Homework 1

Notation: if X is a Markov chain on a states space S with transition probabilities $P = (p_{x,y})_{x,y\in S}$, we denote by $p_{x,y}(n)$, $n \geq 0$, the n-step transition probabilities, i.e.

$$p_{x,y}(0) = \delta_{x,y}, \quad p_{x,y}(1) = p_{x,y}, \quad p_{x,y}(n+1) = \sum_{z\in S} p_{x,z}(n)p_{z,y},$$

and note that $p_{x,y}(n) = P_x[X_n = y]$.

1. A fair six-sided die is rolled repeatedly. Let Y_n denote the outcome of the n-th roll. We assume that Y_n, $n \geq 1$ are independent. Which of the following stochastic processes $(X_n)_{n\in\mathbb{N}}$ are Markov chains? For those that are, determine the state space S, the transition matrix P and in a) additionally the n-step transition probabilitities.

a) Let X_n denote the largest number shown up in n rolls.

b) Let X_n denote the number of sixes in n rolls.

c) Let X_n denote the number of rolls at time n since the most recent six.

2. Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with countable state space E and transition probability $(p_{x,y})_{x,y\in E}$. Let $C \subseteq E$ be such that $E\backslash C$ is finite. Define $p_{x,C}(n) = \sum_{y\in C} p_{x,y}(n)$ (see notation above). Suppose that for each $x \in E\backslash C$ there exists an $n(x)$ such that $p_{x,C}(n(x)) > 0$. Let $H_C = \inf\{n \geq 0 \mid X_n \in C\}$, $\varepsilon = \inf\{p_{x,C}(n(x)) \,:\, x \in E\backslash C\}$, and $N = \sup\{n(x) \mid x \in E\backslash C\}$. Show that for all $k \geq 1$ and $y \in E$,

$$P_y(H_C > kN) \leq (1-\varepsilon)^k.$$

Hint: use the Markov property and induction over k.

3. We use the same notation as in Exercise 2. Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $n \geq 0$, be the canonical filtration, and $A, B \subseteq E$ with $A \cap B = \emptyset$. Suppose that $E\backslash(A\cup B)$ is finite and $P_x(H_{A\cup B} < \infty) > 0$ for all $x \in E\backslash(A\cup B)$.

a) Show that the function h defined as $h(x) = P_x(H_A < H_B)$, $x \in E$, is P-harmonic outside $A \cup B$, i.e. it satisfies

$$h(x) = \sum_{y \in E} p_{x,y} h(y) \quad \text{for all } x \in E \backslash (A \cup B) \qquad\qquad (\star).$$

Hint: condition on $\mathcal{F}_1$.

b) Use exercise 2 to show that $P_x(H_{A\cup B} < \infty) = 1$.

c) Show that if a function h on E satisfies $(\star)$, then

$$E_\mu[h(X_{n \wedge H_{A\cup B}}) \mid \mathcal{F}_{n-1}] = h(X_{(n-1)\wedge H_{A\cup B}}),$$

hence $(h(X_{n\wedge H_{A\cup B}}))_{n\geq 0}$ is a martingale.
Optional: Use this to show that $h(x) = P_x(H_A < H_B)$ is the only solution of $(\star)$ that is 1 on A and 0 on B.

Remark: the function h from part a) is called P-harmonic because one can introduce the (discrete) Laplacian

$$\Delta_P = P - \text{Id},$$

where Id denotes the identity operator and $Pf(x) = \sum_{y\in E} p_{x,y} f(y)$, for suitable $f : E \to \mathbb{R}$ (say, with compact support), whence $(\star)$ asserts that $(\Delta_P h)(x) = 0$, for $x \in E\backslash(A \cup B)$. Along with c), it thus follows that h is the unique solution to the Dirichlet problem

$$\Delta_P h = 0 : \text{ on } E\backslash(A\cup B), \quad \text{with boundary condition } h(x) = \begin{cases} 1, & x \in A \\ 0, & x \in B. \end{cases}$$

Due: Friday April 14th at the beginning of class.