Lambda_N \cap \{\varphi \geq h_1 - \varepsilon\}$ and $k' \geq b$. □

Our last missing ingredient needed prior to proceeding to the proofs of (5.2) and (5.4) is an a-priori estimate for the event $\mathcal{G}_z$ from Definition 5.12 at levels below h_*, which is available by current methods and which we supply next. This a-priori bound will play a role akin to (5.11) in the subcritical case and enable us to initiate the bootstrap argument in Proposition 5.14.

Lemma 5.16 (A-priori estimate)**.** $h_1 < h_2 < h_3 < h_*$, $\varepsilon \in (0, \varepsilon_0 \wedge \frac{1}{3}(h_* - h_3))$. *There exist* $L_0 \geq 1$, $M > 1$ *and* $c_{11} > 0$, *each depending on* $\underline{h} = (h_1, h_2, h_3)$ *and* ε *only, such that for* $\mathcal{G}_0 = \mathcal{G}_0(\tilde{\chi}, L_0, L_n, a = 1, \underline{h}, M)$ *with* $\tilde{\chi} \in \{\tilde{\varphi}, \tilde{\psi}\}$ *and* $\frac{L_n}{L_{n+1}} = c(d)$, *one has*

$$\text{(5.59)} \qquad \lim_{n \to \infty} \frac{1}{L_n^{c_{11}}} \log \mathbb{P}[\mathcal{G}_0^c] < 0.$$

Proof. It suffices to consider the case $\tilde{\chi} = \tilde{\varphi}$. The case $\tilde{\chi} = \tilde{\psi}$ then follows by applying (5.48). The bound (5.59) (with $\tilde{\chi} = \tilde{\varphi}$) will follow by applying results of [8] to the graph $\tilde{\mathbb{Z}}^d$ (with unit weights). For $x \in \mathbb{Z}^d$, we consider the events (at scale L_0, see (4.2) for notation)

$$\text{(5.60)} \qquad A_x^1 \coloneqq \{C_{x,L_0} \xleftrightarrow{\varphi \geq h_3 + \varepsilon} \partial D_{x,L_0}\}$$

$$\text{(5.61)} \qquad A_x^2 \coloneqq \left\{ \begin{array}{c} \text{all clusters of } \{\varphi \geq h_2 - \varepsilon\} \text{ crossing } \tilde{D}_{x,L_0} \setminus \tilde{C}_{x,L_0} \\ \text{are connected inside } D_{x,L_0} \cap \{\varphi \geq h_1 + \varepsilon\} \end{array} \right\}$$

$$\text{(5.62)} \qquad A_x^3 \coloneqq \{\mathcal{M} \supset D_{x,L_0}\}$$

with $\mathcal{M} = \mathcal{M}(\tilde{\varphi})$ as defined in (5.30) and $M := (\log L_0)^2$. For $\lambda = \lambda(d)\,(\geq 100)$ sufficiently large –the choice of λ corresponds to the constant $20c_{18}C_{10}$ appearing e.g. in (8.3) of [8], and in the present case $C_{10} = 1$ and c_{18} is determined by the isoperimetric constant on $\mathbb{Z}^d$) – one then sets, for $x \in \mathbb{Z}^d$, with $\bar{\lambda} = 1.1\lambda$, $\ell_0 = 3^d \vee 12\bar{\lambda}$ and $L_n = \ell_0^n L_0$, for $1 \leq k \leq 3$,

$$
\begin{aligned}
\widetilde{A}^k_{x,0} &= \bigcap_{y \in B_{\lambda L_0}(x)} A^k_y \\
\widetilde{A}^k_{x,n} &= \bigcap_{y,z \in (L_{n-1}\mathbb{Z}^d \cap B_{\bar{\lambda} L_n}(x)):\, d(y,z) \geq L_n} \big(\widetilde{A}^k_{y,n-1} \cup \widetilde{A}^k_{z,n-1}\big), \quad \text{for } n \geq 1.
\end{aligned} \tag{5.63}
$$

Since $h_3 + \varepsilon < h_*$, combining the bounds in (1.6) and applying a second union bound over $B_{\lambda L_0}$, one infers that that $\lim_{L_0} \mathbb{P}[\widetilde{A}^k_{x,0}] = 1$ for $k = 1, 2$. Similarly, one shows that $\lim_{L_0} \mathbb{P}[\widetilde{A}^3_{x,0}] = 1$ using a standard Gaussian tail estimate (note that $\mathrm{var}((A\tilde{\varphi})_0) \geq c$) and applying a union bound over D_{x,L_0} and $B_{\lambda L_0}$. All in all, one obtains that $\lim_{L_0} \mathbb{P}[\widetilde{A}^k_{x,0}] = 1$ for all $1 \leq k \leq 3$. In particular, by choosing $L_0 = L_0(\underline{h}, \varepsilon)$ sufficiently large, one can ensure that the conditions (7.5) and (7.6) in [8] are satisfied, whence Proposition 7.1 therein applies and yields that

$$
\mathbb{P}[(\widetilde{A}^{k\prime}_{x,n})^c] \leq 2^{-2^n}, \text{ for all } 1 \leq k \leq 3 \text{ and } n \geq 0, \tag{5.64}
$$

where the primed events $\widetilde{A}^{k\prime}_{x,n}$ refer to those defined in (5.63), but with sprinkled parameters (h_1, h_2, h_3) in place of $(h_1 + \varepsilon, h_2 - \varepsilon, h_3 + \varepsilon)$ in (5.60)-(5.62). Note to this effect that the event $\widetilde{A}^k_{x,0}$ is measurable with respect to the restriction of $\tilde{\varphi}$ to $\tilde{\mathbb{Z}}^d \cap B_{\bar{\lambda} L_0}(x)$, as required for Proposition 7.1 in [8] to apply, and that a slight extension (of the underlying decoupling inequality (2.20)) is required when $k = 2$, cf. (5.61), in order to take care of the two opposite directions of monotonicity. To conclude, one applies Lemma 8.6 in [8], which implies that whenever $\bigcap_k \widetilde{A}^{k\prime}_{x,n}$ occurs, any two connected sets in $B_{\lambda L_n}$ of diameter at least $(\lambda/20)L_n$ each, are connected by a path $\gamma \subset B_{2\lambda L_n}$ such that $\bigcap_k \widetilde{A}^{k\prime}_{x,0}$ occurs for all $x \in \gamma$. Due to the (primed versions of the) choices (5.60)-(5.62), this event is readily seen to imply $\mathcal{G}_0$ with $L = \lfloor (\lambda/10) L_n \rfloor$. The bound (5.59) then follows from (5.64). □

We are now ready to assemble the pieces and prove (5.2) and (5.4). In view of (5.52), this entails probing into the complements of the events $H^i_{N,L}$, $i = 1, 2$, from (5.51). In the spirit of (5.5), (5.6), for $z \in \mathbb{L} = \mathbb{L}(L)$ we say that

$$
\{z \text{ is } \psi\text{-bad}\} := \mathcal{G}^c_z(\tilde{\psi}), \text{ and} \tag{5.65}
$$

$$
\{z \text{ is } (\xi, i)\text{-bad}\} := \begin{cases} \{\sup_{D_z} |\tilde{\xi}_z| > \varepsilon\}, & \text{if } i = 1 \\ \{\inf_{D_z} \tilde{\xi}_z < -\varepsilon\}, & \text{if } i = 2, \end{cases} \tag{5.66}
$$

whence z is either ψ-bad or (ξ, i)-bad whenever $\mathcal{H}^i_z$ occurs, cf. (5.45), (5.46). By (5.51), it then follows that for any $\rho' \in (0, 1 - \rho)$ and $i \in \{1, 2\}$,

$$
\big(H^i_{N,L}\big)^c \subset E^i_{N,L} \cup F^i_{N,L}, \tag{5.67}
$$

where

$$
E^i_{N,L} := \left\{ \begin{array}{c} \exists\, \mathcal{C} \in \mathcal{A}^i \text{ and } \tilde{\mathcal{C}} \subset \mathcal{C} \text{ with } |\tilde{\mathcal{C}}| = \lceil \rho' |\mathcal{C}| \rceil \\ \text{such that all the sites in } \tilde{\mathcal{C}} \text{ are } \psi\text{-bad} \end{array} \right\},
$$

$$
F^i_{N,L} := \left\{ \begin{array}{c} \exists\, \mathcal{C} \in \mathcal{A}^i \text{ and } \tilde{\mathcal{C}} \subset \mathcal{C} \text{ with } |\tilde{\mathcal{C}}| = |\mathcal{C}| - \lceil \rho |\mathcal{C}| \rceil - \lceil \rho' |\mathcal{C}| \rceil \\ \text{such that all the sites in } \tilde{\mathcal{C}} \text{ are } (\xi, i)\text{-bad} \end{array} \right\}.
$$

At this point we consider the cases $d = 3$ and $d \geq 4$ separately.

Upper bound for $d = 3$. Combining Proposition 5.14, Lemmas 5.16 and 5.11 with a bootstrap argument (similar in spirit to the one leading to the corresponding subcritical upper bound), we proceed to give the

Proof of (5.2). Let $h < h_*$. We assume in the sequel that $\varepsilon \in (0, \frac{h_*-h}{16})$ and set $h_1 = h_* - 12\varepsilon$, $h_2 = h_* - 8\varepsilon$ and $h_3 = h_* - 4\varepsilon$, whence $\varepsilon \in (0, \varepsilon_0 \wedge \frac{1}{3}(h_* - h_3))$ (cf. below (5.46) regarding ε_0).

In the first step, we take the bound given by Lemma 5.16 as our input and improve it (along with the parameter a) via Proposition 5.14 applied with $i = 1$. To this end, we first choose L_0 and M, both depending on h, ε only, such that Lemma 5.16 is in force. Then, applying Proposition 5.14 with these choices for L_0 and M, as well as $K = 100$, $\rho = 1/2$ (see (5.51)) and $a = 1$, the inclusion (5.52) and (5.53) yield that

$$\mathbb{P}[(\mathcal{G}_{0,N})^c] \leq \mathbb{P}[(H^1_{N,L})^c] + \mathbb{P}[C_{0,N} \overset{\varphi \geq h_3 - \varepsilon}{\not\longleftrightarrow} \partial D_{0,N}].$$

for all $L > 2L_0$ and $N \geq CL$, where $\mathcal{G}_{0,N} = \mathcal{G}_0(\tilde{\varphi}, L_0, N, a = \lfloor \frac{cN}{L} \rfloor, h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon)$ and $H^1_{N,L} = H^1_{N,L}(L_0, a = 1, h_1, h_2, h_3, \varepsilon, M, \rho = \frac{1}{2})$. Incorporating (5.67) with the choice $\rho' = \frac{1}{2}$ and the upper bound on disconnection probability given by Theorem 5.5 in [20], we obtain from the previous display, under the same assumptions on L and N that

$$\mathbb{P}[(\mathcal{G}_{0,N})^c] \leq \mathbb{P}[E^1_{N,L}] + \mathbb{P}[F^1_{N,L}] + \mathrm{e}^{-c(h,\varepsilon)N}. \tag{5.68}$$

In view of (5.66), adapting the argument used in the proof of Lemma 5.5 to the present case where $\mathcal{A}^1 = \mathcal{A}^{100}_{N,L}(\tilde{D}_{0,N} \setminus \tilde{C}_{0,N})$, using symmetry and applying a union bound (costing an inconsequential factor $\binom{n}{\lceil n/2 \rceil}$ where $n = |\mathcal{C}|$, $\mathcal{C} \in \mathcal{A}^1$) to get rid of absolute values in (5.66) and $F^1_{N,L}$, we deduce that

$$\mathbb{P}[F^1_{N,L}] \leq \mathrm{e}^{-c(h,\varepsilon)\frac{N}{\log N}}, \text{ for all } N \geq C(h, \varepsilon) \text{ and } (\log N)^3 \leq L \leq L_1(N)$$

(with $L_1(N)$ as fixed above Lemma 5.5). On the other hand, in view of (5.65), retracing the steps that led to the proof of Lemma 5.4 and (5.20), replacing the input bound (5.9) by (5.59) for $\tilde{\chi} = \tilde{\psi}$ (whence $f(L) = c(h, \varepsilon)L^{c_{11}}$), one finds that for all $L = L_n$ as appearing in Lemma 5.16 satisfying $L \in [C(h, \varepsilon)(\log N)^{2/c_{11}}, cN]$ and $N \geq C(h, \varepsilon)$,

$$\mathbb{P}[E^1_{N,L}] \leq \mathrm{e}^{-c(h,\varepsilon)\frac{N}{L^{1-c_{11}}}}.$$

Substituting the estimates for $\mathbb{P}[F^1_{N,L}]$ and $\mathbb{P}[E^1_{N,L}]$ into (5.68) and choosing $L = L_{n_0}$ with $n_0 = n_0(N, h, \varepsilon) := \inf\{n \geq 0 : L_n \geq C(h, \varepsilon)(\log N)^{\max(3, 2/c_{11})}\}$, we obtain for all $N \geq C(h, \varepsilon)$,

$$\mathbb{P}[\mathcal{G}_0(\tilde{\varphi}, L_0, N, a_N, h_1 - \varepsilon, h_2 + \varepsilon, h_3 - \varepsilon, M + \varepsilon)^c] \leq \mathrm{e}^{-f'(N)}, \tag{5.69}$$

where $a_N := \lfloor \frac{N}{(\log N)^{c_{12}}} \rfloor$ and $f'(N) := \frac{c(h,\varepsilon)N}{(\log N)^{C(h,\varepsilon)}}$. This yields the desired improvement, both in terms of a and the probabilistic bound, over the a-priori estimate from Lemma 5.16.

In the second step, we start with the improved bound (5.69) and feed it to Proposition 5.14 (in case $i = 2$) to derive an estimate for the event G_N in (5.32), for a suitable choice of the parameters. Thus applying (5.52) with the height parameters from (5.69) in place of (h_1, h_2, h_3),

$\varepsilon' := h_1 - h - 2\varepsilon (> 0)$ in place of ε and $L_0 = L_0(h,\varepsilon)$ as in the previous step, we obtain, for all $L \geq C(h,\varepsilon)$, $K \geq 100$, $\rho \in (0,1)$ and N such that $\frac{\rho N}{u(KL)} \geq C$, using the decomposition (5.67),

$$\mathbb{P}[G_N(a_L, b, h+\varepsilon, M')^c] \leq \mathbb{P}[E^2_{N,L}] + \mathbb{P}[F^2_{N,L}], \tag{5.70}$$

with b as given by (5.53), $M' = M + h_1 - h - \varepsilon$ and where the events $E^2_{N,L}$, $F^2_{N,L}$ inherit the parameters from $H^2_{N,L} = H^2_{N,L}(a_L, h_1-\varepsilon, h_2+\varepsilon, h_3-\varepsilon, \varepsilon', M+\varepsilon, \rho)$, and depend on an additional $\rho' \in (0, 1-\rho)$. Now, mimicking the arguments of the previous step to bound the probabilities on the right-hand side in (5.70), but this time using f' instead f (as implied by (5.69)) when estimating $\mathbb{P}[E^2_{N,L}]$, and with a view to (5.66) (compare with the definition of $F_{N,L}$ in (5.7) and the proof of (5.16)) when dealing with $\mathbb{P}[F^2_{N,L}]$, one obtains the following for the choice $L := L_{n_1}$ where $n_1(N,h,\varepsilon) := \inf\{n \geq 0 : L_n \geq (\log N)^3\}$, whence $(\log N)^3 \leq L_{n_1} \leq C(h,\varepsilon)(\log N)^3$: For K large enough, σ (in the case $\Lambda_N = B_N \setminus B_{\sigma N}$) and ρ close enough to 0, all depending on h and ε and $\rho' = 1 - 2\rho$,

$$\mathbb{P}[G_N(a'_N, b'_N, h+\varepsilon, M')^c] \leq \mathrm{e}^{-\frac{\pi}{6}(h_1-h-2\varepsilon)^2 \frac{N}{\log N}}, \text{ for all } N \geq C(h,\varepsilon), \tag{5.71}$$

where $a'_N = a_{L_{n_1}}$ and $b'_N = \lfloor \frac{c(h,\varepsilon)N}{L_{n_1}} \rfloor$.

With (5.71) at hand, we now deduce (5.2) and first consider the finite-volume event $\mathrm{LocUniq}(N,h)$. Plugging (5.71) (with $\Lambda_N = B_N \setminus B_{\sigma N}$) into (5.34) with $h' = h+\varepsilon$, we obtain for $N \geq C(h,\varepsilon)$,

$$\begin{aligned}\mathbb{P}[\mathrm{LocUniq}(N,h)^c] &\leq \mathrm{e}^{-\frac{\pi}{6}(h_*-h-C\varepsilon)^2 \frac{N}{\log N}} + \mathbb{P}[B_N \overset{\varphi \geq h'}{\not\longleftrightarrow} \partial B_{2N}] + \mathrm{e}^{-c b'_N a'_N} \\ &\leq 2\mathrm{e}^{-\frac{\pi}{6}(h_*-h-C\varepsilon)^2 \frac{N}{\log N}} + \mathrm{e}^{-c(h')N},\end{aligned} \tag{5.72}$$

where the second line follows by Theorem 5.5 in [20] and since $b'_N a'_N \geq c(h,\varepsilon)\frac{N}{(\log\log N)^{C(h,\varepsilon)}}$. The claim readily follows from (5.72) by taking logarithms, multiplying by $\frac{\log N}{N}$ on both sides, letting $N \to \infty$ and then $\varepsilon \downarrow 0$.

The upper bound in (5.2) for the truncated one-arm event will follow similarly from (5.33) and (5.71) upon supplying a suitable upper bound for the probability of disconnecting $B_{\sigma n}$ from infinity above level $h' = h + \varepsilon$ (here $\sigma = \sigma(h,\varepsilon) > 0$ refers to the choice that leads to (5.71)), which is not readily available for us to use. To circumvent this issue, we combine the upper bound (5.2) for the local uniqueness event derived above and the disconnection upper bound from [20] as follows: consider the sequence of events, for some integer $\overline{M} > 1$,

$$\begin{aligned}&\mathcal{E}_1 := \{B_N \overset{\varphi \geq h'}{\longleftrightarrow} \partial B_{4\overline{M}N}\}, \\ &\mathcal{E}_{2k} := \mathrm{LocUniq}(h', 2^k\overline{M}N) \text{ and } \mathcal{E}_{2k+1} := \{B_{2^k\overline{M}N} \overset{\varphi \geq h'}{\longleftrightarrow} \partial B_{6\cdot 2^k\overline{M}N}\}, \text{ for } k \geq 1.\end{aligned}$$

It readily follows from the definition of LocUniq, see (1.5), that $\{B_N \overset{\varphi \geq h'}{\longleftrightarrow} \infty\} \subset \bigcap_{k\geq 1}\mathcal{E}_k$. Therefore, applying a union bound and subsequently using the bounds from [20] and (5.72) for the respective probabilities, we get for any $\overline{M} > 1$ and $N \geq C(h,\varepsilon,\overline{M})$,

$$\begin{aligned}&\mathbb{P}[B_N \overset{\varphi \geq h'}{\not\longleftrightarrow} \infty] \\ &\quad \leq \sum_{k\geq 1}\mathbb{P}[\mathcal{E}_k^c] \leq \sum_{k\geq 0}\mathrm{e}^{-c(h')2^k\overline{M}N} + \sum_{k\geq 1}\mathrm{e}^{-c(h_*-h')^2\frac{2^k\overline{M}N}{\log(2^k\overline{M}N)}} \leq \mathrm{e}^{-c(h_*-h')^2\frac{\overline{M}N}{\log(N)}}.\end{aligned} \tag{5.73}$$

Since $\overline{M}$ is arbitrary, (5.73) implies that

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[B_N \overset{\varphi\geq h}{\not\longleftrightarrow} \infty] = -\infty. \tag{5.74}$$

for all $h < h_*$. Similarly as with (5.72), (5.2) readily follows from (5.74), (5.71) and (5.33). □

Upper bound for $d \geq 4$. Similarly to the subcritical phase, the proof of upper bounds simplifies in higher dimensions.

Proof of (5.4). Let $h < h_*$. We choose h_1, h_2, h_3 as in the beginning of the proof of (5.2) and simply fix $\varepsilon := (h_* - h)/20$. Following the same line of reasoning that led to the bound (5.69), except for applying Proposition 5.14 with $i = 2$ directly (instead of $i = 1$) with $\sigma = \frac{1}{4}$ in case $\Lambda_N = B_N \setminus B_{\sigma_N}$, and using analogues of Lemmas 5.7 and 5.9 in place of Lemmas 5.4 and 5.5, respectively, to bound $\mathbb{P}[E^1_{N,L}]$ and $\mathbb{P}[F^1_{N,L}]$, thereby choosing $L = C(h)$ large enough, one finds that for all $N \geq C(h)$,

$$\mathbb{P}[G_N(a = 1, b = \lfloor c(h)N \rfloor, h_1 - \varepsilon, M + \varepsilon) \leq \mathrm{e}^{-c'(h)N}. \tag{5.75}$$

Here $M = M(h)$ and $L_0 = L_0(h)$, implicit in the definition G_N, cf. (5.30) and (5.32), are chosen as in the proof of (5.2) when applying Lemma 5.16. Plugging (5.75) for the choice $\Lambda_N = B_{2N} \setminus B_N$ into (5.34) with $h' = h_1 = \varepsilon(> h)$ and using Theorem 5.5 in [20] in order to bound the disconnection probability, we get

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}[\mathrm{LocUniq}(N, h)^c] < 0. \tag{5.76}$$

Regarding (5.4) for the truncated one-arm event, proceeding similarly as in the case $d = 3$, we first derive from (5.76) and the disconnection upper bound from [20] that $\frac{1}{N} \log \mathbb{P}[B_N \overset{\varphi\geq h}{\not\longleftrightarrow} \infty]$ tends to $-\infty$ as $N \to \infty$, from which (5.4) follows upon applying (5.33) and using (5.75) with $\Lambda_N = B_N \setminus B_{N/4}$. □

We conclude with a few comments.

Remark 5.17. 1) Analogues of Remarks 5.6 and 5.10 also hold in the supercritical regime. Namely, at the expense of iterating a few more times, initializing the above scheme does not require the full strength of the stretched exponential a-priori bound provided by Lemma 5.16. Moreover, when $d \geq 4$, replacing the admissible collection $\mathcal{A}^i$ inherent to the definition of $H^i_{N,L}$ in (5.51) by the corresponding collection $(\mathcal{A}')^i$ (cf. Remark 4.7), which employs the coarse-graining strategy used for $d = 3$, one can derive analogues of (5.29) for $\mathrm{LocUniq}(N, h)^c$ or the event in (5.4) when $h < h_*$. The proof essentially follows the line of argument of Remark 5.10, applying Proposition 5.14 with $i = 1$ for all but the k-th step (where $k \geq 1$ refers to any target number of iterated logarithms, cf. (5.29)). The important thing to notice is that the number a of contact points, albeit always growing sublinearly in the macroscopic scale due to the choice of L, improves suitably along with the bound on $\mathcal{G}_0$ through intermediate steps of the iteration.

2) (Two-point functions). We now briefly discuss the amendments to our methods that would be required to obtain the asymptotics (1.11) for the truncated two-point function. Regarding lower bounds, in the arguments leading to (3.1) and (3.2), one would need to 'tilt' the construction, thus following the ℓ^2-geodesic between x and y) rather than a horizontal line, in order to force a connection between x and y. An asymptotic capacity estimate for such a discretized ℓ^2-geodesic similar to (2.12) (when $\lfloor |x-y| \rfloor = N$) is given by Remark 2.3. Corresponding analogues of (2.23) and (2.24) would also be required. The former relies on the visibility Lemma 2.4, which is robust with respect to 'tilting' of the above kind. To adapt the proof of (2.24), one could compare to a random walk on $\mathbb{R}^d$ with Gaussian increments, whose law inherits the symmetries of $\mathbb{R}^d$ and to which the arguments leading to the lower bound in (2.25) can be extended, and then rely on the results of [22] for comparison with X. The relevant upper bound for (1.11) would naturally follow by adapting our coarse-graining and bootstrapping scheme to a framework with Euclidean balls replacing ℓ^∞-ones (although, since $L \ll N$ in practice, one may in fact get away by coarse-graining using ℓ^∞-boxes at scale L).

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