Homework 3

Note: unless stated otherwise, all Brownian motions below are implicitly assumed to start at the origin.

1. Let X be a real valued random variable with standard normal distribution as law and Y a random variable independent of X with law defined by

$$P[Y = 1] = p \quad \text{and} \quad P[Y = -1] = 1 - p, \quad (0 \leq p \leq 1).$$

We define $Z := XY$. What is the law of Z? Is the vector (X, Z) a Gaussian vector?

2. Let W be a Brownian motion on $[0, 1]$ and define the *Brownian bridge* $X = (X_t)_{0 \leq t \leq 1}$ by $X_t = W_t - tW_1$.

a) Show that X is a Gaussian process and calculate its mean and covariance functions. Sketch a typical path of X.

b) Show that X does **not** have independent increments.

3. Let $(B_t)_{t \geq 0}$ be a Brownian motion and denote by $\mathcal{G}_t := \sigma(B_u, u \leq t)$, $t \geq 0$.
Define $\widetilde{R}_0 f(x) = f(x)$ and

$$\widetilde{R}_t f(x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty f(y) \left[\exp\left(-\frac{1}{2t}(y - x)^2 \right) + \exp\left(-\frac{1}{2t}(y + x)^2 \right) \right] dy, \quad t > 0$$

Define the process $(X_t)_{t \geq 0}$ by $X_t := |B_t|$.
Show that

$$E\big[f(X_{t+h}) \,\big|\, \mathcal{G}_t\big] = \widetilde{R}_h f(X_t) \;\; P\text{-a.s. for } \; f \in b\mathcal{B}(\mathbb{R}) \text{ and } t, h \geq 0.$$

4. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$.

a) Show that if the sequence $(X_n)_{n \in \mathbb{N}}$ converges in distribution to a random variable X, then the limits $\mu := \lim_{n \to \infty} \mu_n$ and $\sigma^2 := \lim_{n \to \infty} \sigma_n^2$ exist and $X \sim \mathcal{N}(\mu, \sigma^2)$.

b) Show that if $(X_n)_{n\in\mathbb{N}}$ is a Gaussian process indexed by $\mathbb{N}$ and converges in probability to a random variable X as n goes to infinity, then it converges also in L^2 to X.

5. Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, we define for any $(\mathcal{F}_t)$-stopping time τ the σ-field

$$\mathcal{F}_\tau := \big\{A \in \mathcal{F} \,\big|\, A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\big\}.$$

Let S, T be two $(\mathcal{F}_t)$-stopping times. Show that

a) if $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$ and in general, $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

b) $\{S < T\}$, $\{S \leq T\}$ belong to $\mathcal{F}_S \cap \mathcal{F}_T$.
Moreover, for any $A \in \mathcal{F}_S$, $A \cap \{S < T\}$ and $A \cap \{S \leq T\}$ belong to $\mathcal{F}_{S\wedge T}$.

6. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B_t)_{t\geq 0}$ be a Brownian motion.

a) Show that for P-almost all ω, the path $B_\cdot(\omega)$ changes its sign infinitely many times on any interval $[0, t]$, $t \geq 0$.

b) For any $\omega \in \Omega$ we define the set

$$Z(\omega) := \big\{t \in [0, \infty) \,\big|\, B_t(\omega) = 0\big\}.$$

Show that for P-almost all ω, the set $Z(\omega)$ is closed, has Lebesgue measure 0 and has 0 as an accumulation point. *Hint*: for the last part, consider $E[\text{Leb}(Z)]$.

Due: Monday May 22nd at the beginning of class.