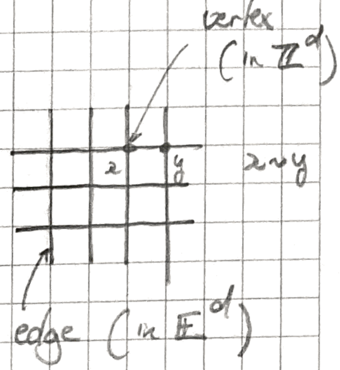
PERCOLATION & RANDOM WALKS

– Mini-course @ Probability-on-Sea, Mexico –

Jan 8 – 12, 2024

0. Introduction

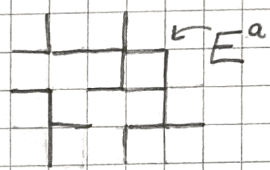


Percolation problem $\quad G = (\mathbb{Z}^d, E^d)$, $d \geq 3$ (here)

$a \in \mathbb{R}$: parameter

Under $\mathbb{P}$ prob. measure: $(E^a)_{a \in \mathbb{R}}$ s.t.

- $E^a \subset G$
- $E^b \subset E^a \quad \forall b > a$
- (• $\mathrm{Law}_{\mathbb{P}}(E^a)$ trans. invariant)



Order parameter:

$$\theta(a) \overset{\text{def.}}{=} \lim_{L \to \infty} \downarrow \underbrace{\mathbb{P}[0 \overset{E^a}{\longleftrightarrow} \partial B_L]}_{\overset{\text{def.}}{=} \theta_L(a)}, \quad a \in \mathbb{R} \quad \text{where}$$

$B_L = B(0, L)$

$B(x, L) = \{y : |y - x|_\infty \leq L\}$

$\partial K = \{x \in K : \exists y \sim x, y \notin K\}$

$\theta(\cdot)$ is decreasing so one can define

$$a_* = a_*(d) = \inf\{a \in \mathbb{R} : \theta(a) = 0\} \quad (\in [-\infty, \infty])$$

convention: $\inf \emptyset = \infty$

Example: Under $\mathbb{P}$: $(\varphi_x)_{x \in \mathbb{Z}^d}$ iid $N(0,1)$,

$E^a = \{x \in \mathbb{Z}^d : \varphi_x \geq a\}$: Bernoulli (site) percolation

supercritical | subcritical
$\theta(a) > 0 \quad a_* \quad \theta(a) = 0$

Typical questions:

- non-trivial, i.e. $a_* \neq \pm\infty$?
- decay of $\theta_L(a)$, $a \geq a_*$?
- classification (regularity of $\theta(\cdot)$) ?

...

Heuristics (for 2^{nd} order PT's)

1) Sharp transition: for $a > a_*$ $L \mapsto \theta_L(a)$ rapidly decaying
   for $a < a_*$ add $\{0 \overset{E^a}{\nleftrightarrow} \infty\}$

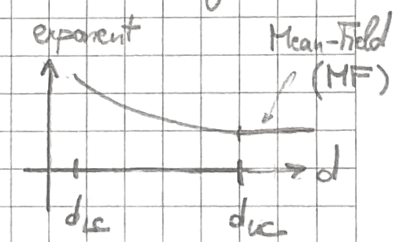
2) Continuity of $\theta(\cdot)$: $\theta_L(a_*) \xrightarrow[L\to\infty]{} 0$

3) Scaling: $\exists\, \nu, \rho \in (0,\infty)$ s.t. $\forall a \in \mathbb{R}$, $L \geq 1$:

$$c L^{-\frac{1}{\rho}} e^{-c' f(\frac{L}{\xi})} \leq \theta_L(a) \leq \tilde{c} L^{-\frac{1}{\rho}} e^{-\tilde{c}' f(\frac{L}{\xi})}$$

where $f(t) \overset{e.g.}{=} t$, $f(0)=0$, $f(\cdot)$ cont., $\lim_{t\to\infty} \frac{f(t)}{\log t} = \infty$,
$\xi = \xi(a) = |a-a_*|^{-\nu}$

ν, ρ: scaling exponents: depend on "key" parameters only,
e.g. dim. d, strength of correlations, ...
$\longrightarrow$ define "universality classes"



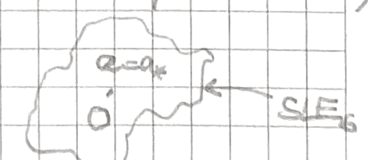
Example (Bernoulli i.i.d.)

1) $\mathbb{Z}^d$, $d \geq 2$: Men'shikov, Aizenman–Barsky, DC–Tassion: $a > a_*$
   Grimmett–Marstrand, Contreras–Martineau–Tassion, $a < a_*$.

2) $\mathbb{Z}^2$-bond or $\triangle$-site: Kesten

3) $\triangle$-site: Russo–Seymour–Welsh, Kesten, Smirnov, Lawler–Schramm–Werner: $\rho = \frac{48}{5}$, $\nu = \frac{4}{3}$ (via identification of scal. limit)

2)+3) $\mathbb{Z}^d$, $d \geq 11$: $\rho = \nu = \frac{1}{2}$ (only for $a > a_*$)



Aizenman–Newman, Hara–Slade, Kozma–Nachmias, Fitzner–van der Hofstad

Here: (some) answers for 1) – 3) when φ "strongly correlated".

I - Some Potential Theory

$G = (\mathbb{Z}^d, E^d)$, $d \geq 3$;

more generally: $G = (V, \lambda)$ transient weighted graph (connected)

E_x / P_x : canonical law of $X = (X_t)_{t \geq 0}$ continuous-time SRW on G, jump rate 1 ; $X_0 = x$ under P_x, $x \in \mathbb{Z}^d$

↑ exp. value

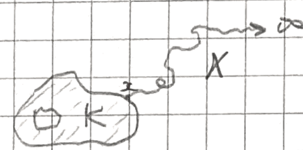
$Z = (Z_n)_{n=0,1,\dots}$ discrete skeleton of X.

For $K \subset \mathbb{Z}^d$

$$H_K = \inf\{t \geq 0 : X_t \in K\} \quad (= \inf\{n \geq 0 : Z_n \in K\})$$

$$\tilde{H}_K = \inf\{t \geq 0 : X_t \in K \text{ and } \exists s \in (0,t) : X_s \neq X_0\}$$

$$(= \inf\{n \geq 1 : Z_n \in K\}).$$



Def: $K \subset \mathbb{Z}^d$,

$e_K(x) \overset{\text{def.}}{=} P_x[\tilde{H}_K = \infty]\, 1\{x \in K\}$, $x \in \mathbb{Z}^d$: equilibrium measure (of K)

$\mathrm{cap}(K) \overset{\text{def.}}{=} \sum_x e_K(x)$: capacity of K

$h_K(x) \overset{\text{def.}}{=} P_x[H_K < \infty]$, $x \in \mathbb{Z}^d$: potential of K

$g_K(x,y) \overset{\text{def.}}{=} \int_0^\infty P_x[X_t = y, t < H_{\mathbb{Z}^d \setminus K}]\, dt$, $x,y \in \mathbb{Z}^d$: Green's function (killed outside K)

$g \overset{\text{not.}}{=} g_{\mathbb{Z}^d}$

Lemma 1 ($K \subset \mathbb{Z}^d$ finite)

(1) "Last exit" decomposition

$$h_K(x) = \sum_y g(x,y)\, e_K(y), \quad x \in \mathbb{Z}^d$$

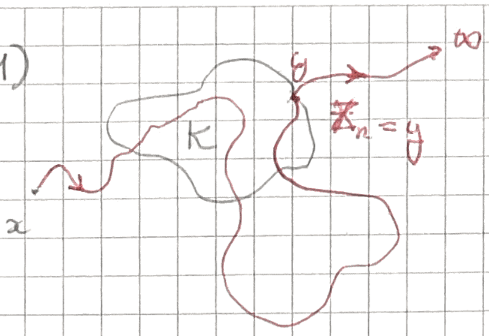
(2) $K \subset K'$, $K' \subset \mathbb{Z}^d$ finite "sweeping"

$$P_{e_{K'}}[H_K < \infty, X_{H_K} = x] = e_K(x), \quad x \in \mathbb{Z}^d$$

where $P_\mu \overset{\text{def.}}{=} \sum_x \mu(x) P_x$

Note: in particular (2) $\Rightarrow \int h_K \, de_{K'} = \mathrm{cap}(K)$

Pf (1)

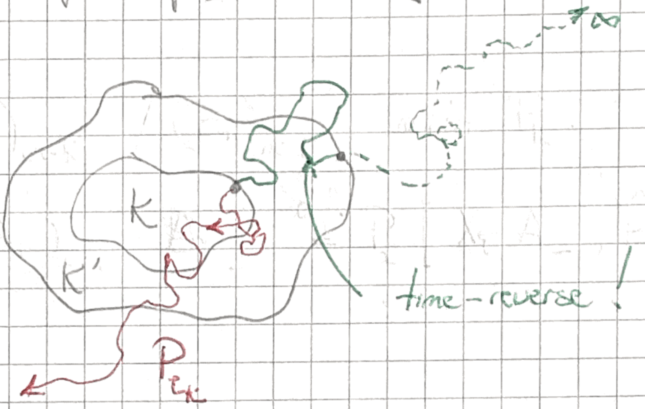


$$h_K(x) = \sum_{\substack{n \geq 0 \\ y \in K}} P_x\left[Z_n = y,\ Z_{k+n} \notin K,\ k = 1, 2, \dots \right]$$

Markov at time n

$$= \sum_{n \geq 0} P_x[Z_n = y]\, P_y[\tilde{H}_K = \infty].$$

(2) Exercise. Sketch:



Gaussian free field on G

Law $\mathbb{P}$: $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$ Gaussian field, $\mathbb{E}[\varphi_x] = 0$, $x \in \mathbb{Z}^d$, and

(3) $$\mathbb{E}[\varphi_x \varphi_y] = g(x,y), \quad x, y \in \mathbb{Z}^d.$$

Note: (3) is well-defined, i.e. $g(\cdot,\cdot)$ defines a positive-definite operator $Gf(x) = \sum_y g(x,y) f(y)$, for suitable $f: \mathbb{Z}^d \to \mathbb{R}$.

Indeed writing $P_t f(x) = \sum_y P_x[X_t = y] f(y)$ so that $P_t \circ P_s = P_{t+s}$ for $t, s \geq 0$, one has

$$\langle f, Gf \rangle_{\ell^2(\mathbb{Z}^d)} = \int_0^\infty \langle f, P_t f \rangle_{\ell^2(\mathbb{Z}^d)}\, dt = \int_0^\infty \| P_{\frac{t}{2}} f \|^2_{\ell^2(\mathbb{Z}^d)}\, dt \geq 0.$$

using that $\langle f, P_t f \rangle = \langle P_t f, f \rangle$.

Meaning of (3): for $K \subset \mathbb{Z}^d$ finite define

$$\mathbb{P}_K(d\varphi) = \frac{1}{Z_K} \exp\left\{ -\frac{1}{2} \mathcal{E}(\varphi, \varphi) \right\} \Big(\prod_{x \in K} d\varphi_x \Big) \Big(\prod_{y \in \mathbb{Z}^d \setminus K} \delta_0(\varphi_y) \Big)$$

($\prod_{x \in K} d\varphi_x$: Lebesgue; δ_0: Dirac-meas.)

where

$$\mathcal{E}(f,f) \overset{\text{def.}}{=} \sum_{|x-y|=1} \frac{1}{2d}\big(f(x)-f(y)\big)^2,$$

for $f: \mathbb{Z}^d \to \mathbb{R}$ s.t. $\mathcal{E}(f,f)<\infty$ (e.g. having finite support)

<u>Exercise</u>: show that $\mathbb{P}_K$ is centered Gaussian with

$$\mathbb{E}_K[\varphi_x \varphi_y] = g_K(x,y), \quad x,y\in\mathbb{Z}^d$$

Deduce that $\mathbb{P}_K \xrightarrow{w} \mathbb{P}$ for $K = B_N$ as $N\to\infty$.

<u>Hint</u>: show the discrete "Gauss-Green" identity, by which

$$\mathcal{E}(f,f) = \langle f, (-\Delta) f\rangle,$$

e.g. for $f: \mathbb{Z}^d \to \mathbb{R}$, $f(x)=0$, $x\notin K$, where

$$(\Delta f)(x) = \sum_{y\sim x} \frac{1}{2d}\big(f(y)-f(x)\big), \quad x\in K.$$

<u>Lemma 2</u> (Markov property of φ)

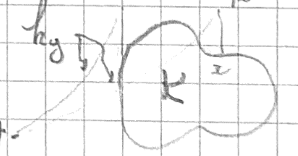
For $K\subset\mathbb{Z}^d$ finite, let

(4) $$h_x \overset{\text{def.}}{=} E_x\left[\varphi_{X_{H_K}} 1\{H_K<\infty\}\right] \left(= \sum_{y\in K} P_x[H_K<\infty, X_{H_K}=y]\,\varphi_y\right)$$

Then defining $\psi_\cdot$ by $\varphi_x = \psi_x + h_x$, $x\in\mathbb{Z}^d$, one has:

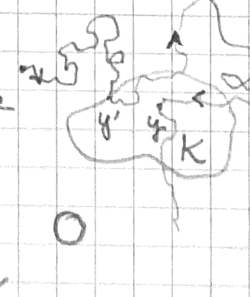
i) h, ψ are centered Gaussian fields

ii) $h \perp\!\!\!\perp \psi$ and $\psi \overset{\text{law}}{=} \mathbb{P}_{\mathbb{Z}^d\setminus K}$



<u>Pf</u> i) for each x, h_x / ψ_x is a finite linear combination of φ_y's. $\qquad \varphi_y = \psi_y + h_y$

ii) For all $y\in K$, $x\in\mathbb{Z}^d$



$$\mathbb{E}[\psi_x\varphi_y] = \mathbb{E}[\varphi_x\varphi_y] - \mathbb{E}[h_x\varphi_y]$$

$$\overset{(3)}{\underset{(4)}{=}} g(x,y) - \sum_{y'\in K} P_x[H_K<\infty, X_{H_K}=y']\, g(x,y') \underset{\text{Markov}}{\overset{\text{strong}}{=}} 0$$

so $\sigma(\psi_x, x\in\mathbb{Z}^d) \perp\!\!\!\perp \sigma(\varphi_x: x\in K) \ni h.$

To identify law (ψ), enough to compute covariance

$$E[\psi_x \psi_y] = E[\psi_x \varphi_y] - E[\psi_x h_y] = E[\psi_x \varphi_y]$$
$$= g(x,y) - \sum_{y' \in K} P_x[H_K < \infty, X_{H_K} = y']\, g(y',y) \overset{(*)}{=} g_{\mathbb{Z}^d \setminus K}(x,y)$$

(*) upon writing $g(x,y) = E_x\left[\int_0^{H_K} 1\{X_t = y\}\,dt\right] + E_x\left[\int_{H_K}^{\infty} 1\{X_t=y\}\,dt\, 1_{\{H_K<\infty\}}\right]$ and applying strong Markov property to second term at time H_K.

□

II – Level-set percolation for φ

Recall: φ GFF, law $\mathbb{P}$

Since $g(x,y) = g(x+z, y+z)$, $x,y,z \in \mathbb{Z}^d$, $\varphi_\cdot \overset{law}{=} \varphi_{\cdot+z}$, $z \in \mathbb{Z}^d$.

For $a \in \mathbb{R}$, let

(5) $$E^a \overset{def.}{=} \{x \in \mathbb{Z}^d : \varphi_x \geq a\}$$

with critical parameter $\left(\theta(a) = \mathbb{P}[0 \overset{E^a}{\longleftrightarrow} \infty]\right)$

$$h_* = h_*(d) = \inf\{a \in \mathbb{R} : \theta(a) = 0\} \quad (\in [-\infty, +\infty])$$

(convention $\inf \emptyset = +\infty$)

Theorem 3 (non-trivial phase transition)

For all $d \geq 3$: $0 \underset{(*)}{\leq} h_*(d) \underset{(**)}{<} \infty$

Remark: For Bernoulli percolation, $(**)$ is essentially trivial. Indeed with E^a as on p.1 and for a s.t. "$p(a)$" $= \mathbb{P}[N(0,1) > a] < (2d-1)^{-1}$

$$\mathbb{P}[0 \overset{E^a}{\longleftrightarrow} \partial B_L] < ((2d-1)p(a))^L \leq e^{-c(a)L} \to 0,$$

whereas $h_* > -\infty$ is more involved (Peierl's argument). For E^a as in (5) it's the opposite: $(*)$ has a soft argument (see below)!

Proposition 4 [RS, PR]

$$\forall a \geq C,\ L \geq 1:\ \theta_L(a) \leq Ce^{-L^c}.$$

Proof (Sketch) $L_n = L_0^n$, $L_0 = 10^3$.

$$q_n(a) \overset{\text{def.}}{=} \mathbb{P}\left[\partial B_{L_n} \overset{E^a}{\longleftrightarrow} \partial B_{2L_n}\right]$$

Enough to argue $q_n(a) \leq 2^{-2^n}$.

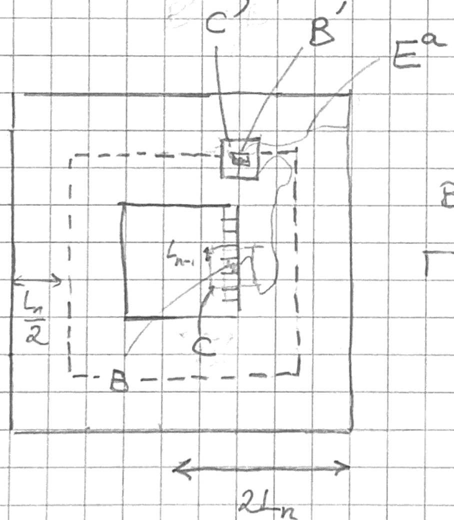
Lemma 5 (bootstrap)

$\forall a \in \mathbb{R}$, $n \geq 1$; $\varepsilon > 0$.

$$(6) \qquad q_n(a) \leq \underbrace{(CL_0)^{2d}}_{\Gamma}\left(q_{n-1}(a-\varepsilon)^2 + \eta_n\right), \quad |\eta_n| \leq C L_n^d e^{-c\varepsilon^2 L_n^{d-2}}$$

If lemma holds then pick $a \geq 1$ large enough s.t. $q_0(a-1) \leq (2\Gamma)^{-C}$ and show using (6) that $q_n\left(a - \sum_{k \geq n} 2^{-(k+1)}\right) \leq 2^{-2^n}$.

Proof of Lemma 5:



$L_n^d e^{-c L_n^{d-2}}$

B, B': translates of $B_{L_{n-1}}$

Γ: union bd over choice of B, B'

$$q_n(a) \leq \Gamma\, \mathbb{P}\left[A_B(\varphi) \cap A_{B'}(\varphi)\right]$$

$$A_B(\varphi) \overset{\text{def.}}{=} \left\{\partial B \overset{\{\varphi \geq a\}}{\longleftrightarrow} \partial C\right\}$$

(C: concentric to B, radius $2L_{n-1}$)

By Markov property, with $K = C'$

$$\varphi = \psi + h, \quad h|_{C'} = \varphi, \quad \psi \perp\!\!\!\perp \varphi$$

$$\mathbb{P}\left[A_B(\varphi) \cap A_{B'}(\varphi)\right] = \mathbb{P}\left[A_B(\psi + h) \cap A_{B'}(h)\right]$$

$$\leq \mathbb{P}\left[A_B(\varphi + \varepsilon)\right] \mathbb{P}\left[A_{B'}(\varphi)\right] + 2\,\mathbb{P}\left[\min_{x \in C} h_x < -\frac{\varepsilon}{2}\right]$$

Apply union bound; h_x is Gaussian, mean zero & for $x \in C$

$$\text{var}(h_x) = \sum_{y, y' \in K} P_x\left[H_K < \infty, X_{H_K} = y\right] g(y, y') P_x\left[H_K < \infty, X_{H_K} = y'\right]$$

$$\overset{\text{str. Markov}}{=} \sum_{y \in K} P_x\left[H_K < \infty, X_{H_K} = y\right] g(x, y) \underset{x \in C,\ d(x,K) \geq L_n/10}{\leq} C L_n^{2-d} \qquad \square$$

Proposition 6 [BLM]

$\forall a>0 \ \exists\, c(a) \in (0,\infty)$ s.t. $\forall L \geq 1$

$$\mathbb{P}\left[\, 0 \overset{E^{-a}}{\not\leftrightarrow} \partial B_L \right] \leq \frac{1}{1+c(a)}$$

Note: Prop. 4 + 6 $\Longrightarrow$ Thm 3.

Proof C = "explored" cluster of ∂B_L in E^{-a}.

$\varphi_{|\partial C} < -a$ pushes down the field at 0 !

Apply Lemma 2 with $K = C^{*}$: at $x=0$

$\varphi_0 = \psi_0 + h_0$ (ψ_0 and h_0 independent, ⫫)

$\psi_0 \sim N(0, \underbrace{g_{\mathbb{Z}^d \setminus K}(0,0)}_{\geq 1})$ non-deg. !

$h_0 = E_0[\varphi_{X_{H_K}}] \leq -a$: on D (using $H_K < \infty$ P_0-a.s. on D, and $\varphi_{|K} \leq -a$)

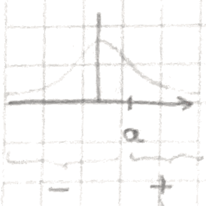
where $D = D(\varphi) \overset{\text{def.}}{=} \left\{ 0 \overset{\{\varphi \geq -a\}}{\not\leftrightarrow} \partial B_L \right\}$

$D(\varphi_\cdot) = D(h_\cdot)$!

$$\Longrightarrow\ 0 = E[\operatorname{sign}(\varphi_0)] = E[\operatorname{sign}(\psi_0 + \underbrace{h_0}_{\leq -a \,:\, \text{on } D}) 1_{D(h)}] + E[\underbrace{\operatorname{sign}(\varphi_0) 1_{D^c}}_{\leq 1}]$$

$$\leq E[\operatorname{sign}(\psi_0 - a) 1_{D(h)}] + \mathbb{P}[D^c]$$

$$\overset{\psi ⫫ h}{=} -\mathbb{P}[|\psi_0| < a]\,\mathbb{P}[D] + 1 - \mathbb{P}[D]$$



Claim follows as $\mathbb{P}[|\psi_0| < a] \geq c(a) > 0$. $\square$

* C is random. To make this precise one disintegrate over all realizations of C and applies Lemma 2. for fixed realization (a "strong Markov" property)

Remarks

1) (Sharp transition) Prop 4 shows $\theta_L(a) \leq Ce^{-L^c}$ for large a. This rapid decay in L holds troughout the subcritical regime $a > a_*$, and for all $a \neq a_*$ when replacing $\theta_L(a)$ by

$$\theta_L^{tr}(a) = \mathbb{P}[0 \overset{E^a}{\longleftrightarrow} \partial B_L,\ 0 \overset{E^a}{\not\longleftrightarrow} \infty];$$

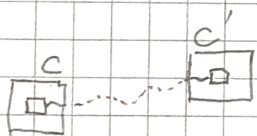
see [DCGRS]. The speed of decay of $\theta_L^{tr}(a)$ was determined in [GRS]. Together these results yield:

Theorem 7 $\forall a \neq a_*$

$d \geq 4$ $\quad e^{-C(a)L} \leq \theta_L^{tr}(a) \leq e^{-c'(a)L}$

$d = 3$ $\quad \lim\limits_{L \to \infty} \frac{3}{\pi} \frac{\log L}{L} \theta_L^{tr}(a) = -\frac{1}{2}(a - a_*)^2.$

For $d=3$, the decay is sub-exponential. In the proof of Prop 4 the stretched exp. decay comes from a too-wasteful coarse-graining in Lemma 5: one forgets a big "chunk" of the connection event:



The large distance btw. C and C' makes η_n "negligible" in (6)

By a more careful coarse-graining, one gets the right decay and η_n contributes to leading exp. order in $d=3$.

2) In terms of 3) on p. 2 the limit suggests for $d=3$ $\xi = |a - a_*|^{-\nu}$ with $\nu = 2 = \frac{2}{d-2}$: Weinrib - Halperin scaling
(exponent of corr. decay → $d-2$)

3) Proof of Prop. 6 is slick but is there a constructive proof of the existence of an ∞ cluster in E^{-a} (→ see below).
In fact one even knows that (see [DPR1])

$$a_*(d) > 0 \quad \forall d \geq 3.$$

4) The information about percolation of E^{-a}, which is very robust (for instance Proposition 6 holds with identical proof on any transient weighted graph) can be used to prove $p_c<1$ for Bernoulli percolation, thus avoiding eg the Peierls' argument; see Duminil-C.-Goswami-Rodriguez-Severo-Yadin.

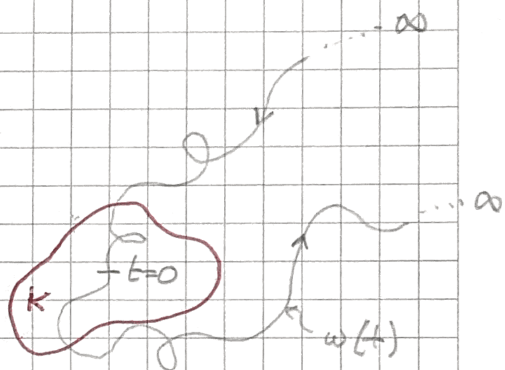
III - Some links with interlacements

Random interlacements "=" stationarized (local) RW trace

Law $\mathbb{P}$ of Poisson process on $\mathbb{R}_+ \times W^*$ ← {bi-infinite transient $\mathbb{Z}^d$-trajectories modulo time-shift}

intensity $du \times \nu$

(du ↑ Lebesgue measure)



a representative of $w^*(t) \in W_K^*$

What is ν?

$W_K^* \subset W^*$: those traj with $H_K < \infty$.

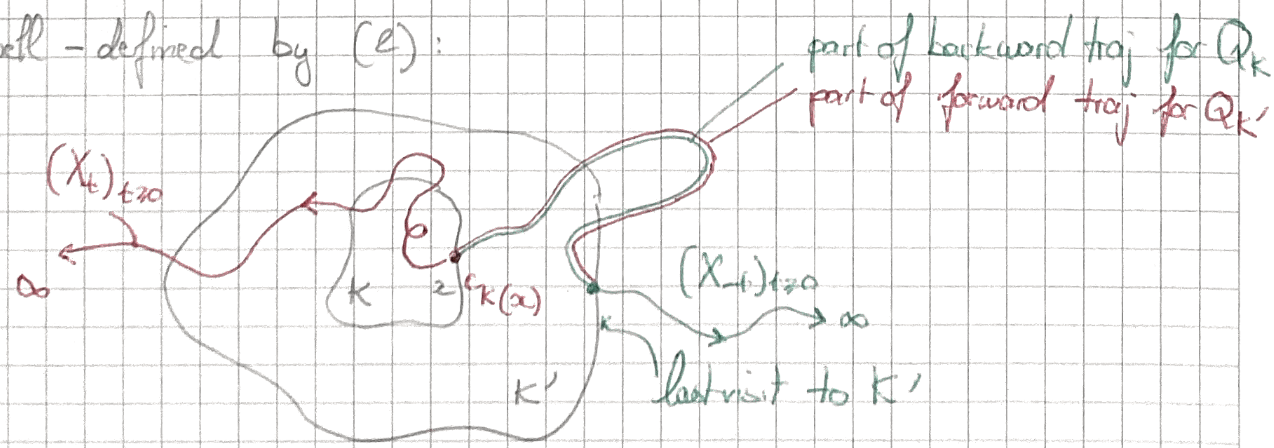
$\pi^* : W \to W^*$ canon. projection (W ← no time shift)

Def: $\nu|_{W_K^*} = \pi^* \circ Q_K$ [Sz0]

where $Q_K\big((X_{-t})_{t\geq 0} \in A_-, (X_t)_{t\geq 0} \in A_+\big)$

$$= \sum_x P_x\big[X \in A_- \,\big|\, \tilde{H}_K = \infty\big]\, e_K(x)\, P_x\big[X \in A_+\big].$$

ν is well-defined by (e):



$$\omega = \sum_i \delta_{(u_i, w_i^*)} \qquad u_i \in \mathbb{R}_+, \; w_i^* \in W^*$$

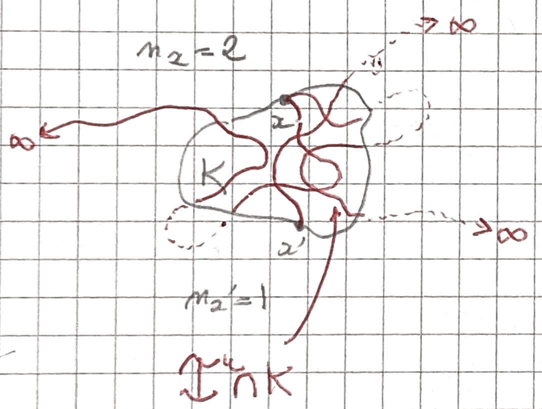
typical realisation of interlacement Poisson random (point) measure, characterized by

(7) $$E\left[\exp\left\{-\int_{\mathbb{R}_+ \times W^*} f \, d\omega\right\}\right] = \exp\left\{\int_{\mathbb{R}_+ \times W^*} (e^{-f}-1) \, d\nu \, du\right\}$$

for suitable $f : \mathbb{R}_+ \times W^* \to \mathbb{R}$ $\qquad E[e^{fX}] = e^{\lambda(e^f-1)}$, $X \sim Po(\lambda)$

Interlacement set : $\mathcal{I}^u = \mathcal{I}^u(\omega) = \bigcup_{i : u_i \leq u} \text{range}(w_i^*)$.

Concretely to sample $\mathcal{I}^u \cap K$, $K \subset \mathbb{Z}^d$ finite



① sample $n_x \stackrel{\text{law}}{=} \text{Poisson}(u e_K(x))$ indep. as x varies

② start n_x indep RWs with law P_x for each x

Occupation times

$$\ell_x^u = \ell_x^u(\omega) = \sum_{i ; u_i \leq u} \int_{-\infty}^{\infty} 1\{w_i^*(t) = x\} \, dt$$

(8) $$\mathcal{I}^u = \{x \in \mathbb{Z}^d : \ell_x^u > 0\}.$$

The set $\mathcal{I}^u$ has remarkable properties:

- its law under P is translation invariant
- it is connected P-a.s. (and unbounded!)
- its density $P[0 \in \mathcal{I}^u]$ can be made arbitrarily small by letting $u \downarrow 0$

Remark : another good example satisfying all three items is presumably the infinite cluster of supercritical Bernoulli percolation, but the last point corresponds to $\theta(p) \downarrow 0$, $p \downarrow p_c$!

Link to φ:

<u>Theorem 8</u> (Ray-Knight theorem for interlacements) [Sgn]

$$\forall u>0: \quad \frac{1}{2}(\tilde{\varphi}+\sqrt{2u})^2 \overset{law}{=} \frac{1}{2}\varphi^2 + \ell^u$$

(under $\mathbb{P}$) (under $\mathbb{P}\otimes P$)

In fact (Lupu, see below) one can couple $(\varphi, \ell^u, \tilde{\varphi})$ s.t. $\varphi \perp\!\!\!\perp \ell^u$, the above equality holds a.s. and

(9) $\quad \operatorname{sign}(\tilde{\varphi}+\sqrt{2u}) = \text{cst.}$ on $\mathcal{I}^u \overset{(8)}{=} \{\ell^u > 0\}$

<u>Remark</u> (9) explains Prop. 6 (i.e. $a_* \geq 0$), see also Remark 3) p 9.
Indeed it implies with $a = \sqrt{2u}$ using the above coupling that

$$\begin{cases} E^{-a} \overset{law}{=} \{\tilde{\varphi}+\sqrt{2u} \geq 0\} \supset \mathcal{I}^u \\ \text{or} \\ E^{-a} \geq_{st} \{\tilde{\varphi}+\sqrt{2u} < 0\} \supset \mathcal{I}^u \end{cases} \leftarrow \text{percolates!}$$

Proof of '$\overset{law}{=}$':

$V: \mathbb{Z}^d \to \mathbb{R}$ finite support & s.t. $\mathbb{E}[e^{\frac{1}{2}\sum_x V(x)\varphi_x^2}] = Z^V < \infty$.

Define $\mathbb{P}^V$: $\frac{d\mathbb{P}^V}{d\mathbb{P}} = (Z^V)^{-1} e^{\frac{1}{2}\sum_x V(x)\varphi_x^2}$

Need to show $\mathbb{E}^V[e^{\sqrt{2u}\langle V,\varphi\rangle}] = E[e^{\langle V,\ell^u\rangle}]e^{-\langle V,u\rangle}$

The LHS is Gaussian and

$$\log \mathbb{E}^V[e^{\sqrt{2u}\langle V,\varphi\rangle}] = \frac{1}{2}(\sqrt{2u})^2 \sum_{x,y} V(x)\mathbb{E}^V[\varphi_x\varphi_y]V(y)$$

$$= u\langle V, (-\Delta-V)^{-1}V\rangle$$

For RHS: $\langle V,\ell^u\rangle \overset{(8)}{=} \sum_{i:u_i\leq u} \int_{-\infty}^{\infty} V(w_i^*(t))dt$ so applying (7) with

$f(v,w^*) = 1_{\{v\leq u\}} \int_{-\infty}^{\infty} V(w^*(t))dt$ gives

$$\rightsquigarrow \log E[e^{\langle V,\ell^u\rangle}] \underset{(K=\operatorname{supp}(V))}{\overset{\text{def of }\nu}{=}} u\, E_{e_K}[e^{\int_0^\infty V(X_s)ds} - 1]$$

$$= u\, E_{e_K}\left[\int_0^\infty dt\, e^{\int_0^t V(X_s)ds} V(X_t)\right]$$

$$E_{e_K}\Big[\int_0^\infty dt\, e^{\int_0^t V(X_s)ds}\, V(X_t)\Big] - \langle 1, V\rangle$$

$$= \Big\langle e_K, \int_0^\infty dt\, e^{t(\Delta+V)}\, V\Big\rangle - \langle 1, V\rangle$$

$$= \langle e_K, -(\Delta+V)^{-1} V\rangle - \langle 1, V\rangle$$

write as $(-\Delta)h_K = e_K$

$$\overset{(1)}{=} \langle -(\Delta+V)h_K + Vh_K, -(\Delta+V)^{-1}V\rangle - \langle 1, V\rangle$$

$h_K = 1$ on $K = \mathrm{supp}(V)$

$$= \langle 1, V\rangle + \langle V, -(\Delta+V)^{-1}V\rangle - \langle 1, V\rangle \qquad \square$$

IV A continuous phase transition

We now consider a modified model $\tilde{E}^a$, for which we prove continuity of the phase transition. Extending $\mathbb{P}$ suitably we assume that, conditionally on φ

(12) $$\eta_e = \begin{cases} 1 & \text{w. prob } 1 - e^{-2(\varphi_x - a)_+(\varphi_y - a)_+} \\ 0, & \text{else} \end{cases} \quad \text{for } e = \{x,y\} \in E^d \text{ indep. as } e \text{ varies}$$

and define

$$\tilde{E}^a = \{x \in \mathbb{Z}^d : \eta_e = 1 \text{ for some } e \ni x\} \quad (\subset E^a)$$

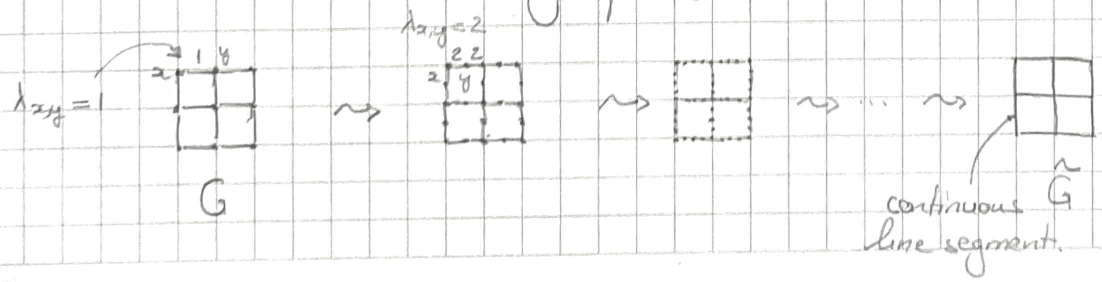
(↑ if $\varphi_x < a$, $\eta_e = 0$ for all $e \ni x$)

$$\tilde{a}_* = \inf\{a \in \mathbb{R} : \underbrace{\mathbb{P}[0 \overset{\tilde{E}^a}{\longleftrightarrow} \infty]}_{\tilde{\theta}(a)} = 0\}$$

Theorem 9 [Lu, DW, DPR2]

For all $d \geq 3$, $\tilde{a}_* = \tilde{a}_*(d) = 0$ and
$$\tilde{\theta}(\tilde{a}_*) = 0.$$

To understand the meaning of (12) introduce the metric graph $\tilde{G}$:



On $\tilde{G}$ one defines a Brownian motion $\tilde{X}$, with Green's function $\tilde{g}(x,y)$ s.t. $\tilde{g}|_{G\times G} = g$. The GFF $\tilde{\varphi}$ on $\tilde{G}$ is the <u>continuous</u> mean zero Gaussian field with $\mathbb{E}[\tilde{\varphi}_x \tilde{\varphi}_y] = \tilde{g}(x,y)$ $x,y \in \tilde{G}$

<u>Note:</u> $\tilde{\varphi}|_G = \varphi$ and one can construct $\tilde{\varphi}$ as follows:
First one samples φ and cond. on φ independent Brownian bridges between φ_x and φ_y on each edge $e=\{x,y\}$.
(The) prob. that $\psi_e = 1$ in (12) corresponds to the probability that the bridge (with final values φ_x, φ_y) stays above the barrier a. Hence

$$\tilde{\theta}(a) = \mathbb{P}\left[0 \overset{\{\tilde{\varphi}\geq a\}}{\longleftrightarrow} \infty\right]$$

All previous results extend to $\tilde{G}$. In particular $\mathcal{I}^u \rightsquigarrow \tilde{\mathcal{I}}^u \subset \tilde{G}$ (replace all RWs X by $\tilde{X}$) and there is a coupling with $\tilde{\varphi}$ s.t.

$$\{x \in \tilde{G} : \tilde{\varphi}_x \geq -a\} \supseteq \tilde{\mathcal{I}}^{\frac{a^2}{2}}$$

$\leftarrow$ percolates

This implies $\tilde{\theta}(-a) > 0$, $a > 0$. To prove Thm 9 it thus remains to argue $\tilde{\theta}(0) = 0$. First isolate

<u>Lemma 10</u> (Differential formula)

$C_L^a \overset{\text{def.}}{=}$ cluster of 0 in $\{\tilde{\varphi} \geq a\} \cap B_L$ $\quad (C_L^a \subset \tilde{G})$.

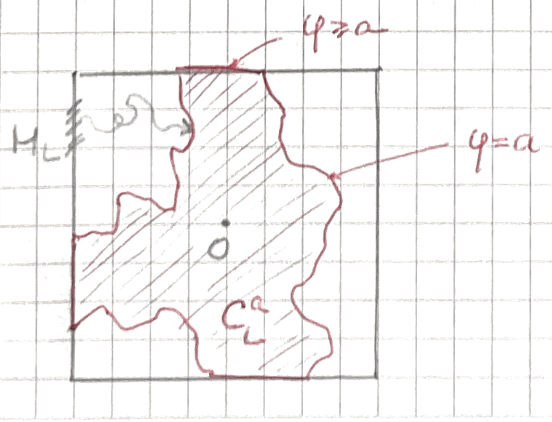
(13) $$-\frac{d}{da}\mathbb{E}[f(C_L^a)] = \mathbb{E}[M_L f(C_L^a)], \quad a \in \mathbb{R}$$

for suitable f,

where

$M_L = \langle e_{B_L}, \varphi\rangle$

$\left(= \sum_x e_{B_L}(x)\varphi_x\right)$



<u>Proof</u> Application of Cameron – Martin

Let $\mathbb{P}_a$: law of $\varphi - a h_{B_L}$ $\left(= \varphi - a \text{ in } B_L \right)$
↑ see p. 3

Then:

$$\frac{d\mathbb{P}_a}{d\mathbb{P}} = \exp\left\{ -aM_L - \frac{a^2}{2} \mathbb{E}[M_L]^2 \right\}$$

Write $\mathbb{E}\left[f(C_L^a) \right] = \mathbb{E}_a\left[f(C_L^0) \right] = \mathbb{E}\left[\frac{d\mathbb{P}_a}{d\mathbb{P}} f(C_L^0) \right]$

and compute $\frac{d}{da} \frac{d\mathbb{P}_a}{d\mathbb{P}}$.

□

Explore C_L^a "from inside". Applying the strong Markov property for $\tilde{\varphi}$ (i.e. formally Lemma 2 with $\tilde{\varphi}$ instead of φ and "$K = C_L^a$"), one gets

$$\mathbb{E}\left[M_L \,\middle|\, \text{"}\varphi_{C_L^a}\text{"} \right] \overset{\text{"}K = C_L^a\text{"}}{=} \sum_x e_{B_L}(x) \Big(\underbrace{\mathbb{E}^{\psi}\left[\psi_x \right]}_{=0} + h_x \Big)$$

$$= \sum_x e_{B_L}(x) \, E_x\Big[\underbrace{\tilde{\varphi}_{X_{H_{C_L^a}}}}_{\geq a} 1\{ H_{C_L^a} < \infty \} \Big]$$

(14) $$\geq a \, P_{e_{B_L}}\left[H_{C_L^a} < \infty \right] \overset{(2)}{=} a \, \mathrm{cap}\left(C_L^a \right) \;!$$

Substitute into (13) and choose $f(C_L^a) = 1\{ \mathrm{cap}(C_L^a) \in (s,t] \}$ and $\int_{a_1}^{a_2}$ to find

(*) $$\mathbb{P}\left[s < \mathrm{cap}(C_L^{a_2}) \leq t \right] \leq \mathbb{P}\left[s < \mathrm{cap}(C_L^{a_1}) \leq t \right] e^{-\frac{a_2^2 - a_1^2}{2} s}$$

First $a_1 = 0$, $a_2 = a$, $t \to \infty$, $L \to \infty$ to get, with C^a = cluster of 0 in $\{\tilde{\varphi} \geq a\}$

$$\mathbb{P}\left[\mathrm{cap}(C^a) > s \right] \leq 1 \cdot e^{-\frac{a^2}{2} s}.$$

Letting $s \to \infty$ gives $\mathbb{P}\left[\mathrm{cap}(C^a) < \infty \right] = 1$ for $a > 0$. Then take $a_1 = 0$, $s = \frac{1}{g(0,0)} \overset{(1)}{=} \mathrm{cap}(\{0\})$, $a_2 = a$ and $L \to \infty$, $t \to \infty$, $a \downarrow 0$:

$\mathrm{LHS}(*) \to \frac{1}{2} = \mathbb{P}\left[\varphi_0 > 0 \right]$, $\mathrm{RHS}(*) \to \mathbb{P}\left[\mathrm{cap}(C^0) < \infty \right] - \mathbb{P}\left[\varphi_0 < 0 \right]$

so overall $\mathbb{P}[\,\mathrm{cap}(\mathcal{C}^0) < \infty\,] = 1$, implying $\tilde{\Theta}(0) = 0$.

□

Remarks

1) One can re-run the proof after establishing Theorem 9 to improve (14) to an equality (in the figure on p.14 $\mathcal{C}_L^a = \mathcal{C}^a$ for L sufficiently large, and $\varphi = a$ on $\partial \mathcal{C}^a$. This yields differential equalities for functionals of $\mathrm{cap}(\mathcal{C}^a)$, which is integrable (see [DPR2] for its law). For instance one finds that for all $d \geq 3$,

(15) $$\mathbb{P}[\,\mathrm{cap}(\mathcal{C}^0) > t\,] \sim t^{-\frac{1}{2}} \quad \text{as } t \to \infty.$$

2) The fact that $\tilde{\Theta}(\tilde{a}_*) = 0$ was first shown on $\mathbb{Z}^d$ in [Lu] by combining an exact formula yielding $\mathbb{P}[0 \overset{\tilde{E}^a}{\longleftrightarrow} x] \to 0$ as $|x| \to \infty$ and uniqueness of the infinite cluster. The argument presented here works on all transient weighted graphs and yields that $\mathbb{P}[\,\mathrm{cap}(\mathcal{C}^0) < \infty\,] = 1$ ($\forall$ transient weighted G).

3) As opposed to the exploration "from "outside"" underlying the proof of Prop. 6 (cf. figure p.8), the proof of Theorem 9 exploits an exploration from "inside" (i.e. 0); see also [DC-T] for applications in the context of Bernoulli percolation leveraging such explorations (the (strong) Markov property in the present context allows to reach stronger conclusions and prove continuity).

One can in fact also devise a proof of Theorem 9 using an exploration "from outside". This proof can be thought of as refining the [BLM] argument underlying Prop 6. when passing from φ to $\tilde{\varphi}$. The key is:

Theorem 8' (signed Ray-Knight thm for interlacements) [Sz]

Under $\mathbb{P}\otimes P$ ($\tilde\varphi$ under $\mathbb{P}$, $\tilde{\mathcal{I}}^u$ under P), let

$\tilde{\mathcal{E}}^u \overset{def}{=}$ the union of the connected components of $\{x\in\tilde{\mathcal{G}} : |\tilde\varphi_x|>0\}$ intersecting $\tilde{\mathcal{I}}^u$.

Then

(16) $$\left(\tilde\varphi_x \mathbb{1}_{\{x\notin\tilde{\mathcal{E}}^u\}} + \sqrt{\tilde\varphi_x^2+\tilde\ell_x^u}\;\mathbb{1}_{\{x\in\tilde{\mathcal{E}}^u\}}\right)_{x\in\tilde{\mathcal{G}}} \overset{law}{=} \left(\tilde\varphi_x+\sqrt{2u}\right)_{x\in\tilde{\mathcal{G}}}$$

(under $\mathbb{P}\otimes P$) (under $\mathbb{P}$)

Theorem 8' implies Theorem 8. In a nutshell (16) is proved by "exploring $\tilde{\mathcal{E}}^u$" and applying the strong Markov property (care is needed though in particular because $\tilde{\mathcal{E}}^u$ is not compact).

Exercise: use (16) to prove Theorem 9.

Hint: recall that $P[\tilde{\mathcal{I}}^u\cap K=\emptyset] = e^{-u\,\mathrm{cap}(K)}$.

¡ Muchas Gracias !

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