Equality of critical parameters for percolation of Gaussian free field level-sets

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Abstract

We consider level-sets of the Gaussian free field on $\mathbb{Z}^d$, for $d \geq 3$, above a given real-valued height parameter h. As h varies, this defines a canonical percolation model with strong, algebraically decaying correlations. We prove that three natural critical parameters associated to this model, namely $h_{**}(d)$, $h_*(d)$ and $\bar{h}(d)$, respectively describing a well-ordered subcritical phase, the emergence of an infinite cluster, and the onset of a local uniqueness regime in the supercritical phase, actually coincide, i.e. $h_{**}(d) = h_*(d) = \bar{h}(d)$ for any $d \geq 3$. At the core of our proof lies a new interpolation scheme aimed at integrating out the long-range dependence of the Gaussian free field. The successful implementation of this strategy relies extensively on certain novel renormalization techniques, in particular to control so-called large-field effects. This approach opens the way to a complete understanding of the off-critical phases of strongly correlated percolation models.

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1 Introduction

1.1 Motivation

Percolation has been at the heart of statistical physics for more than sixty years. Its most studied representative is the so-called Bernoulli (independent) percolation model. While the understanding of its critical phase is still incomplete, its behaviour away from criticality (in the sub- and supercritical phases) has been characterized very precisely, see [4, 40, 33]. Motivated by field theory and random geometry considerations, a whole new class of percolation models, emerging from disordered systems with long-range interactions, has been the object of intense study over the last two decades. A common feature of these models is the strength of the correlations between local observables, which exhibit power law decay like $|x-y|^{-a}$ as $|x-y| \to \infty$ for a certain (small) exponent $a > 0$. This slow decay — often a distinguishing feature of critical phases — is present throughout the entire parameter range, thus making the study of such models very challenging.

A few cases in point are the following: i) random interlacements on $\mathbb{Z}^d$, $d \geq 3$, see [59, 62, 61, 63], which describe the local limit of a random walk trace on $(\mathbb{Z}/N\mathbb{Z})^d$ as $N \to \infty$ and which relate to various covering and fragmentation problems for random walks; cf. for instance [57, 58, 68, 18]; ii) loop-soup percolation [34, 35, 19, 38]; iii) the voter percolation model [36, 39, 50]; iv) level-set percolation of random fields, see [41, 55, 7] and references therein (see also [10, 51, 11, 42] and [43, 16, 56]) for Gaussian ensembles relating to various classes of functions, e.g. randomized spherical harmonics (Laplace eigenfunctions) at high frequencies; v) the massless Gaussian free field φ on $\mathbb{Z}^d$ for $d \geq 3$. This last model, which will be the focus of the present article, was originally investigated by Lebowitz and Saleur in [36] as a canonical percolation model with slow, algebraic decay of correlations. It has received considerable attention since then; see for instance [15, 31, 53, 64, 23, 20, 22, 65, 2, 3], and references below.

All these models have a different behaviour than Bernoulli percolation at criticality. As mentioned above, even their off-critical phases represent a challenge for mathematical physicists and probabilists, since their constructions involve correlations between vertices with slow algebraic decay. A persistent and fundamental question in this context is to assess whether several natural critical parameters (see next section), defining regimes in which renormalization techniques lead to a deep understanding of the model, actually coincide; see [53, Remark 2.8,1] and [24, Remark 2.9], respectively regarding the sub- and supercritical phases of Gaussian free field level-sets, see also [59, Remark 4.4,3)], [58, (0,7)] and [67] for similar questions concerning the vacant set of random interlacements. In the present work, we answer this question affirmatively for the historical example of Gaussian free field level sets; see Theorem 1.1 below. To the best of our knowledge, this is the first instance of a *unified* approach towards the understanding of both sub- and supercritical regimes of percolation models.

The core of our proof is a new and delicate interpolation scheme aimed at removing the long-range (algebraic) dependences intrinsic to the model. This scheme will work in a regime, expected to reduce to criticality, in which connection and disconnection probabilities decay slowly. Under the (a posteriori wrong) assumption that the critical parameters mentioned above do not coincide, this regime fictitiously extends "away from" criticality. As a consequence, our interpolation scheme implies the existence of a percolation model with finite-range dependences for which the corresponding critical parameters do not coincide either. This leads to a contradiction thanks to the main result of [33] combined with recent progress made in the study of such models [27, 29, 28].

A similar yet much simpler interpolation was used in [26] in a different framework. There, results were perturbative in nature, while here we must implement the scheme close to criticality and in the presence of strong correlations. In the current context, a "bridging lemma" for the Gaussian free field (see Lemma 1.6 below) will play a central role in allowing for various path reconstructions, and will be derived by expanding on renormalization ideas from [60, 22]. We regard this step as a key progress in the understanding of percolation models that do not enjoy the so-called (uniform) finite-energy property (in this context due to regions of large field). We believe that our methods will pave the way towards proving equality of these critical parameters for many interesting models in the above class.

In combination with previous results in the literature, our findings have many implications regarding our understanding of the level-set geometry of φ, both in the subcritical and supercritical regimes. We defer a thorough discussion of these matters for a few lines and first describe our results.

1.2   Main result

We consider the massless Gaussian free field (GFF) on $\mathbb{Z}^d$, for $d \geq 3$, which is a centered, real-valued Gaussian field $\varphi = \{\varphi_x : x \in \mathbb{Z}^d\}$. Its canonical law $\mathbb{P}$ is uniquely determined by specifying that φ has covariance function $\mathbb{E}[\varphi_x \varphi_y] = g(x, y)$ for all $x, y \in \mathbb{Z}^d$, where

$$(1.1) \qquad g(x, y) := \sum_{k=0}^{\infty} P_x[X_k = y], \ \ x, y \in \mathbb{Z}^d,$$

denotes the Green function of the simple random walk on $\mathbb{Z}^d$. Here, P_x stands for the canonical law of the discrete-time random walk $\{X_k : k \geq 0\}$ on $\mathbb{Z}^d$ with starting point $X_0 = x \in \mathbb{Z}^d$. For $h \in \mathbb{R}$, we introduce the level-set above height h as $\{\varphi \geq h\} := \{x \in \mathbb{Z}^d : \varphi_x \geq h\}$ and for any $A, B, C \subset \mathbb{Z}^d$, let

$$(1.2) \qquad \{A \xleftrightarrow[C]{\varphi \geq h} B\} := \{A \text{ and } B \text{ are connected in } \{\varphi \geq h\} \cap C\}.$$

The subscript C is omitted when $C = \mathbb{Z}^d$. We also write $\{A \xleftrightarrow{\varphi \geq h} \infty\}$ for the event that there is an infinite connected component (connected components will also be called *clusters*) of $\{\varphi \geq h\}$ intersecting A. Note that all previous events are decreasing in h. We then define the *critical parameter h_** of $\{\varphi \geq h\}$ as

$$(1.3) \qquad h_*(d) := \inf \{h \in \mathbb{R} : \ \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \infty] = 0\}.$$

It is known that $0 < h_*(d) < \infty$ for all $d \geq 3$; see [15, 53, 23]. Thus, in particular, the level sets $\{\varphi \geq h\}$ undergo a (nontrivial) percolation phase transition as h varies. Moreover, for all $h < h_*$, $\{\varphi \geq h\}$ has $\mathbb{P}$-a.s. a unique infinite cluster, whereas for all $h > h_*$, $\{\varphi \geq h\}$ consists a.s. of finite clusters only.

Following [53], we consider an auxiliary critical value $h_{**} \geq h_*$ defined as

$$(1.4) \qquad h_{**}(d) := \inf \{h \in \mathbb{R} : \inf_R \mathbb{P}[B_R \xleftrightarrow{\varphi \geq h} \partial B_{2R}] = 0\},$$

where $B_R := ([-R, R] \cap \mathbb{Z})^d$ and $\partial B_R := \{y \in B_R : y \sim z \text{ for some } z \in \mathbb{Z}^d \setminus B_R\}$ stand for the ball of radius R centered at 0 and its inner boundary, respectively. Here $y \sim z$ means y

and z are nearest-neighbors in $\mathbb{Z}^d$. The quantity h_{**} is well-suited for certain renormalization arguments in the subcritical phase, by which it was shown in [53] that $h_{**}(d)$ is finite for all $d \geq 3$ and that for all $h > h_{**}$, the level-set $\{\varphi \geq h\}$ is in a *strongly non-percolative regime* in the sense that probabilities of connections decay very fast. More precisely, for any $h > h_{**}$ there exist constants c, ρ depending on d and h, such that

$$(1.5) \qquad \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_R] \leq e^{-cR^\rho}.$$

In fact, one can even take $\rho = 1$ for $d \geq 4$, with logarithmic corrections when $d = 3$; see [47, 48]. The arguments of [53] originally required the probability on the right-hand side of (1.4) to decay polynomially in R (along subsequences). It was later shown in [47, 48] that it suffices for the infimum in (1.4) to lie below the value $\frac{7}{2d \cdot 21^d}$ in order to guarantee (stretched) exponential decay of the probability that B_R is connected to ∂B_{2R} in $\{\varphi \geq h\}$ for all larger values of h, implying in particular the equivalence of these definitions with the one in (1.4), which is natural in the present context. It was further shown in [25] that $h_*(d) \sim h_{**}(d) \sim \sqrt{2 \log d}$ as $d \to \infty$, where $\sim$ means that the ratio of the two quantities on either side converges to 1, but little was otherwise known prior to this article about the relationship between h_* and h_{**}.

Another critical parameter $\bar{h} \leq h_*$ was introduced in [22], inspired by similar quantities defined in [24, 64], cf. also [8], which allows to implement certain (static) renormalization arguments in the supercritical phase. As a consequence, the geometry of the level-sets $\{\varphi \geq h\}$ is well-understood at levels $h < \bar{h}$, as will be explained further below. To define $\bar{h}$, we first introduce the events, for $\alpha \in \mathbb{R}$,

$$(1.6) \qquad \text{Exist}(R, \alpha) := \left\{ \begin{array}{c} \text{there exists a connected component in} \\ \{\varphi \geq \alpha\} \cap B_R \text{ with diameter at least } R/5 \end{array} \right\}, \text{ and}$$

$$(1.7) \qquad \text{Unique}(R, \alpha) := \left\{ \begin{array}{c} \text{any two clusters in } \{\varphi \geq \alpha\} \cap B_R \text{ having diameter at} \\ \text{least } R/10 \text{ are connected to each other in } \{\varphi \geq \alpha\} \cap B_{2R} \end{array} \right\},$$

(throughout the article, the diameter of a set is with respect to the sup-norm). We say that φ *strongly percolates up to level* $h \in \mathbb{R}$ if there are constants $c \in (0, \infty)$ and $\rho \in (0, 1]$, possibly depending on d and h, such that for all $\alpha \leq h$ and $R \geq 1$,

$$(1.8) \qquad \mathbb{P}[\text{Exist}(R, \alpha)] \geq 1 - e^{-cR^\rho} \text{ and } \mathbb{P}[\text{Unique}(R, \alpha)] \geq 1 - e^{-cR^\rho}.$$

We then define

$$(1.9) \qquad \bar{h}(d) := \sup \left\{ h \in \mathbb{R} : \ \varphi \text{ strongly percolates up to level } h \right\}.$$

It was proved in [24] that $\bar{h} \ (\leq h_*)$ is non-trivial, i.e. $\bar{h} > -\infty$, and it was recently shown that $\bar{h}(d) > 0$; see [22], which implies in particular that the sign clusters of φ percolate. It is easy to see from (1.8) that $\{\varphi \geq h\}$ has a unique infinite connected component for any $h < \bar{h}$, and one can show that any finite connected component $\{\varphi \geq h\}$ is necessarily tiny: for instance, the radius of a finite cluster has stretched exponential tails for $h < \bar{h}$.

Let us mention that different notions of $\bar{h}$ have been introduced in the literature, e.g. [64, 22, 66]. We chose to consider here the strongest of these notions (resembling the one from [66]) so that our main result directly holds for the other ones as well.

With h_*, h_{**} and $\bar{h}$ given by (1.3), (1.4) and (1.9), our main result is

Theorem 1.1. *For all $d \geq 3$, $\bar{h}(d) = h_*(d) = h_{**}(d)$.*

The following is an important consequence of Theorem 1.1.

Corollary 1.2 (Decay of the truncated two-point function except at criticality)**.** *For all $d \geq 3$ and $\varepsilon > 0$, there exist $c = c(d, \varepsilon) > 0$ and $\rho = \rho(d) \in (0, 1]$ such that for all $h \notin (h_* - \varepsilon, h_* + \varepsilon)$ and $x, y \in \mathbb{Z}^d$,*

$$(1.10) \qquad \tau_h(x, y) := \mathbb{P}[x \xleftrightarrow{\varphi \geq h} y, \ x \xcancel{\xleftrightarrow{\varphi \geq h}} \infty] \leq e^{-c|x-y|^\rho}.$$

For $h > h_{**} \ (= h_*)$, (1.10) follows immediately from (1.5). In fact, as mentioned above, one knows in this case that $\rho(d) = 1$ whenever $d \geq 4$, with logarithmic corrections in dimension 3; see [47, 48]. For $h < \bar{h} \ (= h_*)$, the bound (1.10) directly follows from (1.8) and a straightforward union bound. Moreover, the uniformity over $h < \bar{h} \ (= h_*)$ for ρ in (1.8) (and therefore in (1.10)) is a consequence of our proof, see Remark 4.6. The optimal value of ρ for $h < h_*$, in both (1.8) and (1.10), remains an open problem.

To the best of our knowledge, the only instances in which a full analogue of Theorem 1.1 and Corollary 1.2 is known to hold, in all dimensions greater or equal to three, are the random cluster representation of the Ising model [5, 30, 13] and the aforementioned case of Bernoulli percolation [4, 40, 33]. In particular, the analogue of $\bar{h} = h_*$ for the random cluster model with generic parameter $q \geq 1$ remains open.

We now discuss further consequences of Theorem 1.1. Various geometric properties of the (unique) infinite cluster $\mathscr{C}_\infty^h$ of $\{\varphi \geq h\}$ have been investigated in the regime $h < \bar{h}$, all exhibiting the "well-behavedness" of this phase. For instance, for $h < \bar{h}$, the chemical (i.e. intrinsic) distance ρ on $\mathscr{C}_\infty^h$ is comparable to the Euclidean one, and balls in the metric ρ rescale to a deterministic shape [24]. Moreover, the random walk on $\mathscr{C}_\infty^h$ is known to satisfy a quenched invariance principle [49] and mesoscopic balls in $\mathscr{C}_\infty^h$ have been verified to exhibit regular volume growth and to satisfy a weak Poincaré inequality; see [54]. This condition, originally due to [9], has several important consequences, e.g. it implies quenched Gaussian bounds on the heat kernel of the random walk on $\mathscr{C}_\infty^h$, as well as elliptic and parabolic Harnack inequalities, among other things. It has also been proved that the percolation function giving for each h the probability that 0 is connected to infinity in $\{\varphi \geq h\}$ is C^1 on $(-\infty, \bar{h})$; see [66]. On account of Theorem 1.1, all the above results now hold in the entire supercritical regime $h < h_*$.

The large-deviation problem of disconnection in the supercritical regime has also received considerable attention recently; see [64, 45, 44, 20]. Together with Theorems 2.1 and 5.5 of [64] and Theorems 2.1 and 3.1 of [44] (relying on techniques developed in [45]), Theorem 1.1 yields the following: for $A \subset [-1, 1]^d$ an arbitrary (not necessarily convex) regular compact set A (regular in the sense that A and its interior have the same Brownian capacity), one has

$$(1.11) \qquad \lim_N \frac{1}{N^{d-2}} \log \mathbb{P}[(NA) \cap \mathbb{Z}^d \xcancel{\xleftrightarrow{\varphi \geq h}} \partial B_{2N}] = -\frac{1}{2d}(h_* - h)^2 \mathrm{cap}\,(A), \text{ for all } h \leq h_*,$$

where $\mathrm{cap}(\cdot)$ stands for the Brownian capacity; see also [20] for finer results on the measure $\mathbb{P}$ conditioned on the disconnection event above.

Theorem 1.1 also translates to a finitary setting: consider the zero-average Gaussian free field Ψ on the torus $(\mathbb{Z}/N\mathbb{Z})^d$ as $N \to \infty$; see [1] for relevant definitions. As a consequence of Theorems 3.2 and 3.3 therein and Theorem 1.1 above, one deduces the following with high

probability as $N \to \infty$: $\{\Psi \geq h\}$ only contains connected components of size $o(\log^\lambda N)$ for any $h > h_{**}$ and $\lambda > d$, while $\{\Psi \geq h\}$ has a giant, i.e. of diameter comparable to N, connected component for all $h < h_*$. Plausibly, one could further strengthen these results and determine the size of the second largest component of $\{\Psi \geq h\}$ for all $h < h_*$.

Finally, we briefly discuss the *massive* case, in which $g(\cdot, \cdot)$ in (1.1) is replaced by the Green function of the random walk killed with probability $\theta > 0$ at every step (whence correlations for φ exhibit exponential decay). Let $h_*(\theta)$, $h_{**}(\theta)$ and $\bar{h}(\theta)$ denote the corresponding critical parameters. The techniques we develop here readily apply to prove that $\bar{h}(\theta) = h_*(\theta) = h_{**}(\theta)$. Actually, the equality $h_*(\theta) = h_{**}(\theta)$, for all $\theta > 0$, can be obtained in a simpler fashion: one can apply Lemma 3.2 from [29] directly to the law of $\{\varphi \geq h\}$ (which is monotonic in the sense of [29]) and combine it with Proposition 3.2 in [52] to deduce a suitable differential inequality for the one-arm crossing probability. By current methods, the proof of $h_*(\theta) = \bar{h}(\theta)$ does however require a truncation (as for the case $\theta = 0$, see below).

1.3   Strategy of proof

We now give an overview of our proof of Theorem 1.1. We first introduce an additional critical parameter $\tilde{h}$ for φ which quantifies how small disconnection probabilities are. Formally, let $u(R) := \exp[(\log R)^{1/3}](\ll R)$ and define

$$(1.12) \qquad \tilde{h}(d) := \sup\{h \in \mathbb{R} : \inf_R R^d \, \mathbb{P}[B_{u(R)} \overset{\varphi \geq h}{\longleftrightarrow\!\!\!\!/} \partial B_R] = 0\}.$$

By (1.5) one knows that $\lim_R R^d \mathbb{P}[B_{u(R)} \overset{\varphi \geq h}{\longleftrightarrow} \partial B_R] = 0$ whenever $h > h_{**}$. In view of (1.12), this readily implies that $\tilde{h} \leq h_{**}$. Several reasons motivate the choice of the scale $u(R)$, one of them being the precise form of a certain "reconstruction cost" appearing in Lemma 1.6 below (see (1.17)). We refer to Remarks 4.3 and 5.3 for details on the choice of $u(\cdot)$.

Our proof is organized in three parts, corresponding to Propositions 1.3, 1.4 and 1.5 below: the first two will imply the equality $\tilde{h} = h_{**}$, while the last one will relate $\tilde{h}$ to $\bar{h}$.

We first decompose the GFF into an infinite sum of independent stationary Gaussian fields $(\xi^\ell)_{\ell \geq 0}$ (see Section 3 for precise definitions) with each ξ^ℓ having finite range of dependence (in fact, the range of dependence will be exactly ℓ), and define a *truncated field*

$$(1.13) \qquad \varphi^L := \sum_{0 \leq \ell \leq L} \xi^\ell.$$

The percolation processes $\{\varphi^L \geq h\}$ are natural finite-range approximations for $\{\varphi \geq h\}$. Instead of working directly with those, it turns out to be technically more convenient to use slightly *noised* versions of these approximations, for which a certain finite-energy property plainly holds (along with finite range, this property is crucially needed to deduce the first equality in Proposition 1.3 below in a straightforward way). To this end, we introduce for any $\delta \in (0,1)$ and any percolation configuration $\omega \in \{0,1\}^{\mathbb{Z}^d}$, a new configuration $T_\delta \omega$ where the state of every vertex is resampled independently with probability δ, according to (say) a uniform distribution on $\{0,1\}$ (any non-degenerate distribution on $\{0,1\}$ would do). One can now define the critical parameters $h_*(\delta, L)$, $h_{**}(\delta, L)$ and $\tilde{h}(\delta, L)$ as in (1.3), (1.4) and (1.12), but for the family of processes $\{T_\delta \{\varphi^L \geq h\} : h \in \mathbb{R}\}$ instead of $\{\{\varphi \geq h\} : h \in \mathbb{R}\}$. The next proposition states a sharpness result for these finite-range models.

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Proposition 1.3. *For all $d \geq 3$, $L \geq 0$ and $\delta \in (0,1)$, we have $\tilde{h}(\delta, L) = h_*(\delta, L) = h_{**}(\delta, L)$.*

This proposition is a fairly standard adaptation of known results (a result of Grimmett-Marstrand [33] on one side, and proofs of sharpness using the OSSS-inequality developed in [29, 28, 27] on the other side); see Section 6 for details. Nonetheless, Proposition 1.3 offers a stepping stone for our argument, which, roughly speaking, will consist of carrying over the sharpness for these finite-range models to a sharpness result for level-sets of the full GFF by comparing the two models for parameter values $h \in (\tilde{h}, h_{**})$ (notice that this interval can a priori be empty). The core of our strategy is therefore encapsulated in the following proposition.

Proposition 1.4. *For every $d \geq 3$ and $\varepsilon > 0$, there exist $c, C > 0$, $\delta \in (0,1)$ and an integer $L \geq 1$, all depending on d and ε only, such that for all $h \in (\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon)$ and $R \geq 2r > 0$,*

$$(1.14) \qquad \mathbb{P}[B_r \xleftrightarrow{T_\delta\{\varphi^L \geq h\}} \partial B_R] \geq \mathbb{P}[B_r \xleftrightarrow{\varphi \geq h+\varepsilon} \partial B_R] - C\exp(-e^{c(\log r)^{1/3}}),$$

$$(1.15) \qquad \mathbb{P}[B_r \xleftrightarrow{T_\delta\{\varphi^L \geq h\}} \partial B_R] \leq \mathbb{P}[B_r \xleftrightarrow{\varphi \geq h-\varepsilon} \partial B_R] + C\exp(-e^{c(\log r)^{1/3}}).$$

Proposition 1.4 is truly the heart of the paper. Note that eventually, we show that $\tilde{h} = h_{**}$, so that the interval $(\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon)$ corresponds to a *fictitious regime*, in the sense that the interval in question is in fact empty as a consequence of Theorem 1.1 (a similar fictitious regime was introduced in [27] to study Boolean percolation).

The proof of Proposition 1.4 is based on an interpolation argument, inspired to some extent by [26] and more remotely by [6], enabling us to remove the long-range dependences of the full model at the cost of slightly *varying* the parameter h. More precisely, we will define a family of Gaussian fields χ^t indexed by $t \geq 0$ satisfying the following properties: for each integer $n \geq 0$, the field χ^n will be equal to φ^{L_n} for a certain integer L_n (henceforth referred as the n-th scale, see (1.18) below) and χ^t will interpolate linearly between $\chi^{\lfloor t \rfloor}$ and $\chi^{\lceil t \rceil}$. Then, we will show that the functions

$$f_\pm(t) := \theta(t, h \pm 2e^{-t}, r, R) \mp C\exp(-e^{c(\log r)^{1/3}})e^{-t},$$

where $\theta(t, h, r, R) := \mathbb{P}[B_r \xleftrightarrow{\chi^t \geq h} \partial B_R]$, are increasing and decreasing respectively. This will follow from a careful comparison of the partial derivatives $\partial_t \theta$ and $\partial_h \theta$. One important step in this comparison will be the (re-)construction of suitable "pivotal points" from corresponding coarse-grained ones, cf. Fig. 5, which will involve an instance of a "bridging lemma", akin to Lemma 1.6 below, in order to (re-)construct various pieces of paths in $\{\varphi \geq h\}$ for $h < h_{**}$. The arguments involved in the derivative comparison will repeatedly rely on the assumption that various connection and disconnection events are not too unlikely, as guaranteed by the assumption that $h \in (\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon)$, cf. (1.4) and (1.12). This motivates the introduction of such a (fictitious) regime. A more thorough discussion of the interpolation argument underlying the proof of Proposition 1.4 goes beyond the scope of this introduction and is postponed to Section 5.1.

As a straightforward consequence of Propositions 1.3 and 1.4, one deduces that $\tilde{h} = h_{**}$ for every $d \geq 3$ as follows. On account of the discussion immediately following (1.12), it suffices to argue that $h_{**} \leq \tilde{h}$. Suppose on the contrary that the interval $(\tilde{h}, h_{**})$ is non-empty and consider L and δ provided by Proposition 1.4 with $\varepsilon := (h_{**} - \tilde{h})/8$. It then follows by Proposition 1.4 that the intervals $(h_{**}(\delta, L), \infty)$ and $(\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon)$ have empty intersection. Indeed, otherwise one could pick an $h \in (h_{**}(\delta, L), \infty) \cap (\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon)$ and (1.14) would yield that $\inf_R \mathbb{P}[B_{R/2} \xleftrightarrow{\varphi \geq h+\varepsilon} \partial B_R] = 0$, thus violating the fact that $h + \varepsilon < h_{**}$, cf. (1.4). A similar

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reasoning using (1.15) yields that $(-\infty, \tilde{h}(\delta, L)) \cap (\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon) = \emptyset$. But both $(h_{**}(\delta, L), \infty)$ and $(-\infty, \tilde{h}(\delta, L))$ having empty intersection with $(\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon)$ contradicts the equality $\tilde{h}(\delta, L) = h_{**}(\delta, L)$, which is implied by Proposition 1.3.

All in all, the discussion of the previous paragraph shows that Theorem 1.1 follows immediately from the Propositions 1.3 and 1.4, combined with the following one.

Proposition 1.5. *For all $d \geq 3$, $\tilde{h}(d) \leq \bar{h}(d)$.*

The proof of this proposition will be rather different from that of the previous proposition. Our starting point is a result of Benjamini and Tassion [12], stating that in Bernoulli percolation, for every $\varepsilon > 0$, the probability that a graph spanning the whole box B_R does not become connected after opening every edge independently with probability $\varepsilon > 0$ is extremely small provided that R is sufficiently large. In the present case, for $h < \tilde{h}$, one sees from (1.12) that the probability that every box of size $u(R)$ in B_R is connected to ∂B_R can be taken arbitrarily close to 1 provided that R is chosen large enough. From this, we perform a coarse-grained version of the Benjamini-Tassion argument to prove that the probability of Unique(R, β) converges to 1 (along subsequences) for all $\beta < h$; see Proposition 4.1. Then, we bootstrap this estimate via a renormalization argument to show that the probabilities of Unique(R, α) and Exist(R, α) tend to 1 stretched-exponentially fast for $\alpha < \beta$.

Implementing this scheme will raise a number of difficulties. First, the model has long-range dependence, a fact which forces us to use renormalization techniques, pioneered in [60] (see also references therein) in the context of random interlacements, rather than elementary coarse-graining usually harvested in Bernoulli percolation. Second, the model does not enjoy uniform bounds on conditional probabilities that a vertex is in $\{\varphi \geq h\}$ or not. When conditioning on a portion of $\{\varphi \geq h\} \backslash \{x\}$, the stiffness of the field may force $\varphi_x \geq h$ or $\varphi_x < h$ in a very degenerate fashion. These *large-field effects* are difficult to avoid, as one can see for instance by observing that the probability that $\varphi_0 \geq h$ conditioned on the event that $\varphi_x < h$ for every $x \in B_R \backslash \{0\}$ decays polynomially in R; see [14, 21]. This means that, when implementing a coarse-grained version of the argument from [12], we will rely on yet another "bridge construction" to argue that decreasing h by ε indeed creates connections between nearby clusters.

We now describe more precisely a version of the "bridging lemmas" used in the proofs of both Propositions 1.4 and 1.5, which are needed in order to cope with the large-field effects of φ alluded to above (a glance at Figure 1 in Section 2.1 might also help). For simplicity, we introduce an example of a useful statement asserting that it is still possible, outside of events of stretched-exponentially small probability to connect two large (connected) subsets S_1 and S_2 of B_R at a reasonable cost, even when conditioning on φ_x for $x \notin B_R$ and on $\mathbf{1}_{\varphi_x \geq h}$ for every $x \in S_1 \cup S_2$. We refer to Lemma 3.5 and Remark 3.7 (see also (5.31)) below for results similar to Lemma 1.6 but tailored to the proofs of Propositions 1.5 and 1.4, respectively.

Lemma 1.6 (Bridging lemma). *For every $d \geq 3$ and $\varepsilon > 0$, there exist positive constants $c = c(d, \varepsilon), C = C(d, \varepsilon)$ and $\rho = \rho(d)$ such that for all $R \geq 1$, there are events $\mathcal{G}(S_1, S_2)$ indexed by $S_1, S_2 \subset B_R$ such that*

$$\mathbb{P}\Big[\bigcap_{S_1, S_2} \mathcal{G}(S_1, S_2)\Big] \geq 1 - e^{-cR^\rho} \tag{1.16}$$

and for every $S_1, S_2 \subset B_R$ connected with diameter larger than $R/10$, all $h < h_{**} - 2\varepsilon$, every $D \in \sigma(\mathbf{1}_{\varphi_x \geq h}; \; x \in S_1 \cup S_2)$ and $E \in \sigma(\varphi_x; x \notin B_R)$,

$$\mathbb{P}\Big[S_1 \xleftrightarrow[B_R]{\varphi \geq h - \varepsilon} S_2 \;\Big|\; \begin{matrix} D \cap E \\ \mathcal{G}(S_1, S_2) \end{matrix}\Big] \geq e^{-C(\log R)^2}. \tag{1.17}$$

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Note that the assumption that $h < h_{**}$ is necessary since otherwise, already for the uncon-ditioned measure the probability in (1.17) to connect two sets S_1 and S_2 at a distance of order R of each other is decaying stretched exponentially fast as soon as $h > h_{**}$, cf. the discussion following (1.4).

We now explain the nature of the events $\mathcal{G}(S_1, S_2)$. The argument yielding (1.17) will require (re-)constructing pieces of paths in $\{\varphi \geq h\}$ for $h < h_{**}$ to connect S_1 and S_2 at an affordable cost. The paths in question will be built inside so-called *good bridges*, introduced in Definitions 2.1 and 2.2; see also Fig. 1 below. Roughly speaking, a good bridge is formed by a concatenation of boxes at multiple scales L_n ($n \geq 0$) defined by

$$(1.18) \qquad L_n := \ell_0^n L_0, \text{ for some } L_0 \geq 100, \ell_0 \geq 1000,$$

in which φ has certain desirable (good) properties. Together with the assumption that $h < h_{**}$, these conditions on φ will allow to deduce the bound (1.17). Their precise form may however vary depending on the specific situation in which a bridge construction is applied.

Apart from just connecting two sets of interest, bridges satisfy two important geometric constraints: i) any box at scale L_k which is part of a bridge does not get closer than distance $\approx L_k$ to the two sets connected by the bridge, and ii) a bridge does not involve too many boxes at any scale L_k. The former will allow us to retain some independence when exploring the clusters that need to be connected while the latter is key in order to keep the reconstruction cost under control.

The events $\mathcal{G}(S_1, S_2)$ appearing in Lemma 1.6 then correspond to the existence of a good bridge linking the sets S_1 and S_2. Their likelihood, as implied by (1.16), will follow from Theorem 2.3, derived in the next section. It asserts that good bridges (for a generic underlying notion of goodness, see e.g. (2.4)–(2.5)) can be found with very high probability between any two sufficiently large sets. This result will then be applied in Sections 4 and 5 with different choices of good events involving a decomposition of φ into a sum of independent fields with range L_n for $n \geq 0$, alluded to in (1.13) and introduced in Section 3, to yield (1.16) and (1.17) as a surrogate "finite-energy property" for φ. The definition of bridges as well as the statement of Theorem 2.3 are fairly technical and postponed to Section 2. The proof of Theorem 2.3 is based on renormalization ideas for φ developed in [53, 24, 22]. Interestingly, and in contrast to these works, our main tool in the present context, introduced in the next section, is a geometric object (the good bridge) which involves *all* scales L_k, $0 \leq k \leq n$, for a given macroscopic scale L_n, cf. Fig. 1 in Section 2.1 and Fig. 5 in Section 5.2.

Organization of the paper. Section 2 contains the renormalization scheme and the notion of good bridges that will be used in several places later on. The statements and proofs have been made independent of the model. Section 3 introduces the decomposition of the GFF into finite-range Gaussian processes and presents the proof of Lemma 1.6. Section 4 and Section 5 are respectively devoted to the proofs of Propositions 1.5 and 1.4. The last section contains the proof of Proposition 1.3 and is independent of the rest of the paper.

Notation. For $x \in \mathbb{Z}^d$, let $B_R(x) := x + B_R$ and $\partial B_R(x) := x + \partial B_R$, with B_R and ∂B_R as defined below (1.4). Except otherwise stated, distances are measured using the ℓ^∞-norm, which is denoted by $|\cdot|$. We use $d(U, V)$ to denote the ℓ^∞-distance between sets $U, V \subset \mathbb{Z}^d$.

We write c, c', C, C' for generic numbers in $(0, \infty)$ which may change from line to line. They may depend implicitly on the dimension d. Their dependence on other parameters will always

be explicit. Numbered constants $c_0, c_1, C_0, C_1, \ldots$ refer to constants that are used repeatedly in the text; they are numbered according to their first appearance.

2 Multi-scale bridges

In this section, we introduce the notion of good (multi-scale) bridge which will be later used. The main result is Theorem 2.3, which asserts that good bridges connect any two "admissible" sets with very high probability when certain conditions are met. The proof of Theorem 2.3 appears in Section 2.2. It involves a suitable renormalization scheme, and revolves around Lemma 2.4, which is proved in a separate section (Section 2.3).

2.1 Definition of a bridge and statement of Theorem 2.3

Recall the definition of scales L_n, $n \geq 0$, from (1.18). For $n \geq 0$, let $\mathbb{L}_n := (2L_n + 1)\mathbb{Z}^d$ and call a box of the form $B_{L_n}(x)$ for $x \in \mathbb{L}_n$ the *n-box (attached to x)*. Note that for each $n \geq 0$, every point $y \in \mathbb{Z}^d$ is contained in exactly one n-box. We call a nearest-neighbor path in $\mathbb{L}_n$ any sequence of vertices in $\mathbb{L}_n$ such that any two consecutive elements are at ℓ^1-distance $2L_n + 1$ on $\mathbb{Z}^d$.

We introduce two parameters $\kappa \geq 20$ and $K \geq 100$ which will respectively govern the "separation scale" and the "complexity" of a bridge, see **B3** and **B4** below. These parameters correspond to the geometric features i) and ii) highlighted in the introduction, see below (1.18).

In what follows, for $n \geq 0$, we consider the triplet of domains $(\underline{\Lambda}_n, \Lambda_n, \Sigma_n)$ where $\Sigma_n := B_{9\kappa L_n} \setminus B_{\lfloor 8.5\kappa L_n \rfloor}$ and

$$(2.1) \qquad (\Lambda_n, \underline{\Lambda}_n) := (B_{10\kappa L_n}, B_{8\kappa L_n}) \text{ or } (B_{10\kappa L_n} \setminus B_{\kappa L_n}, B_{8\kappa L_n} \setminus B_{\kappa L_n}).$$

Definition 2.1 (Bridge). For any $S_1, S_2 \subset \Lambda_n$, a bridge between S_1 and S_2 inside Λ_n is a finite collection $\mathcal{B}$ of subsets of Σ_n with the following properties:

B1 Every $B \in \mathcal{B}$ is an m-box, $0 \leq m \leq n$, included in Σ_n and $\bigcup_{B \in \mathcal{B}} B$ is a connected set.

B2 There exist 0-boxes $B_1, B_2 \in \mathcal{B}$ such that $B_i \cap S_i \neq \emptyset$, $i = 1, 2$ and for all $B \in \mathcal{B} \setminus \{B_1, B_2\}$, $B \cap (S_1 \cup S_2) = \emptyset$.

B3 For every m-box $B \in \mathcal{B}$ with $1 \leq m \leq n$, one has $d(B, S_1 \cup S_2) \geq \kappa L_m$.

B4 For every $m \geq 0$, the number of m-boxes in $\mathcal{B}$ is smaller than $2K$.

We now introduce "good events" which will be later chosen according to specific needs. For the remainder of this section, we simply consider, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, families of events $F = \{F_{0,x} : x \in \mathbb{L}_0\}$ and $H_n = \{H_{n,x} : x \in \mathbb{L}_n\}$, for $n \geq 1$.

Definition 2.2 (Good bridge)**.** A bridge as defined above is *good* if

G1 For every 0-box $B_{L_0}(x) \in \mathcal{B}$, the event $F_{0,x}$ occurs,

G2 For every m-box $B_{L_m}(x) \in \mathcal{B}$, $0 \leq m \leq n$, the events $H_{j,y(x,j)}$ occur for every $j \geq m \vee 1$, where $y(x,j)$ is the unique element of $\mathbb{L}_j$ with $x \in B_{L_j}(y(x,j))$.

The property **G2** ensures that every m-box in a good bridge sits inside a "tower" of good events attached to the j-boxes containing its center, for all $j \geq m$. Good bridges will be used to connect a certain class of sets. A set $S \subset \Lambda_n$ (not necessarily connected) is *admissible* if each connected component of S intersects $\partial B_{10\kappa L_n}$ and at least one connected component of S intersects $B_{8\kappa L_n}$. We are interested in the event, for $n \geq 0$,

$$(2.2) \qquad \mathcal{G}_n := \{\text{there exists a good bridge inside } \Sigma_n \text{ between every admissible } S_1, S_2 \subset \Lambda_n\},$$

where inside Σ_n refers to choosing $\Lambda = \Sigma_n$ in Definition 2.1, see also Figure 1 below. We define $\mathcal{G}_{n,x}$, $x \in \mathbb{L}_n$, as the event corresponding to (2.2) when one replaces the triplet $(\underline{\Lambda}_n, \Lambda_n, \Sigma_n)$ in (2.1) and Definitions 2.1 and 2.2 by $(x + \underline{\Lambda}_n, x + \Lambda_n, x + \Sigma_n)$.
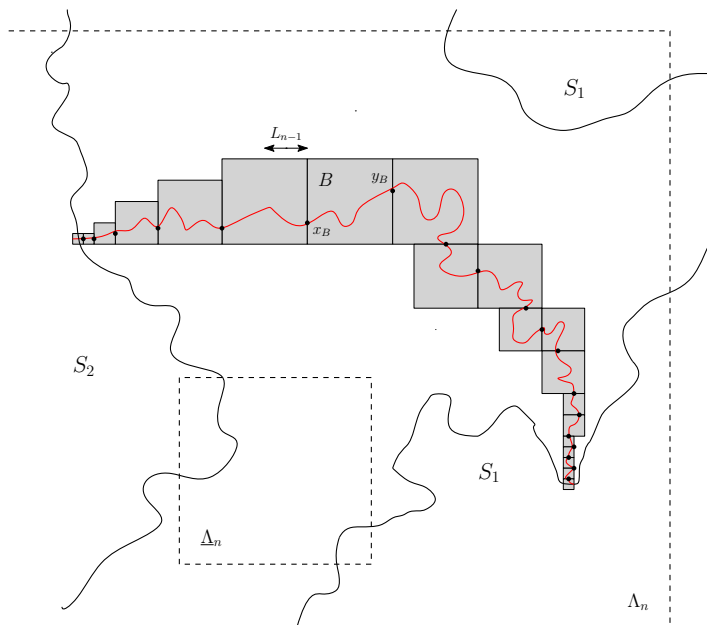

Figure 1 – An illustration of the event $\mathcal{G}_n$: depicted is a pair of admissible sets (S_1, S_2) and the good bridge (in light gray) connecting them. Later in Section 3, the underlying good events will guarantee that the sets S_1 and S_2 can be linked by a certain path (in red) inside a good bridge. Albeit not required by the definition, our construction of a good bridge on certain good events actually yields a "croissant-type" shape. More precisely, one can define two sequences of boxes, starting with the boxes B_1 and B_2, respectively, and corresponding to the two *arches* in the proof of Lemma 2.4, which comprise all but the largest boxes involved in the bridge construction and whose side lengths are non-decreasing.

Recall that the event $\mathcal{G}_n$ depends implicitly on the four parameters L_0, ℓ_0, κ, K, as well as on the choice of families F and H_n, $n \geq 1$. We now state the main result of this section.

Theorem 2.3. *For each $\kappa \geq 20$, $\ell_0 \geq C(\kappa)$, there exist $K = K(\kappa, \ell_0) \geq 100$ and $c_1 = c_1(\kappa, \ell_0) \in (0,1)$ such that for all $L_0 \geq C_1(\kappa, \ell_0)$ the following hold: if the families of events $F = \{F_{0,x} : x \in \mathbb{L}_0\}$ and $H_n = \{H_{n,x} : x \in \mathbb{L}_n\}$ (for $n \geq 1$) satisfy*

(C1)   *the families $F, H_n, n \geq 1$ are independent,*

(C2)   *for any set $U \subset\subset \mathbb{Z}^d$ such that $|y - z| \geq \frac{\kappa}{2}$ for all $y, z \in U$ with $y \neq z$, the events $F_{0,x}$, $x \in (2L_0 + 1)U$, are independent and the events $H_{n,x}$, $x \in (2L_n + 1)U$, are independent.*

(C3)   *for every $x \in \mathbb{L}_0$, $\mathbb{P}\left[F_{0,x}^c\right] \leq c_1$ and for every $n \geq 1$ and $x \in \mathbb{L}_n$, $\mathbb{P}[H_{n,x}^c] \leq c_1 2^{-2^n}$,*

then for every $n \geq 0$ and $x \in \mathbb{L}_n$,

$$(2.3) \qquad\qquad \mathbb{P}\left[\mathcal{G}_{n,x}\right] \geq 1 - 2^{-2^n}.$$

2.2   Proof of Theorem 2.3

The proof involves a multi-scale argument and a corresponding notion of "goodness at level n" for every $n \geq 0$, that we now introduce. For $\Lambda \subset \mathbb{Z}^d$ finite, let $\mathbb{L}_n(\Lambda) := \{x \in \mathbb{L}_n : B_{L_n}(x) \cap \Lambda \neq \emptyset\}$ as well as for $x \in \mathbb{L}_0$,

$$(2.4) \qquad\qquad G_{0,x}(\Lambda) := \bigcap_{y \in \mathbb{L}_0(\Lambda \cap B_{10\kappa L_0}(x))} F_{0,y}$$

and for every $x \in \mathbb{L}_n$ with $n \geq 1$,

$$(2.5) \quad G_{n,x}(\Lambda) := \bigcap_{\substack{y,y' \in \mathbb{L}_{n-1}(\Lambda \cap B_{10\kappa L_n}(x)): \\ |y-y'|_\infty \geq 22\kappa L_{n-1}}} \left(G_{n-1,y}(\Lambda) \cup G_{n-1,y'}(\Lambda)\right) \;\cap\; \bigcap_{z \in \mathbb{L}_n(\Lambda \cap B_{10\kappa L_n}(x))} H_{n,z}.$$

A vertex $x \in \mathbb{L}_n$ will be called *n-good* if the event $G_{n,x}(\Lambda_n)$ occurs with Λ_n given by either choice of $(\Lambda_n, \underline{\Lambda}_n)$ in (2.1), and *n-bad* otherwise. In words, x is n-bad if either $H_{n,z}$ does not occur for some $z \in \mathbb{L}_n\left(\Lambda_n \cap B_{10\kappa L_n}(x)\right)$, or $\mathbb{L}_{n-1}\left(\Lambda_n \cap B_{10\kappa L_n}(x)\right)$ contains two distant vertices that are *both* $(n-1)$-bad.

The reason for introducing the notion of n-goodness is the following key deterministic lemma, which yields that n-goodness implies the occurrence of $\mathcal{G}_{n,0}$.

Lemma 2.4. *For all $\kappa \geq 20$, provided that $\ell_0 \geq C(\kappa)$ and $K \geq C'(\kappa, \ell_0)$, we have that for every $n \geq 0$, $x \in \mathbb{L}_n$ and $L_0 \geq 100$, if the events $H_{m,y(x,m)}$, $m > n$, and $G_{n,x}(x + \Lambda_n)$ all occur, then so does $\mathcal{G}_{n,x}$.*

We will prove Lemma 2.4 in the next section and now focus on the proof of Theorem 2.3.

Proof of Theorem 2.3. For simplicity, we assume that $x = 0$. For a given $\kappa \geq 20$, consider ℓ_0, K, L_0 such that the previous lemma holds true. Now, let

$$(2.6) \qquad\qquad q_n := \sup_{x \in \mathbb{L}_n} \mathbb{P}[x \text{ is } n\text{-bad}], \quad n \geq 0$$

and observe that by the previous lemma, it suffices to show, provided that c_1 such that (**C3**) holds is chosen small enough, that

$$(2.7) \qquad\qquad \sum_{m>n} \mathbb{P}[H_{m,0}^c] \leq \frac{1}{2} 2^{-2^n} \text{ and } q_n \leq \frac{1}{2} 2^{-2^n}$$

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hold. The former is immediate by (**C3**).

With the choice that $\ell_0 \geq 22\kappa$, one deduces from (**C1**)–(**C2**) and the definition of goodness that the events $G_{n-1,y}(\Lambda_n)$ and $G_{n-1,y'}(\Lambda_n)$ are independent for any $|y - y'|_\infty \geq 22\kappa L_{n-1}$. Hence, the definition of goodness and the union bound yield that

$$(2.8) \qquad q_n \leq |\mathbb{L}_{n-1}(\Lambda_n)|^2 q_{n-1}^2 + |\mathbb{L}_n(\Lambda_n)| \sup_x \mathbb{P}[H_{n,x}^{\mathsf{c}}] \leq \tfrac{1}{4}\Gamma^2 q_{n-1}^2 + \tfrac{1}{2}\Gamma c_1 2^{-2^n},$$

for all $n \geq 1$, where in the second inequality we introduced $\Gamma = \Gamma(\kappa, \ell_0) \geq 2|\mathbb{L}_{n-1}(\Lambda_n)|$ (for all n), and we used (**C3**). Since (**C3**) implies that $q_0 \leq \tfrac{1}{2}\Gamma c_1$, a simple induction using (2.8) implies that for every $n \geq 1$, $q_n \leq c_1 \Gamma 2^{-2^n} \leq \tfrac{1}{2} 2^{-2^n}$ as soon as $\tfrac{1}{4} c_1 \Gamma^3 \leq \tfrac{1}{2}$ and $c_1 \Gamma \leq \tfrac{1}{2}$. $\qquad\square$

Remark 2.5. 1) Careful inspection of the proofs of this section and the next reveals that one could in fact replace **G1** and **G2** by the property that for any box $B = B_{L_n}(x) \in \mathcal{B}$, any $n \leq m \leq n_{\max}$, where

$$(2.9) \qquad n_{\max} = n_{\max}(\Lambda) := \max\{k : B_{L_k}(x) \subset \Lambda \text{ for some } x \in \mathbb{L}_k\},$$

and any $y \in \mathbb{L}_m$ such that $x \in B_{L_m}(y)$, the event $G_{m,y}(\Lambda_{n_{\max}})$ occurs, together with the events $H_{m,y(x,m)}$ for $m > n_{\max}$, and the conclusions of Theorem 2.3 continue to hold.

2) One could also replace the condition of admissibility of the sets $S_1, S_2 \subset \Lambda_n$ by the requirement that S_1 and S_2 are any two connected subsets of $\underline{\Lambda}_n$ of diameter at least κL_n and connect them by a good bridge in Λ_n, with Σ_n replaced by Λ_n in Definition 2.1.

For later reference, we also collect the following consequence of the above setup. For $\Sigma \subset \Lambda_n = B_{10\kappa L_n}$, define $\mathcal{B}^0$ to be a 0-bridge inside Σ between two sets $S_1, S_2 \subset \Lambda_n$ if $\mathcal{B}^0$ consists of 0-boxes only and $\bigcup_{B \in \mathcal{B}^0} B$ is a connected subset of Σ intersecting both S_1 and S_2, and call $\mathcal{B}^0$ good if for every $B = B_{L_0}(x) \in \mathcal{B}^0$, **G1** and **G2** occur. Let
(2.10)
$$\mathcal{G}_n^0 := \{\text{there is a good 0-bridge inside } \widetilde{\Sigma}_n \text{ between any two admissible sets } S_1, S_2 \subset \Lambda_n\}$$

where "admissible" can be either i) as defined above (2.2), in which case one chooses $\widetilde{\Sigma}_n := \Sigma_n$, or ii) as defined in the previous paragraph with $\widetilde{\Sigma}_n := \Lambda_n$.

Corollary 2.6. *Under the assumptions of Theorem 2.3,* $\mathbb{P}\left[\mathcal{G}_n^0\right] \geq 1 - 2^{-2^n}$, *for all* $n \geq 0$.

We simply sketch the argument. Using Theorem 2.3, one may obtain $\mathcal{B}^0$ from the good bridge $\mathcal{B}$ as follows: one replaces each box $B = B_{L_k}(x) \in \mathcal{B}$ for $k \geq 1$ by the set

$$B^0 := B \setminus \left(\bigcup_{0 \leq k' < k} \quad \bigcup_{y \in \mathbb{L}_{k'}(B) : G_{k',y}(\Lambda_n)^c \text{ occurs}} B_{L_{k'}}(y) \right)$$

and verifies by induction over k that the 0-boxes forming B^0 contain a set $\mathcal{B}^0$ with the desired properties.

Alternatively, one can also prove Corollary 2.6 directly, i.e. without resorting to the existence of $\mathcal{B}$, by following the lines of the proof of Lemma 8.6 in [22], with suitable modifications (in particular, involving a different notion of n-goodness, due to the presence of H_n, $n \geq 1$ in (2.5) replacing a sprinkling of the parameters to define the cascading events in (7.3) of [22]).

2.3 Proof of Lemma 2.4

We now present the proof of Lemma 2.4, which could be skipped at first reading. We insist on the fact that this proof is purely deterministic. We distinguish two cases depending on whether S_1 and S_2 are close to each other, i.e. at a distance at most $c(\kappa)L_{n-1}$, or not. The former case can be dealt with inductively (over n) and one can in fact create a good bridge involving k-boxes at levels $k \leq n-2$ only, essentially by recreating the picture of $\mathcal{G}_n$ at level $n-1$ well inside Σ_n. The case where S_1 and S_2 are further apart requires more work. In this case, the good bridge is constructed by concatenating three pieces: a "horizontal" *deck* and two *arches* (the terms will be introduced in the course of the proof). Roughly speaking, the deck consists of good boxes at level $n-1$ only, which goes most of the distance between S_1 and S_2, leaving only two "open" ends. The ends are filled by two *arches* joining S_1 and S_2, respectively, to a nearby good $(n-1)$-box from the deck. The arches are constructed hierarchically and consist of boxes at lower levels, which, among other things, need to satisfy the conditions **B3** and **B4**. This requires a good deal of care.

Throughout the proof, we set $\ell = 22\kappa$ and assume for simplicity that $x = 0$. Also, we introduce the notation $B_L(A) := \cup_{x \in A} B_L(x)$ for any subset A of $\mathbb{Z}^d$. Recall that $d(\cdot, \cdot)$ refers to the ℓ^∞-distance between sets and let $\mathrm{diam}(\cdot)$ denote to the ℓ^∞-diameter of a set. We first observe that, since the events $H_{m,0}$, $m > n$ occur by assumption in Lemma 2.4 it is sufficient to build a good bridge as in Definition 2.2 but with **G2** only required to hold for all m satisfying $n \vee 1 \leq m \leq n_{\max}$, rather than all $n \vee 1 \leq m$ (cf. (2.9) for the definition of $n_{\max}$ and note that $n_{\max}(\Lambda) = n$ when $\Lambda = \Lambda_n$).

Case 1: *the $n = 0$ case.*

Proof. For admissible $S_1, S_2 \subset \Lambda_0$, consider a nearest-neighbor path $\gamma \subset \{x \in \mathbb{L}_0 : B_{L_0}(x) \subset \Sigma_0\}$ of minimal length with starting point y such that $B_{L_0}(y) \cap S_1 \neq \emptyset$ and endpoint z such that $B_{L_0}(z) \cap S_2 \neq \emptyset$. The collection $\mathcal{B} = \{B_{L_0}(x) : x \in \gamma\}$ plainly satisfies **B1-B3**, and **B4** holds for all $K \geq c\kappa^d$, thus $\mathcal{B}$ is a bridge between S_1 and S_2 in Λ_0. Moreover if 0 is 0-good, i.e. $G_{0,0}(\Lambda_0)$ occurs, then by (2.4) $F_{0,x}$ occurs for each box $B_{L_0}(x) \in \mathcal{B}$, whence **G1** follows, and **G2** holds trivially since $n_{\max} = 0$. This concludes this case. $\square$

We now proceed by induction. From now on, we suppose that $n \geq 1$ and that the conclusion of the lemma is true for $n-1$ and consider any two admissible sets $S_1, S_2 \subset \Lambda_n$ (and assume for simplicity that $x = 0$). Define the random set

$$(2.11) \qquad \mathrm{Bad} := \mathrm{Fill}\Big(\bigcup_{x \in \mathbb{L}_{n-1}(\Lambda_n) : G^{\mathsf{c}}_{n-1,x}(\Lambda_n) \text{ occurs}} B_{L_{n-1}}(x) \Big) \subset \mathbb{Z}^d,$$

where for any $U \subset \mathbb{Z}^d$, $\mathrm{Fill}(U)$ refers to the smallest set $V \supseteq U$ such that for every point $z \in \partial V$, there exists an unbounded nearest-neighbor path in $\mathbb{Z}^d \setminus V$ starting in z. Let

$$(2.12) \qquad \begin{aligned} V_1 &:= \bigcup_{x : |x| = \lfloor 8.5\kappa L_n \rfloor + 10\ell L_{n-1}} B_{\ell L_{n-1}}(x), & V_1' &:= B_{\ell L_{n-1}}(V_1), \\ V_2 &:= \bigcup_{x : |x| = \lfloor 8.5\kappa L_n \rfloor + 20\ell L_{n-1}} B_{\ell L_{n-1}}(x), & V_2' &:= B_{\ell L_{n-1}}(V_2). \end{aligned}$$

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Provided ℓ_0 is large enough, we may ensure that $0.5\kappa\ell_0 \geq 30\ell$, so that V_1' and V_2' are both subsets of Σ_n, and $d(V_1', V_2') \geq 5\ell L_{n-1}$. Thus, if 0 is n-good, By definition of goodness $\mathrm{diam}(\mathrm{Bad}) \leq (\ell+2)L_{n-1}$ so that there exists $V \in \{V_1, V_2\}$ such that for every $x \in \mathbb{L}_{n-1}(V)$, the event $G_{n-1,x}(\Lambda_n)$ occurs.

Case 2: $n \geq 1$, $d(S_1 \cap V, S_2 \cap V) < 15\kappa L_{n-1}$.

Proof. By considering a path γ of $(n-1)$-boxes intersecting V joining $S_1 \cap V$ and $S_2 \cap V$ of minimal length, we find a box $B = B_{L_{n-1}}(x) \in \gamma$ with $x \in \mathbb{L}_{n-1}(V)$ such that $\underline{\widetilde{\Lambda}}_{n-1} := B_{8\kappa L_{n-1}}(x)$ intersects both S_1 and S_2. Let $\widetilde{\Lambda}_{n-1} := B_{10\kappa L_{n-1}}(x)$.

Since $G_{n-1,x}(\Lambda_n)$ occurs for every $x \in \mathbb{L}_{n-1}(V)$ and since $G_{n,x}(\Lambda) \subseteq G_{n,x}(\Lambda')$ whenever $\Lambda' \subseteq \Lambda$ (this can be checked easily by induction), if 0 is n-good then the event $G_{n-1,x}(\widetilde{\Lambda}_{n-1}) \supseteq G_{n-1,x}(\Lambda_n)$ occurs, hence the induction assumption implies that there exists a good bridge $\mathcal{B}$ between $\widetilde{S}_1$ and $\widetilde{S}_2$ in $\widetilde{\Sigma}_{n-1}$, where $\widetilde{S}_i = S_i \cap \widetilde{\Lambda}_{n-1}$, $i = 1, 2$, and $\widetilde{\Sigma}_{n-1} = B_{9\kappa L_{n-1}}(x) \setminus B_{\lfloor 8.5\kappa L_{n-1}\rfloor}(x)$ (to apply the induction hypothesis, one observes that the sets $\widetilde{S}_1$ and $\widetilde{S}_2$ are admissible for $(\widetilde{\Lambda}_{n-1}, \underline{\widetilde{\Lambda}}_{n-1}, \widetilde{\Sigma}_{n-1})$).

We proceed to verify that the bridge $\mathcal{B}$ hereby constructed is in fact a good bridge between S_1 and S_2 in Σ_n. First, as we now explain, $\mathcal{B}$ is a bridge between S_1 and S_2. Indeed, **B1**, **B2** and **B4** are easy to check. For **B3**, since $B \in \widetilde{\Sigma}_{n-1}$ for any $B \in \mathcal{B}$, it follows that

$$d\big(B, \bigcup_{i=1,2}(S_i \setminus \widetilde{S}_i)\big) \geq d\big(\widetilde{\Sigma}_{n-1}, \widetilde{\Lambda}_{n-1}^{\mathrm{c}},\big) \geq \kappa L_{n-1},$$

hence "adding back" $\bigcup_{i=1,2}(S_i \setminus \widetilde{S}_i)$ to form $S_1 \cup S_2$ does not produce additional constraints on the size of the boxes $B \in \mathcal{B}$ in **B3**. Thus $\mathcal{B}$ is a bridge between S_1 and S_2 inside Σ_n since $G_{n-1,x}(\Lambda_n)$ occurs for every $x \in \mathbb{L}_{n-1}(V)$.

It remains to argue that $\mathcal{B}$ is good. By definition of $G_{n,0}(\Lambda_n)$, the event $H_{n,z}$ occurs for the unique $z \in \mathbb{L}_n(\Lambda_n)$ such that $x \in B_{n,z}$. Together with the induction assumption, this implies **G1** and **G2**. This yields that $\mathcal{G}_n$ occurs as soon as 0 is n-good and concludes this case. $\square$

Case 3: $n \geq 1$, $d(S_1 \cap V, S_2 \cap V) \geq 15\kappa L_{n-1}$.

Proof. In this case, we have that if $W := V \setminus \bigcup_{x \in \partial V} B_{3\kappa L_{n-1}}(x)$ and $\widehat{S}_i := B(S_i \cap V, \kappa L_{n-1})$ (note that $\widehat{S}_i \cap W \neq \emptyset$ for $i = 1, 2$), then $d(\widehat{S}_1 \cap W, \widehat{S}_2 \cap W) \geq 10\kappa L_{n-1}$. Using the fact that $G_{n-1,x}(\Lambda_n)$ occurs for every $x \in V$ and that $G_{n,x}(\Lambda) \subseteq G_{n,x}(\Lambda')$ whenever $\Lambda' \subseteq \Lambda$, on the event $G_{n,0}(\Lambda_n)$ we can find a nearest-neighbor path $\gamma = (\gamma_1, \ldots, \gamma_N)$ of vertices in $\mathbb{L}_{n-1}(W)$ of minimal length such that, if $B_i := B_{L_{n-1}}(\gamma_i)$,

P1 $\bigcup_i B_i$ is connected, $B_i \cap (\widehat{S}_1 \cup \widehat{S}_2) = \emptyset$ for all $1 \leq i \leq N$, and $d(B_1, \widehat{S}_1), d(B_N, \widehat{S}_2) \leq 3L_{n-1}$,

P2 the events $G_{n-1,\gamma_i}(B_{3\kappa L_{n-1}}(\gamma_i))$ occur for all $1 \leq i \leq N$.

It remains to construct two suitable connections joining S_1 to B_1 and S_2 to B_N, respectively. This will be done via two (good) *arches*, defined below, whose existence is shown in Lemma 2.7. Together with the path γ, these will then yield the existence of a bridge $\mathcal{B}$ with the desired properties.

Let B be a k-box. We say that a collection $\mathcal{A}$ of n-boxes with $0 \leq n \leq k$ is an *arch* between U and B in Σ if **B1** holds with Σ in place of Σ_n, $B \in \mathcal{A}$ is the only k-box in $\mathcal{A}$, **B2** and **B3**

both hold with $S_1 = S_2 = U$ and $B_1 = B_2$ (and with $\mathcal{A}$ instead of $\mathcal{B}$), and **B4** holds with K in place of $2K$. An arch $\mathcal{A}$ will be called *good* if **G1** and **G2** hold (with $\mathcal{A}$ in place of $\mathcal{B}$).

Set $L_k^+ := L_k + L_{(k-1)\vee 0}$. The following lemma yields the existence of good arches.

Lemma 2.7. *For every $k \geq 0$ and $z \in \mathbb{L}_k$, if $G_{k,z}(B_{3\kappa L_k}(z))$ occurs, then with $B = B_{L_k}(z)$, for any set U with the property that*

$$(2.13) \quad \kappa L_k \leq d(U, B) \leq (\kappa + 3)L_k \text{ and every connected component of } U \text{ intersects } \partial B_{2\kappa L_k^+}(z),$$

there exists a good arch between U and B in $B_{2\kappa L_k}(z)$.

Assuming Lemma 2.7 holds, we first complete the proof of Lemma 2.4 in Case 3 (and with it that of Theorem 2.3). One applies Lemma 2.7 twice for $k = n - 1$, with $B = B_1$ and $U = U_1 := S_1 \cap V \cap \tilde{B}_1$ where $\tilde{B}_1 := B_{2\kappa L_k^+}(\gamma_1)$, respectively $B = B_N$ and $U = U_2 := S_2 \cap V \cap \tilde{B}_2$ where $\tilde{B}_2 := B_{2\kappa L_k^+}(\gamma_N)$. This is justified since the events $G_{k,\gamma_1}(B_{3\kappa L_k}(\gamma_1))$, $G_{k,\gamma_N}(B_{3\kappa L_k}(\gamma_N))$ occur by **P2** and both U_1, U_2 satisfy (2.13) due to **P1**, and the admissibility of the sets S_1 and S_2. Thus, Lemma 2.7 yields the existence of a good arch $\mathcal{A}_1$ between U_1 and B_1, as well as a good arch $\mathcal{A}_2$ between U_2 and B_N. Let

$$\mathcal{B} := \{B_j : 1 \leq j \leq N\} \cup \mathcal{A}_1 \cup \mathcal{A}_2.$$

We proceed to check that the collection $\mathcal{B}$ is the desired good bridge between S_1 and S_2. Properties **B1** and **B2** follow immediately from the corresponding properties of the arches $\mathcal{A}_1$, $\mathcal{A}_2$ and the definition of $\mathcal{B}$; in particular, $\bigcup_{B \in \mathcal{B}} B$ is connected. Property **B4** holds in the same way, noting that the number of $(n-1)$-boxes in $\mathcal{B}$ equals $N \leq |\mathbb{L}_{n-1}(W)| \leq c(\kappa \ell_0)^d \leq K$ provided K is chosen large enough (as a function of κ and ℓ_0).

We now turn to **B3**. The $(n-1)$-boxes $\{B_j : 1 \leq j \leq N\}$ in $\mathcal{B}$ are at a distance greater than κL_{n-1} from $(S_1 \cup S_2) \cap V$ thanks to **P1** and from $(S_1 \cup S_2) \setminus V$ due to the fact that, by definition of W, $d(B_i, (S_1 \cup S_2) \setminus V) > \kappa L_{n-1}$ for all $1 \leq i \leq N$. The boxes at lower levels inherit the corresponding property from the arch they belong to, as we now explain. Consider a box $B \in \mathcal{A}_i \setminus \{B_j, 1 \leq j \leq N\}$. Thus B is a k-box for some $k \leq n - 2$. But $U_i = (S_1 \cup S_2) \cap \tilde{B}_i$ since $\tilde{B}_i \subset V$ (as follows from the definitions), and $S_j \cap \tilde{B}_i = \emptyset$ for $j \neq i$ since $\tilde{B}_i \cap S_i \cap V \neq \emptyset$, $\tilde{B}_i$ has radius smaller than $3\kappa L_{n-1}$ and $d(S_1 \cap V, S_2 \cap V) \geq 15\kappa L_{n-1}$. Thus,

$$d(B, U_i) = d(B, (S_1 \cup S_2) \cap \tilde{B}_i) \overset{(2.13)}{\geq} \kappa L_k$$

and since $B \subset B_{2\kappa L_{n-1}}(\gamma_1) \cup B_{2\kappa L_{n-1}}(\gamma_N)$ by construction $d(B, (S_1 \cup S_2) \setminus \tilde{B}_i) \geq \kappa L_{n-2}$. It follows that $\kappa L_k \leq d(B, S_1 \cup S_2)$ as desired.

Finally, **G1**-**G2** are a consequence of the corresponding properties for the arches $\mathcal{A}_1$ and $\mathcal{A}_2$, **P2**, and the fact that $G_{n,0}(\Lambda_n)$ occurs (the latter to deduce that all the relevant events in H_n also do). This completes the proof of the third case, and therefore of Theorem 2.3 (subject to Lemma 2.7). $\qquad\square$

We conclude this section with the proof of Lemma 2.7.

Proof of Lemma 2.7. We assume for simplicity that $z = 0$. Set $B := B_{L_k}$, $\Sigma := B_{2\kappa L_k}$ and $\tilde{B} := B_{2\kappa L_k^+}$. We proceed by induction over k.

For $k = 0$, the collection $\mathcal{A}$ of 0-boxes corresponding to any nearest-neighbor path of 0-boxes in Σ joining U to B is an arch between U and B (note that $\Sigma \cap U \neq \emptyset$ by (2.13)). Moreover, in view

of (2.4), since $G_{0,z}(B_{3\kappa L_0})$ occurs by assumption, all the events $F_{0,x}$, $x \in \mathbb{L}_0(\Sigma)$ simultaneously occur, so that **G1** is satisfied. Since **G2** holds trivially (as $n_{\max} = 0$), $\mathcal{A}$ is a good arch.

We now assume that $k \geq 1$, and that the conclusions of the lemma hold for any $(k-1)$-box. Define $\overline{B} = B_{\kappa(2L_k - 3L_{k-1})}$, so that $B \subset \overline{B} \subset \Sigma$, and

$$(2.14) \qquad \mathrm{Bad}_{\overline{B}} := \mathrm{Fill}\Big(\bigcup_{y \in \mathbb{L}_{k-1}(\overline{B}):\, G^{\mathsf{c}}_{k-1,y}(B_{3\kappa L_{k-1}}(y)) \text{ occurs}} B_{L_{k-1}}(y) \Big).$$

Bounding the diameter of Bad as we did in Case 2, and noting that $B_{3\kappa L_{k-1}}(y) \subset B_{3\kappa L_k}$ for any $y \in \mathbb{L}_{k-1}(\overline{B})$, one deduces from (2.5) that $\mathrm{diam}(\mathrm{Bad}_{\overline{B}}) \leq (\ell+2)L_{k-1}$ on the event $G_{k,0}(B_{3\kappa L_k})$. For U satisfying (2.13), consider the disjoint sets

$$(2.15) \qquad \begin{aligned} V_1 &:= B, \\ V_2 &:= ((B_{(\kappa+3)L_{k-1}}(U) \setminus B_{\kappa L_{k-1}}(U)) \cap (\overline{B} \setminus \mathrm{Bad}_{\overline{B}}). \end{aligned}$$

The upper bound on $\mathrm{diam}(\mathrm{Bad}_{\overline{B}})$ implies that, whenever $G_{k,0}(B_{3\kappa L_k})$ occurs (which will henceforth be assumed implicitly), $\overline{B} \setminus \mathrm{Bad}_{\overline{B}}$ contains a connected component that intersects both V_1 and V_2 (for the latter, note that $\mathrm{diam}(V_i) \geq L_k$ for $i = 1, 2$ thanks to (2.13)). Hence, by (2.14), there exists a path γ in $\mathbb{L}_{k-1}(\overline{B} \setminus \mathrm{Bad}_{\overline{B}})$ such that $\bigcup_{y \in \gamma} B_{L_{k-1}}(y)$ intersects both V_1 and V_2 and the events $G_{k-1,y}(B_{3\kappa L_{k-1}}(y))$ occur for $y \in \gamma$. By choosing γ to have minimal length, none of the boxes $B_{L_{k-1}}(y)$ with $y \in \gamma$ intersect $B_{\kappa L_{k-1}}(U)$. For later purposes, record the collection

$$(2.16) \qquad \mathcal{A}' := \{V_1\} \cup \{B_{L_{k-1}}(y) : y \in \gamma\}.$$

Now, fix a vertex $y_0 \in \gamma$ such that $B' := B_{L_{k-1}}(y_0) \cap V_2 \neq \emptyset$ and consider $\tilde{B}' := B_{2\kappa L_{k-1}^+}(y_0)$. Since $y_0 \in \overline{B}$, we obtain that $\tilde{B}' \subset \Sigma$. The set $U' := U \cap \overline{B}'$ is easily seen to satisfy (2.13) with $k-1$ in place of k and B' replacing B. Because $G_{k-1,y_0}(B_{3\kappa L_{k-1}}(y_0))$ occurs, the induction assumption implies the existence of a good arch $\mathcal{A}''$ connecting $U'(\subset U)$ and B' inside $\overline{B}'(\subset \Sigma)$.

We claim that $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$ has the desired properties, i.e. it is a good arch between U and B inside Σ. Accordingly, we now argue that the (modified) conditions **B1**-**B4** and **G1**-**G2** for arches hold. Condition **B1** is immediate by construction. So is **B2** since $\mathcal{A}''$ is a good arch between U' and B', $U' \subset U$ and any box $B \in \mathcal{A}'$ does not intersect $B_{\kappa L_{k-1}}(U)$. Condition **B3** follows readily from the induction assumption (applied to the boxes in $\mathcal{A}''$) and the fact that, except for $B = V_1$ which is at the correct distance from U, $\mathcal{A}'$, cf. (2.16), only consists of $(k-1)$-boxes, none of which intersects $B_{\kappa L_{k-1}}(U)$, by definition of V_2 in (2.15) and construction of γ. For **B3**, the bound on N_m, $m \leq k-2$ follows by the induction assumption, and $N_{k-1} = |\gamma| \leq K$ provided K is chosen large enough, where we used that the boxes in γ are all contained in $\overline{B}$.

Finally, the modified conditions **G1** and **G2** for $n \leq k-2$ and $m \leq k-1$ are immediate (by the induction hypothesis), and the remaining cases i) $n = k-1$, $m = k-1$, and ii) $m = k$ (and n arbitrary) for **G2** follow from the occurrence of the events $G_{k-1,y}(B_{3\kappa L_{k-1}}(y))$, $y \in \gamma$, and $G_{k,z}(B_{3\kappa L_k})$. Overall, $\mathcal{A}$ is a good arch between U and B inside $\tilde{B}$, which completes the proof. $\qquad\square$

3 Decomposition of φ and "bridging lemma"

In this section, we gather several results that will be needed for both the proofs of Proposition 1.5 (Section 4) and Proposition 1.4 (Section 5). Among other things, we set up a certain

decomposition of the free field φ (Lemma 3.1) which will be used throughout, and prove a modified form of the "gluing" Lemma 1.6 (Lemma 3.5). The likelihood of the notion of "goodness" involved in the statement, see (3.15), will be guaranteed by an application of Theorem 2.3.

3.1 Decomposition of φ

Consider the graph with vertex set $\widetilde{\mathbb{Z}}^d = \mathbb{Z}^d \cup \mathbb{M}^d$, where $\mathbb{M}^d$ denotes the set of midpoints $\frac{x+y}{2}$, for $x, y \in \mathbb{Z}^d$ neighbors, with an edge joining every midpoint $m \in \mathbb{M}^d$ to each of the two vertices in $\mathbb{Z}^d$ at distance $\frac{1}{2}$ from m (each original edge is thereby split into two). Note that $\widetilde{\mathbb{Z}}^d$ is bipartite. Let $\widetilde{Q}$ be the transition operator (acting on $\ell^2(\widetilde{\mathbb{Z}}^d)$) for the simple random walk on $\widetilde{\mathbb{Z}}^d$, with transition kernel

$$\tilde{q}(\tilde{x}, \tilde{y}) = \frac{1}{|\{\tilde{z} \in \widetilde{\mathbb{Z}}^d : \tilde{z} \sim \tilde{x}\}|} 1\{\tilde{x} \sim \tilde{y}\}, \tag{3.1}$$

for $\tilde{x}, \tilde{y} \in \widetilde{\mathbb{Z}}^d$, where $\tilde{x} \sim \tilde{y}$ means that $\tilde{x}$ and $\tilde{y}$ are neighbors in $\widetilde{\mathbb{Z}}^d$, and write $\tilde{q}_\ell(\tilde{x}, \tilde{y}) = (\widetilde{Q}^\ell 1_{\tilde{y}})(\tilde{x})$, for $\ell \geq 0$.

Let $\mathbf{Z} = \{\mathbf{Z}_\ell(\tilde{z}) : \tilde{z} \in \widetilde{\mathbb{Z}}^d, \ell \geq 0\}$, denote a family of independent, centered, unit variance Gaussian random variables under the probability measure $\mathbb{P}$. For later reference, let τ_x, $x \in \mathbb{Z}^d$, denote the shifts on this space induced by $(\tau_x \mathbf{Z}_\ell)(\tilde{z}) = \mathbf{Z}_\ell(x + \tilde{z})$, for $\tilde{z} \in \widetilde{\mathbb{Z}}^d$, $\ell \geq 0$. We define the processes ξ^ℓ, $\ell \geq 0$ (and φ) alluded to in the introduction in terms of $\mathbf{Z}$ as follows. For each $\ell \in \{0, 1, 2, \dots\}$ and $x \in \mathbb{Z}^d$, let

$$\xi_x^\ell := c(\ell) \sum_{\tilde{z} \in \widetilde{\mathbb{Z}}^d} \tilde{q}_\ell(x, \tilde{z}) \mathbf{Z}_\ell(\tilde{z}), \tag{3.2}$$

with $c(\ell) = \sqrt{d/2}$ if ℓ is odd and $c(\ell) = \sqrt{1/2}$ if ℓ is even. Note that $\tilde{q}_0(\tilde{y}, \tilde{z}) = \delta(\tilde{y}, \tilde{z})$ so $\xi^0_\cdot = \mathbf{Z}_0(\cdot)/\sqrt{2}$ is an i.i.d. field indexed by $\mathbb{Z}^d$. The fields ξ^ℓ, $\ell \geq 0$ are independent, translation invariant centered Gaussian fields that have finite range:

$$\mathbb{E}[\xi_x^\ell \xi_y^\ell] = 0 \text{ for any } x, y \in \mathbb{Z}^d \text{ with } |x - y| > \ell, \tag{3.3}$$

which follows readily from (3.2). Our interest in ξ^ℓ stems from the following orthogonal decomposition of φ.

Lemma 3.1. *For every $\ell \geq 1$ and $x \in \mathbb{Z}^d$, $\mathrm{Var}(\xi_x^\ell) \leq C\ell^{-\frac{d}{2}}$. In particular, the series*

$$\varphi := \sum_{\ell \geq 0} \xi^\ell \tag{3.4}$$

converges in $L^2(\mathbb{P})$. Moreover, the convergence also holds $\mathbb{P}$-a.s. and the field φ is a Gaussian free field under $\mathbb{P}$.

Proof. One verifies that $\mathbb{E}[\xi_x^\ell \xi_y^\ell] = \frac{1}{2}\tilde{q}_{2\ell}(x, y)$, for all $\ell \geq 0$ and $x, y \in \mathbb{Z}^d$, using in case ℓ is odd that $\widetilde{Q}\widetilde{Q}^* = \frac{1}{d}\widetilde{Q}^2$, where $\widetilde{Q}^*$ denotes the adjoint of $\widetilde{Q}$, with kernel $q^*(\tilde{x}, \tilde{y}) = q(\tilde{y}, \tilde{x})$. One naturally identifies $q_\ell(x, y) := \tilde{q}_{2\ell}(x, y)$, for $x, y \in \mathbb{Z}^d$ as the transition kernel of a lazy simple random walk on $\mathbb{Z}^d$, which stays put with probability $\frac{1}{2}$ and otherwise jumps to a uniformly chosen neighbor at every step. One knows from the local central limit theorem that

$$q_\ell(x, x) \leq \frac{C}{\ell^{d/2}}, \quad \text{for all } \ell \geq 0 \text{ and } x \in \mathbb{Z}^d, \tag{3.5}$$

which implies the convergence in $L^2(\mathbb{P})$ in (3.4). The $\mathbb{P}$-a.s. convergence is then standard (e.g. as a consequence of Kolmogorov's maximal inequality). Finally, the previous observation also implies that

(3.6)
$$g(x,y) = \tfrac{1}{2} \sum_{\ell \geq 0} q_\ell(x,y) \overset{(3.4)}{=} \mathbb{E}[\varphi_x \varphi_y], \text{ for all } x,y \in \mathbb{Z}^d$$

with $g(\cdot,\cdot)$ as defined in (1.1), so φ defined by (3.4) is indeed a Gaussian free field. $\qquad\square$

We will tacitly work with the realization of φ given by (3.4), (3.2) throughout the remainder of this article. We now gather a few elementary properties of this setup. Denote the sequence of partial sums of ξ^ℓ's as

(3.7)
$$\varphi^L := \sum_{0 \leq \ell \leq L} \xi^\ell$$

and define for $\Lambda \subset \mathbb{Z}^d$,

(3.8)
$$\mathbf{Z}(\Lambda) := \big\{ \mathbf{Z}_\ell(\tilde{z}) : (\ell, \tilde{z}) \text{ s.t. } \tilde{q}_\ell(x, \tilde{z}) \neq 0 \text{ for some } x \in \Lambda \big\}.$$

By (3.2) and (3.4), $(\varphi_x)_{x \in \Lambda}$ is measurable with respect to $\mathbf{Z}(\Lambda)$. Moreover, on account of (3.3), for any $L \geq 0$,

(3.9)
$$(\varphi_x^L)_{x \in U} \text{ is independent of } \mathbf{Z}(V) \text{ whenever } d(U,V) > L.$$

We state below two simple lemmas which will be used repeatedly afterwards. The first one says that, up to a certain scale, it is easy to compare φ and φ^L; while the second gives a lower bound for point-to-point connections in a box at levels below h_{**}.

Lemma 3.2. *There exist $c, C > 0$ such that for every $\varepsilon > 0$ and $L, R \geq 1$,*

(3.10)
$$\mathbb{P}[|\varphi_x - \varphi_x^L| < \varepsilon , \ \forall x \in B_R] \geq 1 - C R^d e^{-c \varepsilon^2 L^{\frac{d-2}{2}}}.$$

Proof. Since $\varphi_x - \varphi_x^L$ is a centered Gaussian variable with variance at most $C L^{-\frac{d-2}{2}}$, the result follows from a simple union bound and a standard Gaussian tail estimate. $\qquad\square$

Lemma 3.3. *For every $h < h_{**}$, there exist $C_2 = C_2(d) > 0$ and $c_2 = c_2(d,h) > 0$ such that for every $L \geq 1$ and $x, y \in B_L$,*

(3.11)
$$\mathbb{P}[x \xleftrightarrow[B_{2L}]{\varphi^L \geq h} y] \geq c_2 L^{-C_2}.$$

Proof. For arbitrary $h < h_{**}$, let $\varepsilon := (h_{**} - h)/2$. By definition of h_{**}, see (1.4), we have

$$\mathbb{P}[B_L \xleftrightarrow{\varphi \geq h+\varepsilon} \partial B_{2L}] \geq c(h) > 0 \quad \forall L \geq 1.$$

A union bound over $x \in B_L$ and translation invariance thus imply that

$$\mathbb{P}[0 \xleftrightarrow{\varphi \geq h+\varepsilon} \partial B_L] \geq c'(h) L^{-(d-1)} \quad \forall L \geq 1.$$

By using arguments akin to those appearing in the proof of [17, Lemma 6.1], which involve only the FKG inequality and the invariance under reflections and permutation of coordinates, we deduce that

$$(3.12) \qquad \mathbb{P}[x \xleftrightarrow[B_{2L}]{\varphi \geq h+\varepsilon} y] \geq c''(h)L^{-C_2} \quad \forall x, y \in B_L.$$

Finally, Lemma 3.2 enables us to replace $\{\varphi \geq h + \varepsilon\}$ by $\{\varphi^L \geq h\}$ provided L is chosen large enough. This concludes the proof. $\qquad \square$

Remark 3.4. Following the same lines as the proof above, one can show that for any percolation model ω satisfying an FKG inequality and invariance under reflections and permutation of coordinates, if one has $\mathbb{P}[B_L \xleftrightarrow{\omega} \partial B_{2L}] \geq a > 0$ for all $L \leq R$, then for all $x, y \in B_L$ and $L \leq R$,

$$(3.13) \qquad \mathbb{P}[x \xleftrightarrow[B_{2L}]{\omega} y] \geq c_3(a)L^{-C_2}.$$

This will be useful in Section 5.

3.2 The "bridging lemma"

We now borrow the notation from Section 2. Recall the definition of the scales L_n, $n \geq 0$, from (1.18). We first choose $\kappa = 20$ and ℓ_0, K with $\ell_0 \geq 10\kappa$ large enough such that the conclusions of Theorem 2.3 hold whenever $L_0 \geq C_1(\kappa, \ell_0)$. The parameters κ, ℓ_0 and K will remain fixed throughout the remainder of this article. This will guarantee that all exponents ρ appearing in the following statements depend on d only.

For the rest of this section, we use the notation $\Lambda_n := B_{10\kappa L_n}$ as appearing in (2.1), along with the corresponding notion of admissible sets, see above (2.2). The use of annuli will not be necessary until Section 5. We now prove a result which is slightly different from Lemma 1.6 (see Remark 3.6, 2) below for a comparison between the two) and tailored to our later purposes.

Lemma 3.5 (Bridging). *For every $\varepsilon > 0$ and $L_0 \geq C_3(d, \varepsilon)$, there exist positive constants $\rho = \rho(d) > 0$, $c_4 = c_4(d, \varepsilon, L_0) > 0$ and $C_4 = C_4(d, \varepsilon, L_0) > 0$ such that the following holds. For all $n \geq 0$, there is a family of events $\mathcal{G}(S_1, S_2)$ indexed by $S_1, S_2 \subset \Lambda_n$, measurable and increasing with respect to $\mathbf{Z}(\Lambda_n)$, such that*

$$(3.14) \qquad \mathbb{P}\Big[\bigcap_{S_1, S_2} \mathcal{G}(S_1, S_2) \Big] \geq 1 - e^{-c_4 L_n^\rho},$$

and for every $h \leq h_{**} - 2\varepsilon$, every admissible S_1, S_2 and all events $D \in \sigma(1_{\varphi_x \geq h}; \ x \in S_1 \cup S_2)$ and $E \in \sigma(\mathbf{Z}(\Lambda_n^c))$,

$$(3.15) \qquad \mathbb{P}\Big[S_1 \xleftrightarrow[\Lambda_n]{\varphi \geq h-\varepsilon} S_2 \ \Big| \ \begin{matrix} D \cap E \\ \mathcal{G}(S_1, S_2) \end{matrix} \Big] \geq e^{-C_4 (\log L_n)^2}.$$

Remark 3.6. 1) We will apply Lemma 3.5 in Section 4 in order to connect, after sprinkling, two families of clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ inside of a ball Λ_n whenever the event $\mathcal{G}(S_1, S_2)$ occurs; cf. also Fig. 1. In this context, S_1 and S_2 will represent the explored regions of Λ_n when discovering $\mathcal{C}_1$ and $\mathcal{C}_2$ (i.e. $S_i = B(\mathcal{C}_i, 1) \cap \Lambda_n$, $i = 1, 2$); D will represent the information discovered inside this region; and E will represent all the information outside of Λ_n.

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2) One can derive Lemma 1.6 by following the proof of Lemma 3.5, with minor modifications. The differences between the two are the following: in Lemma 3.5, we require i) a certain measurability and monotonicity property of the events $\mathcal{G}(S_1, S_2)$ with respect to the σ-algebra $\mathbf{Z}(\Lambda_n)$ and ii) the bound (3.15) on the connection probability to hold for admissible sets, rather than sets with large diameter, cf. Remark 2.5, 2).

Proof. We start by defining events $\mathcal{G}(S_1, S_2)$ for which (3.14) holds. Let $\varepsilon > 0$ and $M, L_0 \geq 1$ to be chosen later. In the framework of Section 2, consider the event $\mathcal{G}_n$ (see (2.2)) given by the following choice of families of events H and F:

$$(3.16) \qquad F_{0,x} := \{\varphi_y^{L_0} - \varphi_y^0 \geq -M + \varepsilon, \quad \forall y \in B_{L_0}(x)\},$$

$$(3.17) \qquad H_{m,x} := \Big\{\varphi_y^{L_m} - \varphi_y^{L_{m-1}} \geq -\frac{6\varepsilon}{(\pi m)^2}, \quad \forall y \in B_{2L_m}(x)\Big\}.$$

Now, for any pair of admissible subsets S_1, S_2 of Λ_n, define

$$(3.18) \qquad \mathcal{G}(S_1, S_2) := \bigcup_{\mathcal{B}} \{\mathcal{B} \text{ is good}\},$$

where the union is taken over all the bridges between S_1 and S_2 inside Σ_n, see Definition 2.2 and around (2.1) for the relevant notions. For non-admissible S_1, S_2, set $\mathcal{G}(S_1, S_2) = \Omega$ (the full space on which $\mathbb{P}$ is defined). The events $\mathcal{G}(S_1, S_2)$ have the desired monotonicity property. In view of (2.2) and for later reference, we note that

$$(3.19) \qquad \bigcap_{S_1, S_2} \mathcal{G}(S_1, S_2) = \mathcal{G}_n$$

for the choice of families F and H in (3.16)–(3.17). We then assume (tacitly from here on) that $L_0 \geq C_1 \vee C(\varepsilon)$, so that the bounds in (**C3**) are respectively satisfied with $M = (\log L_0)^2$; to bound the probability of $H_{m,x}$, one simply applies a union bound and uses (3.5), or one uses (3.10) twice. Any choice of $M = M(L_0)$ yielding (**C3**) for the collection F would work. It follows from (3.2) and (3.3) that the families F and H in (3.16) and (3.17) satisfy (**C1**) and (**C2**) (recall that $\kappa = 20$). Hence, Theorem 2.3 applies and yields (3.14).

The choices (3.16), (3.17) and (3.18), along with **G1**, **G2** in Definition 2.2 imply the following property, which will be used repeatedly in the sequel. For any good bridge $\mathcal{B}$ and any m-box $B = B_{L_m}(x) \in \mathcal{B}$ ($m \geq 0$), the following holds:

$$(3.20) \qquad \begin{aligned} \varphi_z - \varphi_z^0 &\geq -M, \quad \forall z \in B_{L_m}(x), \quad \text{if } m = 0, \\ \varphi_z - \varphi_z^{L_m} &\geq -\varepsilon, \quad \forall z \in B_{2L_m}(x), \quad \text{if } m \geq 1. \end{aligned}$$

We now turn to the proof (3.15). Consider a bridge $\mathcal{B}$ between a pair of (admissible) sets S_1 and S_2 in Λ_n. It follows directly from Definition 2.1 that one can find vertices $s_1 \in S_1 \cap B_1$, $s_2 \in S_2 \cap B_2$ (recall B_1 and B_2 from **B2**) and $x_B, y_B \in B$ for each $B \in \mathcal{B}$ so that for any family of paths $(\pi_B)_{B \in \mathcal{B}}$ between x_B and y_B, the union of s_1, s_2 and $(\pi_B)_{B \in \mathcal{B}}$ forms a path connecting S_1 and S_2, cf. also Fig. 1. We can further impose that, for $B \in \{B_1, B_2\}$, the vertices x_B, y_B are chosen in such a way that there exists a path $\pi_B \subset B \setminus (S_1 \cup S_2)$ between x_B and y_B (in particular, $x_B, y_B \notin S_1 \cup S_2$). For each $B = B_{L_m}(x)$, $x \in \mathbb{L}_m$, consider the event

$$(3.21) \qquad A_B := \begin{cases} \{x_B \xleftrightarrow[B_{L_m}(x)]{\varphi^0 \geq h + M} y_B\}, & \text{if } m = 0, \\[2mm] \{x_B \xleftrightarrow[B_{2L_m}(x)]{\varphi^{L_m} \geq h + \varepsilon} y_B\}, & \text{if } m \geq 1. \end{cases}$$

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Then it follows directly from (3.20) that x_B and y_B are connected in $\{\varphi \geq h\} \cap B_{2L_m}(x)$ if $B \in \mathcal{B}$ and A_B occurs.

By these observations, we deduce that for any pair of admissible sets S_1, S_2, any events D, E as above (3.15), and any bridge $\mathcal{B}$ inside Σ_n between S_1 and S_2,

(3.22)
$$\mathbb{P}\Big[\{S_1 \xleftrightarrow[\Lambda_n]{\varphi \geq h - \varepsilon} S_2\} \cap D \cap E \cap \mathcal{G}(S_1, S_2)\Big] \geq \mathbb{P}\Big[\{S_1 \xleftrightarrow[\Lambda_n]{\varphi \geq h - \varepsilon} S_2\} \cap D \cap E \cap \{\mathcal{B} \text{ is good}\}\Big]$$
$$\geq \mathbb{P}\Big[D \cap E \cap \{\mathcal{B} \text{ is good}\} \cap \{\varphi_{s_1}, \varphi_{s_2} \geq h - \varepsilon\} \bigcap_{B \in \mathcal{B}} A_B\Big]$$

(also, note for the last inequality that the path in $\{\varphi \geq h - \varepsilon\}$ connecting S_1 and S_2 in the second line is indeed contained in Λ_n since $\mathcal{B}$ itself lies in Σ_n, cf. (2.1) and (3.21)). Conditioning on all the random variables $\mathbf{Z}_\ell(\tilde{z})$, $\tilde{z} \in \widetilde{\mathbb{Z}}^d$, $\ell \geq 0$, except $\mathbf{Z}_0(s_1)$ (proportional to $\varphi_{s_1}^0$) and $\mathbf{Z}_0(s_2)$, and noticing that $\mathcal{B}$ being good implies that $\varphi_{s_i} - \varphi_{s_i}^0 \geq -M$ for $i = 1, 2$, one easily deduces that

(3.23)
$$\mathbb{P}\Big[D \cap E \cap \{\mathcal{B} \text{ is good}\} \cap \{\varphi_{s_1}, \varphi_{s_2} \geq h - \varepsilon\} \bigcap_{B \in \mathcal{B}} A_B\Big] \geq c\mathbb{P}\Big[D \cap E \cap \{\mathcal{B} \text{ is good}\} \bigcap_{B \in \mathcal{B}} A_B\Big],$$

where
$$c = c(d, \varepsilon, L_0) := \inf_{a \geq -M} \inf_{h \leq h_{**}} \mathbb{P}[\varphi_0^0 + a \geq h - \varepsilon | \varphi_0^0 + a < h]^2 > 0.$$

Now, by (3.2), (3.7) and (3.20), (3.21), conditionally on $\mathbf{Z}(\Lambda_n^c \cup S_1 \cup S_2)$, the events $\{\mathcal{B} \text{ is good}\}$ and $(A_B)_{B \in \mathcal{B}}$ are all increasing in the remaining random variables from $\mathbf{Z}$. Also, $D \cap E$ is measurable with respect to $\mathbf{Z}(\Lambda_n^c \cup S_1 \cup S_2)$ and A_B is independent of $\mathbf{Z}(\Lambda_n^c \cup S_1 \cup S_2)$ for all $B \in \mathcal{B} \backslash \{B_1, B_2\}$ because of **B2** and **B3**. Together with the FKG inequality for the i.i.d. random variables in $\mathbf{Z}$, these observations imply that for suitable c', c'' depending on d, ε, and L_0,

$$\mathbb{P}\Big[D \cap E \cap \{\mathcal{B} \text{ is good}\} \bigcap_{B \in \mathcal{B}} A_B\Big]$$
$$= \mathbb{E}\Big[1_{D \cap E} \, \mathbb{P}\Big[\{\mathcal{B} \text{ is good}\} \bigcap_{B \in \mathcal{B}} A_B \Big| \mathbf{Z}(\Lambda_n^c \cup S_1 \cup S_2)\Big]\Big]$$

(3.24)
$$\geq \mathbb{E}\Big[1_{D \cap E} \mathbb{P}[\mathcal{B} \text{ is good}|\mathbf{Z}(\Lambda_n^c \cup S_1 \cup S_2)] \prod_{B \in \mathcal{B}} \mathbb{P}[A_B|\mathbf{Z}(\Lambda_n^c \cup S_1 \cup S_2)]\Big]$$
$$\geq c'\mathbb{P}[D \cap E \cap \{\mathcal{B} \text{ is good}\}] \prod_{B \in \mathcal{B} \backslash \{B_1, B_2\}} \mathbb{P}[A_B]$$
$$\geq c'' e^{-C(\varepsilon)(\log L_n)^2} \, \mathbb{P}[D \cap E \cap \{\mathcal{B} \text{ is good}\}],$$

where in the fourth line, we used that $\prod_{i=1,2} \mathbb{P}[A_{B_i}|\mathbf{Z}(\Lambda_n^c \cup S_1 \cup S_2)] \geq c'$, which follows from the existence of a path $\pi_{B_i} \subset B_i \setminus (S_1 \cup S_2)$ between x_{B_i} and y_{B_i} together with the fact that φ^0 is an i.i.d. field. In the last line we used that for any m-box B with $m \geq 1$ one has

$$\mathbb{P}[A_B] \geq c_2 (L_m)^{-C_2} = c_2 e^{-C_2(\log L_m)},$$

where $c_2 = c_2(d, h_{**} - \varepsilon) > 0$ and $C_2 > 0$ are given by Lemma 3.3 (remind that $h + \varepsilon \leq h_{**} - \varepsilon$); while for $m = 0$, we simply bounded $\mathbb{P}[A_B] \geq c(L_0, d, \varepsilon) > 0$ using the finite energy of φ^0. The

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last line of (3.24) then follows from the fact that

$$\sum_{\substack{B \in \mathcal{B} \\ m\text{-box}}} C_2 \log L_m \leq 2C_2 K \sum_{0 \leq m \leq n} (\log L_m) \leq C(\log L_n)^2,$$

which relies on **B4**. Combining (3.22), (3.23) and (3.24), we conclude that

$$\mathbb{P}[\{S_1 \xleftrightarrow[\Lambda_n]{\varphi \geq h-\varepsilon} S_2\} \cap D \cap E \cap \mathcal{G}(S_1, S_2)] \geq e^{-C(\log L_n)^2} \, \mathbb{P}[D \cap E \cap \{\mathcal{B} \text{ is good}\}],$$

where C depends on d, ε and L_0. Summing this inequality over the at most

$$\prod_{0 \leq k \leq n} (CL_n/L_k)^{2dK} \leq \exp[C'(\log L_n)^2]$$

possible bridges $\mathcal{B}$ between S_1 and S_2 gives

$$\mathbb{P}[\{S_1 \xleftrightarrow[\Lambda]{\varphi \geq h-\varepsilon} S_2\} \cap D \cap E \cap \mathcal{G}(S_1, S_2)] \geq e^{-C''(\log L_n)^2} \, \mathbb{P}[D \cap E \cap \mathcal{G}(S_1, S_2)],$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.7. Retracing the steps of the above proof and imposing the occurrence of $\bigcap_{B \in \mathcal{B}} A_B$ but not of $\{\varphi_{s_1}, \varphi_{s_2} \geq h - \varepsilon\}$ implies a connection between the 1-neighborhoods of S_1 and S_2 *without* an ε-sprinkling. In other words, Lemma 3.5 continues to hold with (3.15) replaced by

$$(3.25) \qquad \mathbb{P}\Bigg[\bigcup_{\substack{s_1 \in S_1 \\ s_2 \in S_2}} \{\mathcal{N}(s_1) \xleftrightarrow[\Lambda_n \backslash (S_1 \cup S_2)]{\varphi \geq h} \mathcal{N}(s_2)\} \cap \mathcal{H}_{s_1} \cap \mathcal{H}_{s_2} \,\Big|\, F \cap \mathcal{G}(S_1, S_2) \Bigg] \geq e^{-C_4(\log L_n)^2},$$

for all $h < h_{**} - 2\varepsilon$, $L_0 \geq C(\varepsilon)$, and $F \in \sigma(\mathbf{Z}(\Lambda_n \cup S_1 \cup S_2))$, where $\mathcal{H}_y := \{\varphi_y - \varphi_y^0 \geq -M\}$ with $M = M(L_0)$ suitably large (as chosen below (3.18)) and $\mathcal{N}(x) := \{y \in \mathbb{Z}^d : |y - x|_1 \leq 1\}$. We will use the kind of events appearing in (3.25) in Section 5.

4 Local uniqueness regime

This section deals with Proposition 1.5, whose proof is split into three parts. In Section 4.1 we prove Proposition 4.1, which roughly asserts that for $h < \tilde{h}$ with $\tilde{h}$ given by (1.12), crossing clusters inside an annulus are typically connected in $\{\varphi \geq h - \varepsilon\}$. This is then used in Section 4.2 to trigger a renormalization and thereby deduce that $\{\varphi \geq h\}$ has a "ubiquitous" cluster inside large boxes for all values of $h < \tilde{h}$ with very high probability, see Proposition 4.4. Finally in Section 4.3, we use the previous result in order to conclude the proof of Proposition 1.5 by proving the desired stretched-exponential decay of the probabilities defining $\bar{h}$ in (1.8). Some care is needed because one ultimately wants to avoid any sprinkling for the local uniqueness event. The last part of the argument would simplify if one worked with a weaker notion of $\bar{h}$ as in [64] involving sprinkling for the uniqueness event (1.7), see Remark 4.5 below.

4.1 From connection to local uniqueness

We start by defining a certain "unique crossing" event $\mathcal{E}$: given any $\alpha > \beta$, let
(4.1)

$$\mathcal{E}(N, \alpha, \beta) := \{B_N \xleftrightarrow{\varphi \geq \alpha} \partial B_{6N}\} \cap \left\{ \begin{array}{c} \text{all clusters in } \{\varphi \geq \alpha\} \cap B_{4N} \text{ crossing } B_{4N} \setminus B_{2N} \\ \text{are connected to each other in } \{\varphi \geq \beta\} \cap B_{4N} \end{array} \right\}.$$

Notice that unlike Unique$(2N, \alpha)$ defined in (1.7), the corresponding event in $\mathcal{E}(N, \alpha, \beta)$ involves a sprinkling.

Proposition 4.1. *For every $\varepsilon > 0$ one has*

(4.2)
$$\limsup_{N \to \infty} \inf_{h \leq \tilde{h} - 2\varepsilon} \mathbb{P}[\mathcal{E}(N, h, h - \varepsilon)] = 1.$$

The idea of the proof is roughly the following. We first require that all the balls of size $B_{u(N)}$ inside B_{4N} are connected to distance N, which happens with probability converging to 1 along a subsequence of values of N since $h < \tilde{h}$. On this event, the picture we see at level h inside the ball B_N is that of an "almost everywhere percolating" subgraph: every vertex is at distance at most $u(N) \ll N$ from some macroscopic cluster in B_N. In other words, the union of all macroscopic clusters form a $u(N)$-dense subset of the ball B_{4N}. The goal is then to adapt the techniques from [12] in order to show that after an ε-sprinkling, all such clusters will be connected together. In order to implement this adaptation we need some kind of "sprinkling property" stating that conditionally on the configuration at level h, there is a decent probability of making extra connections at level $h - \varepsilon$. As explained in the introduction, the level sets of φ do *not* have such property and this issue will be overcome by applying Lemma 3.5.

Proof of Proposition 4.1. Fix $\varepsilon > 0$ and take any $h \leq \tilde{h} - 2\varepsilon$. We import the notation and definitions from Section 3. We fix $L_0 = L_0(\varepsilon)$ large enough such that the conclusions of Lemma 3.5 hold (recall that $h \leq \tilde{h} - 2\varepsilon \leq h_{**} - 2\varepsilon$). We say that a ball $\Lambda_n = B_{10\kappa L_n}(x)$ for some $x \in \mathbb{Z}^d$ is *good* if the event $\mathcal{G}_{n,x} := \tau_x \mathcal{G}_n$ happens with $\mathcal{G}_n$ as in (3.19). Throughout the remainder of this section, all constants c, C may depend implicitly on ε.

Let $n_0 := \min\{n : L_n \geq u(10N)\}$, $h \in I$ and consider the events

(4.3)
$$A := \{B_{L_{n_0}}(x) \xleftrightarrow{\varphi \geq h} \partial B_{6N} \text{ for all } x \text{ s.t. } B_{L_{n_0}}(x) \subset B_{4N}\},$$

(4.4)
$$G := \{B_{10\kappa L_{n_0}}(x) \text{ is good for all } x \in \mathbb{L}_{n_0}(B_{4N})\}$$

(see above (2.4) for notation). Clearly, if A does not occur, then there must be $x \in B_{4N}$ such that $B_{u(10N)}(x)$ is not connected to ∂B_{6N} in $\{\varphi \geq h\}$. Now since $B_{6N} \subset B_{10N}(x)$ for any $x \in B_{4N}$, it follows directly that $B_{u(10N)}(x)$ is not connected to $\partial B_{10N}(x)$ for some $x \in B_{4N}$ on the complement of A. Consequently, by translation invariance,

(4.5)
$$\mathbb{P}[A] \geq 1 - CN^d \mathbb{P}[B_{u(10N)} \xcancel{\xleftrightarrow{\varphi \geq h}} \partial B_{10N}].$$

At the same time, it follows from Lemma 3.5 that for any N sufficiently large,

(4.6)
$$\mathbb{P}[G] \geq 1 - e^{-c'_4 u(N)^\rho}.$$

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Define the collection

(4.7) $\qquad \mathcal{C} := \{C \subset B_{4N} : C \text{ a cluster in } \{\varphi \geq h\} \cap B_{4N} \text{ intersecting } \partial B_{4N}\},$

and for any percolation configuration $\omega \in \{0,1\}^{\mathbb{Z}^d}$ with $\{\omega = 1\} \supset \{\varphi \geq h\}$, let

(4.8) $\qquad\qquad\qquad C \sim_\omega C' \text{ if } C \overset{\omega}{\longleftrightarrow} C', \text{ for } C, C' \in \mathcal{C}.$

The relation $\sim_\omega$ defines an equivalence relation on any $\tilde{\mathcal{C}} \subset \mathcal{C}$. The elements of $\tilde{\mathcal{C}}/\sim_\omega$ thus form a partition of $\tilde{\mathcal{C}}$, whereby clusters of $\tilde{\mathcal{C}}$ which are connected in the configuration ω get grouped. For every $0 \leq i \leq \lfloor 2\sqrt{N} \rfloor$, let $V_i := B_{4N - i\sqrt{N}}$. We will study the sets

(4.9) $\qquad\qquad\qquad \mathcal{U}_i(\omega) := \{C \in \mathcal{C} : C \cap V_{2i} \neq \emptyset\}/\sim_\omega$

for $0 \leq i \leq \lfloor \sqrt{N} \rfloor$ and $\{\omega = 1\} \supset \{\varphi \geq h\}$. We denote by $U_i(\omega) = |\mathcal{U}_i(\omega)|$ and will frequently rely on the fact that $U_i(\omega)$ is decreasing in both ω and i, as apparent from (4.9). We will use $\mathscr{C}$ in the sequel to denote groups of clusters of $\mathcal{C}$, e.g. elements of $\mathcal{U}_i(\omega)$, and more generally of $2^{\mathcal{C}}$. It will be convenient to write $\text{supp}(\mathscr{C}) = \bigcup_{C \in \mathscr{C}} C \subset \mathbb{Z}^d$, for $\mathscr{C} \in 2^{\mathcal{C}}$. Now, for $0 \leq i \leq \lfloor \sqrt{N} \rfloor$, introduce the percolation configurations

(4.10) $\qquad\qquad \omega_0 \leq \omega_1 \leq \dots, \quad \text{where } \omega_i = \omega_i(\varphi) := \begin{cases} 1_{\{\varphi \geq h\}}, & x \in V_{2i}, \\ 1_{\{\varphi \geq h - \varepsilon\}}, & x \notin V_{2i}, \end{cases}$

so $\omega_i \in \{0,1\}^{\mathbb{Z}^d}$ corresponds to a partial sprinkling outside of V_{2i}, and set

(4.11) $\qquad\qquad\qquad \mathcal{U}_i := \mathcal{U}_i(\omega_i), \quad U_i := |\mathcal{U}_i|, \quad 0 \leq i \leq \lfloor \sqrt{N} \rfloor.$

Note that U_i is decreasing in i. In view of (4.1), (4.3) and (4.9)–(4.11), the event $\mathcal{E}(N, h, h - \varepsilon)$ occurs as soon as A does and $U_{\lfloor \sqrt{N} \rfloor} = 1$. Hence, (4.5) and (4.6) give that

(4.12)
$$\mathbb{P}[\mathcal{E}(N, h, h - \varepsilon)^c] \leq \mathbb{P}[A^c] + \mathbb{P}[G^c] + \mathbb{P}[A \cap G \cap \{U_{\lfloor \sqrt{N} \rfloor} > 1\}]$$
$$\leq CN^d \sup_{h \in I} \mathbb{P}[B_{u(10N)} \overset{\varphi \geq h}{\longleftrightarrow\!\!\!\!/} \partial B_{10N}] + e^{-c_4' u(N)^\rho} + \mathbb{P}[A \cap G \cap \{U_{\lfloor \sqrt{N} \rfloor} > 1\}].$$

By the definition of $\tilde{h}$ and the monotonicity of the disconnection event with respect to h, the first two terms on the right-hand side of (4.12) converge to 0 uniformly in $h \leq \tilde{h} - 2\varepsilon$ along a subsequence $N_k \to \infty$. As a consequence, it suffices to prove that the last term tends to 0 uniformly in $h \leq \tilde{h} - 2\varepsilon$ as $N \to \infty$ to conclude. The proof will be based on the following lemma.

Lemma 4.2. *There exists a constant $c = c(\varepsilon) > 0$ such that for any $h \leq \tilde{h} - 2\varepsilon$, $N \geq 1$, any $a \in \mathbb{N}$ with $4 \leq a \leq N^{1/4}$ and any $0 \leq i \leq \lfloor \sqrt{N} \rfloor - a$,*

(4.13) $\qquad\qquad \mathbb{P}[A \cap G \cap \{U_{i+a} > 1 \vee 2U_i/a\}] \leq \exp(-cN^{1/4}).$

Admitting Lemma 4.2, we first finish the proof of Proposition 4.1. Observe that, if the event $\bigcap_{0 \leq k < M} \{U_{(k+1)a} \leq 1 \vee \frac{2U_{ka}}{a}\}$ occurs for some $M \geq 1$ and an a as appearing in Lemma 4.2 with $aM \leq \lfloor \sqrt{N} \rfloor$, then either

(4.14) $\qquad U_{\lfloor \sqrt{N} \rfloor} \leq U_{Ma} \leq \frac{2}{a} U_{(M-1)a} \leq \dots \leq \left(\frac{2}{a}\right)^M U_0 = \left(\frac{2}{a}\right)^M |\mathcal{C}| \overset{(4.7)}{\leq} C\left(\frac{2}{a}\right)^M N^{d-1},$

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or $U_{(k+1)a} \leq 1$ for some $0 \leq k < M$, in which case $U_{\lfloor\sqrt{N}\rfloor} \leq U_{(k+1)a} \leq 1$ by monotonicity. Thus, letting $M = \lfloor (C' \log N / \log a) \rfloor$, with $C' = C'(d)$ chosen large enough so that the right-hand side of (4.14) is bounded by 1, we deduce that $A \cap G \cap \{U_{\lfloor\sqrt{N}\rfloor} > 1\}$ implies the event

$$\bigcup_{0 \leq k < M} A \cap G \cap \{U_{(k+1)a} \geq 1 \vee 2U_{ka}/a\}.$$

Applying a union bound over k, choosing say, $a = 4$, (4.13) readily yields that

$$(4.15) \qquad \mathbb{P}\big[A \cap G \cap \{U_{\lfloor\sqrt{N}\rfloor} > 1\}\big] \leq M \exp(-cN^{1/4}).$$

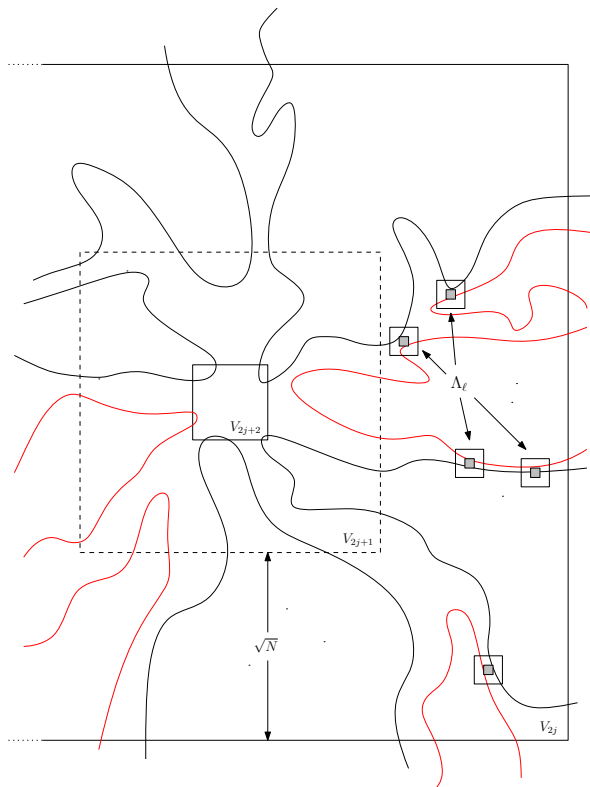Proposition 4.1 then follows immediately from (4.12) and (4.15). $\qquad\qquad\square$

Figure 2 – Some of the clusters in $\mathcal{C}$. On the event A, these clusters are $u(N)$-dense in the annulus $V_{2j} \setminus V_{2j+2}$. Each of the boxes $\tilde{\Lambda}_\ell$ (grey) is intersected by both the clusters in the support of $\widetilde{\mathcal{U}}_1$ (black) and $\widetilde{\mathcal{U}}_2$ (red), see (4.20); the picture corresponds to Case 2, i.e. $|\mathcal{U}_{j+\frac{1}{2},j+1}(\omega_j)| \neq 0$ in the arguments following (4.19)–(4.20). When G occurs, each box Λ_ℓ provides the opportunity to link two admissible pieces of $\widetilde{C}_1$ and $\widetilde{C}_2$ using a good bridge at a cost given by Lemma 3.5.

It remains to give the proof of Lemma 4.2.

Proof of Lemma 4.2. We begin with a reduction step. For $\{\omega = 1\} \supset \{\varphi \geq h\}$, $0 \leq i < \lfloor\sqrt{N}\rfloor$ and $k \in \{0, \frac{1}{2}\}$, let

$$(4.16) \qquad \mathcal{U}_{i+k,i+1}(\omega) := \{\mathscr{C} \in \mathcal{U}_i(\omega) : \operatorname{supp}(\mathscr{C}) \cap V_{2(i+1)} = \emptyset,\ \operatorname{supp}(\mathscr{C}) \cap V_{2(i+k)} \neq \emptyset\},$$

and set $U_{i,i+1}(\omega) := |\mathcal{U}_{i,i+1}(\omega)|$. Note that $U_{i,i+1}(\omega)$ is decreasing in ω. Moreover, since the elements of $\mathcal{U}_i(\omega) \setminus \mathcal{U}_{i,i+1}(\omega)$ each contain a cluster $C \in \mathcal{C}$ intersecting $V_{2(i+1)}$, the map $\psi : \mathcal{U}_i(\omega) \setminus \mathcal{U}_{i,i+1}(\omega) \to \mathcal{U}_{i+1}(\omega)$ given by $\psi(\mathscr{C}) := \mathscr{C} \setminus \{C \in \mathcal{C} : C \cap V_{2(i+1)} = \emptyset\}$ is a bijection. Hence, $U_i(\omega) = U_{i+1}(\omega) + U_{i,i+1}(\omega)$ and by iteration

$$U_{i+a}(\omega_i) + \sum_{j : i \leq j < i+a} U_{j,j+1}(\omega_i) = U_i(\omega_i) = U_i,$$

whence $U_{j,j+1}(\omega_i) \leq U_i/a$ for some j with $i \leq j < i+a$. Together with a union bound, we see that (4.13) follows at once if we can show that

$$(4.17) \qquad \mathbb{P}\big[A \cap G \cap \{U_{j+1} > 1 \vee (U_j/a + U_{j,j+1}(\omega_j))\}\big] \leq \exp(-c'N^{1/4}),$$

for all $0 \leq i \leq \lfloor \sqrt{N} \rfloor - a$ and $i \leq j < i+a$. To see this, simply notice that $U_{i+a} = U_{i+a}(\omega_{i+a}) \leq U_{j+1}(\omega_{j+1}) = U_{j+1}$ by monotonicity since $i + a \geq j + 1$, and similarly that $U_i \geq U_j$ and $U_{j,j+1}(\omega_i) \geq U_{j,j+1}(\omega_j)$, for all $i \leq j$.

We now prove (4.17) for all $0 \leq j < \lfloor \sqrt{N} \rfloor$. Fix any such j and let E denote the event on the left-hand side of (4.17). Recalling (4.16), we introduce

$$(4.18) \qquad \widetilde{\mathcal{U}}(\omega_j) := \begin{cases} \mathcal{U}_j(\omega_j) \setminus \mathcal{U}_{j,j+1}(\omega_j), & \text{if } \mathcal{U}_{j+\frac{1}{2},j+1}(\omega_j) = \emptyset, \\ (\mathcal{U}_j(\omega_j) \setminus \mathcal{U}_{j,j+1}(\omega_j)) \cup \{\widetilde{\mathscr{C}}\}, & \text{otherwise,} \end{cases}$$

where $\widetilde{\mathscr{C}} := \{C : C \in \mathscr{C} \text{ for some } \mathscr{C} \in \mathcal{U}_{j,j+1}(\omega_j)\}$ is obtained by merging the elements of $\mathcal{U}_{j,j+1}(\omega_j)$. We drop the argument ω_j in the sequel and proceed to verify that $\widetilde{\mathcal{U}}$ has the following properties: on the event E,

$$(4.19) \qquad \begin{aligned} &\text{for } A_j = V_{2j} \setminus V_{2j+1} \text{ or } A_j = V_{2j+1} \setminus V_{2j+2}, \text{ each of the sets } \text{supp}(\mathscr{C}), \text{ with } \mathscr{C} \in \widetilde{\mathcal{U}}, \\ &\text{crosses } A_j \text{ and their union intersects all the balls of radius } L_{n_0} \text{ contained in } A_j, \end{aligned}$$

$$(4.20) \qquad \exists \text{ a non-trivial partition } \widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}_1 \sqcup \widetilde{\mathcal{U}}_2 \text{ s.t. } |\{\mathscr{C} : \mathscr{C} \in \widetilde{\mathcal{U}}_1\}| \leq a \text{ and } \{\widetilde{C}_1 \overset{\omega_{j+1}}{\nleftrightarrow} \widetilde{C}_2\},$$

where $\widetilde{C}_i = \bigcup_{\mathscr{C} \in \widetilde{\mathcal{U}}_i} \text{supp}(\mathscr{C})$. We first check that (4.19) holds with the choice $A_j = V_{2j+1} \setminus V_{2j+2}$ when $|\mathcal{U}_{j+\frac{1}{2},j+1}| = 0$ (henceforth referred to as Case 1) and $A_j = V_{2j} \setminus V_{2j+1}$ when $|\mathcal{U}_{j+\frac{1}{2},j+1}| \neq 0$ (Case 2). Indeed, in either case each $\mathscr{C} \in \mathcal{U}_j \setminus \mathcal{U}_{j,j+1}$ contains a cluster C crossing $V_{2j} \setminus V_{2(j+1)}$, see (4.9) and (4.16). Moreover, the assumption $|\mathcal{U}_{j+\frac{1}{2},j+1}| \neq 0$ of Case 2 implies that $\widetilde{\mathscr{C}}$ defined below (4.18) contains a cluster C crossing $V_{2j} \setminus V_{2j+1}$.

To conclude that (4.19) holds, it thus remains to check that all the L_{n_0}-balls in A_j are intersected by the set $\bigcup_{\mathscr{C} \in \widetilde{\mathcal{U}}} \text{supp}(\mathscr{C})$. First note that on the event $E \subset A$ (recall (4.3)), by definition of $\mathcal{U}_j$ each such ball is intersected by $\text{supp}(\mathscr{C})$, for some $\mathscr{C} \in \mathcal{U}_j$. Since each $\mathscr{C} \in \mathcal{U}_j$ belongs to a group of $\widetilde{\mathcal{U}}$ in Case 2, the claim immediately follows. In Case 1, the assumption $|\mathcal{U}_{j+\frac{1}{2},j+1}(\omega_j)| = 0$ implies that none of the sets $\text{supp}(\mathscr{C})$, $\mathscr{C} \in \mathcal{U}_{j,j+1}$, intersects $V_{2j+1} \setminus V_{2j+2} = A_j$ and (4.19) follows as well.

We now argue that (4.20) holds. It suffices to show that in either case,

$$(4.21) \qquad \text{on } E, \quad |\widetilde{\mathcal{U}}/ \sim_{\omega_{j+1}}| \geq \frac{|\widetilde{\mathcal{U}}|}{a},$$

where, with hopefully transparent notation, we extend the relation (4.8) by declaring $\mathscr{C} \sim_\omega \mathscr{C}'$ for arbitrary $\mathscr{C}, \mathscr{C}' \in 2^{\mathcal{C}}$ if $C \overset{\omega}{\leftrightarrow} C'$ for some $C \in \mathscr{C}$ and $C' \in \mathscr{C}'$. Indeed if (4.21) holds then at

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least one element of $\widetilde{\mathcal{U}}/\sim_{\omega_{j+1}}$ is obtained by merging at most a elements of $\widetilde{\mathcal{U}}$, and its clusters are not connected to their complement in $\widetilde{\mathcal{U}}$ by definition of $\sim_{\omega_{j+1}}$. In Case 1 we simply use that $U_{j+1} \geq U_j/a$ on E by (4.17), from which (4.21) follows because $U_j = |\mathcal{U}_j| \geq |\widetilde{\mathcal{U}}|$, cf. (4.18), and

$$|\widetilde{\mathcal{U}}/\sim_{\omega_{j+1}}| = |\mathcal{U}_{j+1}(\omega_j)/\sim_{\omega_{j+1}}| = |\mathcal{U}_{j+1}(\omega_{j+1})| = U_{j+1}$$

(regarding the first of these equalities, see the discussion following (4.16) and the definition below (4.21)). In Case 2 we deduce (4.21) from $U_{j+1} > |\widetilde{\mathcal{U}}|/a + U_{j,j+1}(\omega_j)$, which holds on E due to (4.17) and (4.18), together with the inequality $|\widetilde{\mathcal{U}}/\sim_{\omega_{j+1}}| \geq U_{j+1} - U_{j,j+1}(\omega_j)$.

To see the latter, one thinks of $\mathcal{U}_{j+1}(\omega_{j+1})$, which has U_{j+1} elements, as obtained from $\mathcal{U}_j(\omega_j)$ by first forming $\mathcal{U}_j(\omega_j)/\sim_{\omega_{j+1}}$ and then removing the clusters $C \in \mathcal{C}$ not intersecting $V_{2(j+1)}$ from the resulting groups. As $\widetilde{\mathcal{U}}$ is formed from $\mathcal{U}_j(\omega_j)$ by merging the elements of $\mathcal{U}_{j,j+1}(\omega_j)$, the quotient $\widetilde{\mathcal{U}}/\sim_{\omega_{j+1}}$ will cause at most $U_{j,j+1}(\omega_j)$ of the elements in $\mathcal{U}_j(\omega_j)/\sim_{\omega_{j+1}}$ to merge, yielding the desired inequality.

As a consequence of (4.19) and (4.20), we deduce that on E, there exist $k := \lceil \sqrt{N}/(100\kappa L_{n_0}) \rceil$ disjoint balls $\Lambda_1, \ldots, \Lambda_k$ of radius $10\kappa L_{n_0}$ centered in $\mathbb{L}_{n_0}$ and contained in A_j such that each $\tilde{\Lambda}_\ell$ intersects both sets $\widetilde{C}_1$ and $\widetilde{C}_2$ defined in (4.20), where $\tilde{\Lambda}_\ell$ denotes the ball of radius $8\kappa L_{n_0}$ with the same center as Λ_ℓ. One constructs the balls $\tilde{\Lambda}_\ell$, $1 \leq \ell \leq k$, for instance as follows: consider the shells $\mathbb{S}_\ell := \partial B(V, \ell \cdot 50\kappa L_{n_0})$ with $V = V_{2j+1}$ or V_{2j+2} depending on A_j, so that $\mathbb{S}_\ell \subset A_j$ for all $1 \leq \ell \leq k$. Color a vertex $x \in \mathbb{S}_\ell$ black if $B_{L_{n_0}}(x)$ intersects $\widetilde{C}_1$ and red if it intersects $\widetilde{C}_2$. By (4.19), each vertex in $\mathbb{S}_\ell$ is black or red (or both) and $\mathbb{S}_\ell$ contains at least one black and one red vertex (each possibly carrying the other color as well), as $\widetilde{C}_1$ and $\widetilde{C}_2$ both cross $\mathbb{S}_\ell$. In particular, there exists a pair of neighboring vertices in $\mathbb{S}_\ell$ carrying a different color. The ball $\tilde{\Lambda}_\ell$ centered at the closest vertex in $\mathbb{L}_{n_0}$ from this pair will then have the desired properties.

Finally, we note that if $E \subset G$ happens, cf. (4.4), then each ball Λ_ℓ is good. Now, conditioning on the possible realizations $\{\mathscr{C}\}$ of $\widetilde{\mathcal{U}}$ and applying a union bound on the partition $\{\mathscr{C}\} = \{\mathscr{C}\}_1 \sqcup \{\mathscr{C}\}_2$ provided by (4.20) (where $\{\mathscr{C}\}_i$ corresponds to the realization of $\widetilde{\mathcal{U}}_i$ on the event $\{\widetilde{\mathcal{U}} = \{\mathscr{C}\}\}$), we get, with $C_i = \bigcup_{\mathscr{C} \in \{\mathscr{C}\}_i} \text{supp}(\mathscr{C})$,

$$(4.22) \quad \mathbb{P}[E] \leq \sum_{\{\mathscr{C}\}} \mathbb{P}[\widetilde{\mathcal{U}} = \{\mathscr{C}\}] \sum_{\substack{\{\mathscr{C}\}=\{\mathscr{C}\}_1 \sqcup \{\mathscr{C}\}_2 \\ |\{\mathscr{C}\}_1| \leq a}} \mathbb{P}\Big[\bigcap_{\ell \leq k} \{\Lambda_\ell \text{ is good}\} \cap \{C_1 \overset{\varphi \geq h-\varepsilon}{\underset{\Lambda_\ell}{\longleftrightarrow}} C_2\} \Big| \widetilde{\mathcal{U}} = \{\mathscr{C}\}\Big].$$

Notice that the subsets of Λ_ℓ defined by $S_i^\ell := B(C_i, 1) \cap \Lambda_\ell$, $i = 1, 2$, are admissible for all $1 \leq \ell \leq k$. We deduce that

$$\mathbb{P}\Big[\bigcap_{\ell \leq k} \{\Lambda_\ell \text{ is good}\} \cap \{C_1 \overset{\varphi \geq h-\varepsilon}{\underset{\Lambda_\ell}{\longleftrightarrow}} C_2\}\Big| \widetilde{\mathcal{U}} = \{\mathscr{C}\}\Big]$$

$$\leq \mathbb{P}\Big[\bigcap_{\ell \leq k} \mathcal{G}(S_1^\ell, S_2^\ell) \cap \{S_1^\ell \overset{\varphi \geq h-\varepsilon}{\underset{\Lambda_\ell}{\longleftrightarrow}} S_2^\ell\}\Big| \widetilde{\mathcal{U}} = \{\mathscr{C}\}\Big]$$

$$(4.23) \qquad \leq \mathbb{P}\Big[\bigcap_{\ell \leq k} \{S_1^\ell \overset{\varphi \geq h-\varepsilon}{\underset{\Lambda_\ell}{\longleftrightarrow}} S_2^\ell\}\Big| \widetilde{\mathcal{U}} = \{\mathscr{C}\}, \bigcap_{\ell \leq k} \mathcal{G}(S_1^\ell, S_2^\ell)\Big]$$

$$= \prod_{\ell \leq k} \mathbb{P}\Big[S_1^\ell \overset{\varphi \geq h-\varepsilon}{\underset{\Lambda_\ell}{\longleftrightarrow}} S_2^\ell\Big| \widetilde{\mathcal{U}} = \{\mathscr{C}\}, \bigcap_{j \leq k} \mathcal{G}(S_1^j, S_2^j), \bigcap_{j < \ell} \{S_1^j \overset{\varphi \geq h-\varepsilon}{\underset{\Lambda_j}{\longleftrightarrow}} S_2^j\}\Big]$$

$$\leq (1 - e^{-c_4(\log u(N))^2})^k,$$

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where in the last line we used Lemma 3.5 in order to bound each one of the k terms in the product by $1 - e^{-c_4(\log u(N))^2}$ (one can easily check that the event in the conditioning can indeed be written as $D \cap E \cap \mathcal{G}(S_1^\ell, S_2^\ell)$, with $D \in \sigma(1_{\varphi_x \geq h}; \; x \in S_1^\ell \cup S_2^\ell)$ and $E \in \sigma(\mathbf{Z}(\Lambda_\ell^c)))$. Now combining (4.22), (4.23) and the fact that, as $|\{\mathscr{C}\}| \leq |\partial B_{4N}| \leq CN^{d-1}$, the number of partitions $\{\mathscr{C}\} = \{\mathscr{C}\}_1 \sqcup \{\mathscr{C}\}_2$ with $|\{\mathscr{C}\}_1| \leq a$ is at most $(CN^{d-1})^a$, we get

$$(4.24) \qquad \mathbb{P}[E] \leq (CN^{d-1})^a (1 - e^{-c_4(\log u(N))^2})^k \leq \exp(-c'N^{1/4}),$$

recalling that $k \geq c\sqrt{N}/u(N)$ and $a \leq N^{1/4}$, as required by (4.17). This completes the proof of Lemma 4.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 4.3. The estimate (4.24) imposes a constraint on the choice of $u(\cdot)$ in the definition of $\tilde{h}$ of the form $(\log u(N))^2 \ll \log N$ (in particular, any choice $\log u(N) = (\log N)^{1/(2+\delta)}$, $\delta > 0$ would be sufficient, but $u(N)$ needs to be subpolynomial). This constraint is indirectly caused by the lower bound derived in (3.15).

4.2 Renormalization

In the sequel, consider a fixed $\varepsilon > 0$ and any $h \leq \tilde{h} - 6\varepsilon$. We will eventually show that φ strongly percolates at level h to deduce Proposition 1.5. For $L_0 \geq 100$ and $x \in \tilde{\mathbb{L}}_0 := L_0\mathbb{Z}^d$, define x to be $(L_0$-$)good$ if (see (4.1) for notation) the translate by x of $\mathcal{E}(L_0, h + 2\varepsilon, h + \varepsilon)$ occurs and $\sup_{B_{6L_0}(x)} |\varphi - \varphi^0| \leq M$ with $M = M(L_0) := (\log L_0)^2$. Whenever $x \in \tilde{\mathbb{L}}_0$ is good, the set $B_{L_0}(x)$ will be called a (L_0)-*good box*. Finally, let $\mathscr{S}_N$ be the L_0-neighborhood of the connected component of good vertices in $\tilde{\mathbb{L}}_0(B_{2N})$ intersecting $B_{N/2}$ with largest diameter (if there is more than one such component, choose the smallest in some deterministic order). Notice that $\mathscr{S}_N$ is measurable with respect to the sigma-algebra

$$(4.25) \qquad \mathcal{F} := \sigma\big(\varphi_x - \varphi_x^0, \; 1\{\varphi_x \geq h + m\varepsilon\}, \; m = 1, 2, \; x \in \mathbb{Z}^d\big).$$

It will be important below that $\mathcal{F}$ does not completely determine φ^0, i.e. φ^0 is not $\mathcal{F}$-measurable.

The following result asserts that with very high probability, $\mathscr{S}_N$ (under $\mathbb{P}$) is ubiquitous at a mesoscopic scale of order $N^{1/2}$ inside B_N, for all sufficiently large N.

Proposition 4.4. *For every $\varepsilon > 0$, there exist constants $\rho = \rho(d) \in (0, \frac{1}{2})$, $L_0 = L_0(d, \varepsilon) \geq 100$ and $c_5 = c_5(d, \varepsilon) > 0$ such that for all $h < \tilde{h} - 6\varepsilon$ and $N \geq 1$,*

$$(4.26) \qquad \mathbb{P}\left[\begin{array}{c} \mathscr{S}_N \text{ intersects every connected set} \\ S \subset B_N \text{ with } \operatorname{diam}(S) \geq N^{1/2} \end{array} \right] \geq 1 - \exp(-c_5 N^\rho).$$

Proof. Consider the event $F_{0,x} := F_x^{(1)} \cap F_x^{(2)} \cap F_x^{(3)}$ defined for $x \in \tilde{\mathbb{L}}_0$, where

$$F_x^{(1)} = \{B_{L_0}(x) \xleftarrow{\{\varphi^{L_0} \geq h + 3\varepsilon\}} \partial B_{6L_0}(x)\},$$

$$F_x^{(2)} = \left\{ \begin{array}{c} \text{all the clusters in } \{\varphi^{L_0} \geq h + \frac{5}{3}\varepsilon\} \cap B_{4L_0}(x) \text{ crossing the annulus} \\ B_{4L_0}(x) \setminus B_{2L_0}(x) \text{ are connected to each other in } \{\varphi^{L_0} \geq h + \frac{4}{3}\varepsilon\} \cap B_{4L_0}(x) \end{array} \right\},$$

$$F_x^{(3)} = \{ \sup_{y \in B_{6L_0}(x)} |\varphi_y^{L_0} - \varphi_y^0| \leq M(L_0) - 1\}.$$

These choices are motivated by (4.29) below. The events $F_{0,x}$, $x \in \widetilde{\mathbb{L}}_0$, are typical, meaning that

$$(4.27) \qquad \limsup_{L_0 \to \infty} \mathbb{P}[F_{0,0}] = 1.$$

Indeed, since $h + 4\varepsilon < \tilde{h} - 2\varepsilon$ and

$$F_0^{(1)} \supset \mathcal{E}(L_0, h + 4\varepsilon, h + 3\varepsilon) \cap \{\sup_{B_{6L_0}} |\varphi - \varphi^{L_0}| \le \varepsilon\},$$
$$F_0^{(2)} \supset \mathcal{E}(L_0, h + \tfrac{14}{9}\varepsilon, h + \tfrac{13}{9}\varepsilon) \cap \{\sup_{B_{6L_0}} |\varphi - \varphi^{L_0}| \le \tfrac{\varepsilon}{9}\},$$

(the first of these inclusions only requires the existence part of the event $\mathcal{E}$, the second only the uniqueness part, cf. (4.1)) it follows using Proposition 4.1 and Lemma 3.2 that

$$\limsup_{L_0 \to \infty} \mathbb{P}[F_0^{(1)} \cap F_0^{(2)}] = 1.$$

The fact that $\mathbb{P}[F_0^{(3)}]$ tends to 1 as $L_0 \to \infty$ follows immediately from a union bound and a standard Gaussian tail estimate, since $\mathbb{E}[(\varphi_0^{L_0} - \varphi_0^0)^2]$ is bounded uniformly in L_0.

We now aim at applying Corollary 2.6. For a sequence of length scales $(L_n)_{n \ge 0}$ as in (1.18) (recall that ℓ_0 has been fixed at the beginning of Section 3.2), let

$$(4.28) \qquad H_{n,x} = \Big\{ \sup_{y \in B_{L_n}(x)} |\varphi_y^{L_n} - \varphi_y^{L_{n-1}}| \le \frac{2\varepsilon}{(\pi n)^2} \Big\}, \quad x \in \widetilde{\mathbb{L}}_n, \ n \ge 1.$$

Together, (4.28) and the definitions of $F_x^{(1)}$, $F_x^{(2)}$ and $F_x^{(3)}$ yield that

$$(4.29) \qquad \Big(F_{0,x} \cap \bigcap_{n \ge 1} H_{n,x}\Big) \subset \{x \text{ is good}\}, \quad x \in \widetilde{\mathbb{L}}_0,$$

with the notion of goodness introduced above (4.25). Using Lemma 3.2 and (4.27), we then choose L_0 large enough such that the following hold: the probabilities of the events $H_{n,x}$ and of the seed events $F_{0,x}$ satisfy the bounds in (**C3**). It follows that all the assumptions of Corollary 2.6 (with the notion of admissibility given by Remark 2.5, 2)) are in force. With all parameters fixed, consider any $N \ge L_1$ and choose $n := \max\{k : L_{k+1} \le N^{1/2}\}$, so that

$$(4.30) \qquad \frac{N^{1/2}}{\ell_0^2} < L_n \le \frac{N^{1/2}}{\ell_0}.$$

We now define a random set $\tilde{\mathscr{S}}_N$ which will soon be shown to satisfy $\tilde{\mathscr{S}}_N \subset \mathscr{S}_N$ and to have similar connectivity properties as those required of $\mathscr{S}_N$ in (4.26) with very high probability. For a vertex $x \in \widetilde{\mathbb{L}}_n$, we write $\mathrm{Comp}(x)$ for the largest (in diameter) connected component in $\widetilde{\mathbb{L}}_0$ of vertices $y \in (\widetilde{\mathbb{L}}_0 \cap B(x, 4\kappa L_n))$ such that $F_{0,y} \cap \bigcap_{n \ge 1} H_{n,y}$ occurs (if several such components exist, choose the one containing the smallest vertex for some given deterministic ordering of the vertices in $\widetilde{\mathbb{L}}_0$). We then set $\overline{\mathrm{Comp}}(x)$ to be the connected component of $\mathrm{Comp}(x)$ inside $\{y \in \widetilde{\mathbb{L}}_0 \cap B(x, 10\kappa L_n) : F_{0,y} \cap \bigcap_{n \ge 1} H_{n,y} \text{ occurs}\}$ and define

$$(4.31) \qquad \tilde{\mathscr{S}}_N = \bigcup_{x \in \widetilde{\mathbb{L}}_n \cap B_N} \bigcup_{y \in \overline{\mathrm{Comp}}(x)} B_{L_0}(y).$$

Since $\tilde{\mathscr{S}}_N \subset B_{N+10\kappa L_n+L_0}$, it follows on account of (4.30) that $\tilde{\mathscr{S}}_N \subset B_{2N}$ whenever $N \geq C(h)$, which will be tacitly assumed.

We then consider, with $\mathcal{G}^0_{n,x}$ being the translate of $\mathcal{G}^0_n$ by x (see (2.10) for the definition of $\mathcal{G}^0_n$),

$$(4.32) \qquad G_N := \bigcap_{x \in \widetilde{\mathbb{L}}_n \cap B_N} \mathcal{G}^0_{n,x}$$

and proceed to verify that G_N is contained in the event appearing on the left-hand side of (4.26) with $\tilde{\mathscr{S}}_N$ in place of $\mathscr{S}_N$. The lower bound asserted in (4.26) then follows by applying a union bound over x, using (4.30) and Corollary 2.6.

We now check the desired inclusion. First, by (4.29) and (4.31), $\tilde{\mathscr{S}}_N$ is the L_0-neighborhood of a set of good vertices. Moreover, the set $\tilde{\mathscr{S}}_N$ is connected on G_N, as we now explain. To this end, it suffices to argue that on G_N, for any two points $x, y \in \widetilde{\mathbb{L}}_n \cap B_N$ with $|x - y| = L_n$,

$$(4.33) \qquad \Big(\bigcup_{z \in \overline{\mathrm{Comp}}(x)} B_{L_0}(z) \Big) \cap \Big(\bigcup_{z \in \overline{\mathrm{Comp}}(y)} B_{L_0}(z) \Big) \neq \emptyset.$$

To see this, first note that $\mathrm{diam}(S_x) \geq \kappa L_n$ where $S_x := \bigcup_{z \in \overline{\mathrm{Comp}}(x)} B_{L_0}(z)$. Indeed, any two fixed opposite faces of the box $B_{\kappa L_n}(x)$ form two sets of diameter larger than κL_n in $B_{8\kappa L_n}(x)$, which are connected on the event $G_N \subset \mathcal{G}^0_{n,x}$ by the L_0-neighborhood of a path consisting of vertices $v \in \widetilde{\mathbb{L}}_0 \cap B_{10\kappa L_n}(x)$ such that $F_{0,v} \cap \bigcap_{n \geq 1} H_{n,v}$ occurs. In particular, $\mathrm{diam}(S_x \cap B_{4\kappa L_n}(x)) \geq \kappa L_n$. One deduces in the same way that $\mathrm{diam}(S_y) \geq \kappa L_n$. Since both $S_x, S_y \subset B_{8\kappa L_n}(x)$, a similar reasoning using $\mathcal{G}^0_{n,x}$ implies (4.33).

Now we show that every connected set $S \subset B_N$ with $\mathrm{diam}(S) \geq N^{1/2}$ intersects $\tilde{\mathscr{S}}_N$. By (4.30), $\mathrm{diam}(S) \geq \ell_0 L_n \geq 10\kappa L_n$ (recall that $\ell_0 \geq 10\kappa$), whence, by considering an $x \in \widetilde{\mathbb{L}}_n \cap B_N$ such that $B_{L_n}(x) \cap S \neq \emptyset$, it immediately follows that $B_{4\kappa L_n}(x) \cap S$ has a connected component with diameter at least κL_n. The event $\mathcal{G}^0_{n,x}$ then ensures as in the previous paragraph that $S \cap \bigcup_{y \in \overline{\mathrm{Comp}}(x)} B_{L_0}(y) \neq \emptyset$. Thus S intersects $\tilde{\mathscr{S}}_N$.

It is an easy consequence of the two previous paragraphs that $\mathscr{S}_N$ contains $\tilde{\mathscr{S}}_N$ and therefore intersects every connected set $S \subset B_N$ with $\mathrm{diam}(S) \geq N^{1/2}$. The claim (4.26) follows. $\qquad \square$

Remark 4.5. Following [64], one may define a weaker definition of $\bar{h}$ by considering instead of $\mathrm{Unique}(R, \alpha)$ in (1.7) the event $\mathrm{Unique}(R, \alpha, \beta)$ which allows to connect the clusters of $\{\varphi \geq \alpha\} \cap B_R$ of interest with a sprinkling, i.e. in $\{\varphi \geq \beta\} \cap B_{2R}$, for $\alpha > \beta$. One then introduces a corresponding notion of strong percolation at levels (α, β) by requiring bounds analogous to (1.8) and defines

$$\bar{h}' := \sup \big\{ h \in \mathbb{R} : \ \varphi \text{ strongly percolates at levels } (\alpha, \beta) \text{ for all } \beta < \alpha < h \big\}.$$

Then Proposition 4.1 and the renormalization scheme from the proof of Proposition 4.4 (using Corollary 2.6) readily yield the proof of Proposition 1.5 for this alternative (and weaker) definition of $\bar{h}$. The additional arguments of Section 4.3 are needed to deal with the (stronger) case $\alpha = \beta$.

4.3 Proof of Proposition 1.5

In view of (1.8) and (1.9), in order to conclude the proof of Proposition 1.5, it suffices to show that there exists $c = c(\varepsilon) > 0$ such that for all $h \leq \tilde{h} - 6\varepsilon$,

$$(4.34) \qquad \mathbb{P}[\text{Exist}(N, h)^c] \leq \exp(-cN^\rho) \quad \text{and} \quad \mathbb{P}[\text{Unique}(N, h)^c] \leq \exp(-cN^\rho),$$

where ρ is given by Proposition 4.4. Fix $h \leq \tilde{h} - 6\varepsilon$ and let A be the event on the left-hand side of (4.26). As a consequence of the definition of good boxes, see above (4.25), for configurations in the event A, there is a cluster $\mathscr{C}$ of $\{\varphi \geq h + \varepsilon\}$ intersecting every L_0-box of $\mathscr{S} = \mathscr{S}_N$, which in turn intersects every connected subset of B_N with diameter at least $N^{1/2}$. In particular one has $A \subset \text{Exist}(N, h + \varepsilon) \subset \text{Exist}(N, h)$, and the first inequality of (4.34) follows directly from (4.26). We now focus on the second one.

By definition of good boxes, we have that $|\varphi_x - \varphi_x^0| \leq M$ for every $x \in \mathscr{S}$ (the parameter L_0 was fixed below (4.29)). We now argue that conditionally on $\mathcal{F}$ (defined in (4.25)), the level set $\{\varphi \geq h\}$ is simply an independent site percolation with certain inhomogeneous parameters $\mathbf{p} = (\mathbf{p}_x)_{x \in \mathbb{Z}^d}$. Indeed, for every $x \in \mathbb{Z}^d$ conditionally on $\mathcal{F}$ we know the precise value of $\varphi_x - \varphi_x^0$ and that φ_x^0 lies in a prescribed interval (depending only on the value of $\varphi_x - \varphi_x^0$ and on whether x is in $\{\varphi \geq h + m\varepsilon\}$, $m = 1, 2$, or not). Since φ^0 is an i.i.d. field, the claim follows. Furthermore, on A we know that $\mathscr{C}$ is in $\{\varphi \geq h + \varepsilon\}$ and $|\varphi_x - \varphi_x^0| \leq M$ for every $x \in \mathscr{S}$, so we easily infer that

$$(4.35) \qquad \mathbf{p}_x = 1 \text{ for all } x \in \mathscr{C} \text{ and } \mathbf{p}_x \geq c_6 > 0 \text{ for all } x \in \mathscr{S},$$

where

$$c_6 = c_6(\varepsilon) := \inf_{|t| \leq M} \inf_{h \leq \tilde{h}} \mathbb{P}[\varphi_0^0 \geq h - t \mid \varphi_0^0 < h + \varepsilon - t] > 0.$$

By these observations and Proposition 4.4, we obtain

$$(4.36)$$
$$\mathbb{P}[\text{Unique}(N, h)^c] \leq \mathbb{P}[A^c] + \mathbb{E}[1_A \mathbb{P}[\text{Unique}(N, h)^c | \mathcal{F}]] \leq \exp(-c_5 N^\rho) + \mathbb{E}[1_A \mathbf{P}_\mathbf{p}[\text{Unique}(N)^c]],$$

where $\mathbf{P}_\mathbf{p}$ represents the independent site percolation with parameters $\mathbf{p}$, and $\text{Unique}(N)$ is the event that any two clusters in B_N having diameter at least $N/10$ are connected to each other in B_{2N}. Thus, to complete the proof of (4.34), it is enough to show that

$$(4.37) \qquad \mathbf{P}_\mathbf{p}[\text{Unique}(N)^c] \leq \exp(-cN^\rho)$$

uniformly over all families of parameters $\mathbf{p}$ satisfying the properties (4.35) and all pair of sets $\mathscr{C}$ and $\mathscr{S}$ as above. First notice that by (4.35),

$$(4.38) \qquad \mathbf{P}_\mathbf{p}[\text{Unique}(N)^c] \leq \sum_{x \in B_N} \mathbf{P}_\mathbf{p}[\text{diam}(C(x)) \geq N/10, C(x) \cap \mathscr{C} = \emptyset],$$

where $C(x)$ denotes the cluster of x in B_N (under $\mathbf{P}_\mathbf{p}$). We bound the summands on the right individually. In order to do that, we explore the cluster of $C(x)$ vertex by vertex starting from x, in a canonical way, i.e. checking at each step the state of some unexplored vertex in the exterior neighborhood of the currently explored piece of $C(x)$. We do so until the first time we discover some vertex $y_1 \in C(x)$ which is in the exterior neighborhood of some L_0-box $B_1 \in \mathscr{S}$ (recall that $\mathscr{S}$ is determined since we have conditioned on $\mathcal{F}$). At this point, we explore the state of

every vertex in B_1. By definition, $\mathscr{C}$ intersects B_1. We stop the exploration if at some point we discover that some vertex of $\mathscr{C} \cap B_1$ lies in $C(x)$, which occurs for example if all the vertices of B_1 belong to $C(x)$. Otherwise we continue exploring $C(x)$ until we discover some vertex $y_2 \in C(x)$ in the exterior neighborhood of some L_0-box $B_2 \in \mathscr{S} \setminus B_1$ which was not visited by the exploration yet. As before, we then explore the state of every vertex in B_2, stopping the exploration if $C(x)$ intersects $\mathscr{C}$ in that box and continuing otherwise. We proceed like this until we either find that $C(x) \cap \mathscr{C} \neq \emptyset$ or we discover the whole cluster $C(x)$. In the process, we are going to explore a certain (random) number n of boxes $B_1, B_2, \ldots, B_n \in \mathscr{S}$. Notice that by (4.35), every time we discover some box B_i, we stop the exploration with probability at least $c' = c'(\varepsilon, L_0) > 0$. Finally, since $\mathscr{S}$ intersects every connected set of diameter at least $N^{1/2}$, we have that on the event $\{\mathrm{diam}(C(x)) \geq N/10, C(x) \cap \mathscr{C} = \emptyset\}$ the exploration runs until fully discovering $C(x)$ and in addition $n \geq N^{1/2}/20 \geq N^\rho$, for $N \geq C$. As a consequence, we deduce that

$$\mathbf{P_p}\big[\mathrm{diam}(C(x)) \geq N/10, \ C(x) \cap \mathscr{C} = \emptyset\big] \leq (1 - c')^{N^\rho},$$

as desired.

Remark 4.6. As follows from (4.34), the exponent ρ governing the rate of decay for the bound in (1.8) and (1.10), which originates from Proposition 4.4, is in fact uniform in $h < \tilde{h} = \bar{h}$.

5 Interpolation scheme

This section is devoted to the proof of Proposition 1.4, which will be split into several steps, as explained in the next paragraph. To lighten the notation we often use $E[X; \mathrm{F}]$ to denote the expectation $E[X\mathbf{1}_F]$ when X is a random variable and F is an event. Throughout the whole section, we assume that $\tilde{h} < h_{**}$ and that $\varepsilon > 0$ is chosen such that $6\varepsilon < h_{**} - \tilde{h}$. Constants c, C may depend implicitly on ε (and d). We recall the notation $L_n = \ell_0^n L_0$ from (1.18), with ℓ_0 as fixed at the beginning of Section 3.2. The parameter L_0 will be chosen following (5.23).

We decompose this section into three subsections. In Section 5.1, we explain the proof of Proposition 1.4 for $\delta = 0$, i.e. the existence of $L = L(\varepsilon)$ large enough (L will be of the form L_n for some n) so that uniformly in $h \in (\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon)$ and $r \geq 1$, $R \geq 2r$,

$$(5.1) \qquad \mathbb{P}[B_r \xleftrightarrow{\varphi \geq h-\varepsilon} \partial B_R] \geq \mathbb{P}[B_r \xleftrightarrow{\varphi^L \geq h} \partial B_R] - C\exp(-\mathrm{e}^{c(\log r)^{1/3}}), \text{ and}$$

$$(5.2) \qquad \mathbb{P}[B_r \xleftrightarrow{\varphi \geq h+\varepsilon} \partial B_R] \leq \mathbb{P}[B_r \xleftrightarrow{\varphi^L \geq h} \partial B_R] + C\exp(-\mathrm{e}^{c(\log r)^{1/3}}),$$

provided that one is given a certain decoupling result, see Lemma 5.1 below. The proof of this lemma is then presented separately in Section 5.2. This proof is the core of the section. It relies on a multi-scale analysis which is the most technical and innovative part of the paper. Finally, in Section 5.3 we explain how to add the noise parameter $\delta > 0$ into the game to obtain (1.14) and (1.15).

5.1 Setting of the proof

The main difficulty in proving Proposition 1.4 is the long-range dependence of φ. To overcome this problem, we will go from $\omega = \{\varphi \geq h\}$ to $\omega^L = \{\varphi^L \geq h\}$, cf. (3.7), step by step by interpolating between fields of *comparable* ranges and allowing h to vary slightly (we refer to

this as the sprinkling). More precisely, extend our notation to non-integer t by setting $L_t := L_{\lceil t \rceil}$ and define

$$(5.3) \qquad \chi^t := \varphi^{L_{\lfloor t \rfloor}} + (t - \lfloor t \rfloor)\psi^t \quad \text{where} \quad \psi^t := \varphi^{L_t} - \varphi^{L_{\lfloor t \rfloor}}.$$

By definition, $\chi^n = \varphi^{L_n}$ and χ^t interpolates between φ^{L_n} and $\varphi^{L_{n+1}}$ when $n \leq t \leq n + 1$. Introduce the function

$$(5.4) \qquad \theta(t, h, r, R) := \mathbb{P}[B_r \xleftrightarrow{\chi^t \geq h} \partial B_R].$$

We now sketch the argument. Neglecting the additive error terms on the right of (5.1) and (5.2), we roughly aim at proving that the functions $f_\pm(t) := \theta(t, h \pm Ce^{-t}, r, R)$ are increasing (for $+$) and decreasing (for $-$), for t large enough, a fact which is implied by

$$(5.5) \qquad |\partial_t \theta| \leq -Ce^{-t}\partial_h \theta.$$

Let us now look at the probabilistic statement that (5.5) corresponds to. Since the process χ^t is non-degenerate, the partial derivatives of θ exist for all h and all non-integer t and take the form

$$(5.6) \qquad \partial_h \theta = -\sum_{x \in \mathbb{Z}^d} \mathbb{P}[\text{Piv}_x | \chi_x^t = h]p_t(h) \quad \text{and} \quad \partial_t \theta = -\sum_{x \in \mathbb{Z}^d} \mathbb{E}[\psi_x^t; \text{Piv}_x | \chi_x^t = h]p_t(h),$$

where Piv_x is the event that B_r and ∂B_R are connected in $\{\chi^t \geq h\} \cup \{x\}$ but not in $\{\chi^t \geq h\} \backslash \{x\}$ (we call such a vertex x *pivotal*), and $p_t(\cdot)$ is the density of χ_x^t. The dependence of Piv_x on the parameters r, R, t and h is omitted in order to lighten the notation and will always be obvious from the context. Note that the sums in (5.6) are effectively over a finite set and that $(\partial_t \theta)(t, h)$ can be extended to a continuous function on any strip $\mathbb{S}_n := \{(t, h) : h \in \mathbb{R}, n \leq t \leq n + 1\}$, for $n \in \mathbb{N}$.

Suppose for a moment that we were working with Bernoulli percolation. In this case, the pivotality at a vertex x would be independent of the value of the field at x, so that

$$(5.7) \qquad \mathbb{E}[|\psi_x^t|; \text{Piv}_x | \chi_x^t = h] = \mathbb{E}[|\psi_x^t| \, | \chi_x^t = h]\,\mathbb{P}[\text{Piv}_x | \chi_x^t = h].$$

As a consequence, the proof would follow from the fact that $\mathbb{E}[|\psi_x^t| \, | \chi_x^t = h]$ is quite small and that the quantity can be taken smaller than εe^{-t} by choosing L large enough.

In our case, the range of χ^t is L_t so we must make several adjustments to the plan stated above. First of all, we might want to replace Piv_x with a weaker "coarse pivotality" event that is supported outside $B_{L_{\lceil t \rceil}}(x)$, thus allowing us to achieve a *decoupling* with $|\psi_x^t|$ as in the last display. Then the task becomes, roughly speaking, to reconstruct a pivotal vertex from a coarse one. This is the content of Lemma 5.1 below. Its proof, which spans Section 5.2, will involve showing that conditionally on the coarse pivotality, the probability that there are pivotal vertices is not too small. Lemma 5.1 will then allow us to deduce a differential inequality similar to (5.5), see (5.14) below.

As we shall see in detail in Section 5.2, the estimate derived in Lemma 5.1 hinges on *a priori* lower bounds on the disconnection and connection probabilities for $\{\chi^t \geq h\}$ similar to those available for $\{\varphi \geq h\}$ when $h \in (\tilde{h}, h_{**})$ (hence the restriction on the value of h in Proposition 1.4). Set

$$(5.8) \quad c_7 := \tfrac{1}{2}\inf\left\{n^d\mathbb{P}[B_{u(n)} \xcancel{\xleftrightarrow{\varphi \geq h}} B_n], \, \mathbb{P}[B_{\lceil n/2 \rceil} \xleftrightarrow{\varphi \geq h} \partial B_n] : n \geq 1, \, h \in (\tilde{h} + \varepsilon, h_{**} - \varepsilon)\right\} > 0$$

and introduce the convenient notation

$$(5.9) \qquad q_N(t,h) := \inf \{ n^d \mathbb{P}[B_{u(n)} \overset{\chi^t \geq h}{\longleftrightarrow\!\!\!\!\!/} B_n], \ \mathbb{P}[B_{\lceil n/2 \rceil} \overset{\chi^t \geq h}{\longleftrightarrow} \partial B_n] : 1 \leq n \leq N \}.$$

Note that q_N is decreasing in N. We now state the main technical result.

Lemma 5.1 (Decoupling)**.** *For any $\varepsilon > 0$, there exist positive constants C_5, C_6, c_8 and L_0 (depending on ε and d only) such that, for every $h \in (\tilde{h} + 2\varepsilon, h_{**} - 2\varepsilon)$, $C_5 \leq r \leq R/2$ and $t, R \geq 1$ such that $q_R(t,h) \geq c_7$,*

$$\sum_{x \in \mathbb{Z}^d} \mathbb{E}[f(\psi_x^t); \mathrm{Piv}_x \,|\, \chi_x^t = h] \leq \mathbb{E}[f(\psi_0^t)|\chi_0^t = h]\Big(\mathrm{e}^{C_6 t^{18}} \sum_{x \in \mathbb{Z}^d} \mathbb{P}[\mathrm{Piv}_x|\chi_x^t = h] + \exp[C_6 t^3 - r^{c_8}\mathrm{e}^{-C_6 t^3}]\Big)$$

for any non-negative function f such that $\mathbb{E}[f(\psi_0^t)|\chi_0^t = h] < \infty$.

Remark 5.2. The proof of Lemma 5.1 entails a construction like in Section 3.2; hence the "additive" error term $\exp[C_6 t^3 - r^{c_8}\mathrm{e}^{-C_6 t^3}]$ (cf. Lemma 1.6 or Lemma 3.5) which has contributions from all the vertices in the annulus $B_R \setminus B_{r-1}$. The "correction" term $\mathrm{e}^{C_6 t^3}$, on the other hand, appears only to offset for the vertices close to B_r.

We postpone the proof of this lemma until the next section and first show how to obtain (5.1) and (5.2).

Proofs of (5.1) *and* (5.2). Let L_0 be given by Lemma 5.1. Throughout the proof, we tacitly assume that r and R satisfy $C_5 \leq r \leq R/2$. The remaining cases, i.e. $r < C_5$, can be accommodated by adapting the constant C in (5.1) and (5.2). Recalling the definition of χ^t from (5.3) and noting that

$$\mathrm{Var}(\varphi_0^{L_{\lfloor t \rfloor}}) \geq 1/2 \ \text{ and } \ \mathrm{Var}(\psi_0^t) \leq L_t^{-\frac{d-2}{2}} \leq \exp[-(\tfrac{1}{2}\log \ell_0)\, t]$$

(where $\ell_0 \geq 1000 > \mathrm{e}^6$), a standard Gaussian bound gives

$$(5.10) \qquad \alpha(t) := \sup\{\mathbb{E}[|\psi_0^t|; |\psi_0^t| \geq \mathrm{e}^{-t} \,|\, \chi_0^t = h]; \ h \in (\tilde{h}, h_{**})\} \leq C \exp(-L_t^{1/6}).$$

We can therefore fix $C_7(d, \varepsilon)$ large enough such that for all $t \geq C_7$, all $h \in (\tilde{h} + 2\varepsilon, h_{**} - 2\varepsilon)$ and every $r \geq 1$,

$$(5.11) \qquad\qquad\qquad \mathrm{e}^{C_6 t^{18}}\alpha(t) \leq \mathrm{e}^{-t} \leq \varepsilon/2,$$

$$(5.12) \qquad\qquad \exp[C_6 t^3 - r^{c_8}\mathrm{e}^{-C_6 t^3}]\alpha(t)p_t(h) \leq \exp[-\mathrm{e}^{c_9(\log r)^{1/3}}]\mathrm{e}^{-t}.$$

Now recalling the formulas from (5.6), we can write, for any integer n and $(t,h) \in \mathbb{S}_n$,

$$|\partial_t\theta(t,h,r,R)| \leq \sum_{x \in \mathbb{Z}^d} \big(\mathbb{E}[|\psi_x^t|\mathbf{1}_{|\psi_x^t| \leq \mathrm{e}^{-t}}; \mathrm{Piv}_x \,|\, \chi_x^t = h] + \mathbb{E}[|\psi_x^t|\mathbf{1}_{|\psi_x^t| \geq \mathrm{e}^{-t}}; \mathrm{Piv}_x \,|\, \chi_x^t = h] \big)p_t(h)$$

$$(5.13) \qquad\qquad \leq \mathrm{e}^{-t}(-\partial_h\theta) + \sum_{x \in \mathbb{Z}^d} \mathbb{E}[|\psi_x^t|\mathbf{1}_{|\psi_x^t| \geq \mathrm{e}^{-t}}; \mathrm{Piv}_x \,|\, \chi_x^t = h]p_t(h).$$

In order to bound the second term in terms of $-\partial_h\theta$ (up to an additive error), we will apply Lemma 5.1 to the function $f(x) = |x|\mathbf{1}_{|x|\geq e^{-t}}$. But we are only allowed to do so as long as $q_R(t,h) \geq c_6$. To this end let us define

$$t_* = t_*(R) := \sup\{t \geq C_7 : q_R(t,h) < c_7 \text{ for some } h \in (\tilde{h} + 2\varepsilon + 2e^{-t}, h_{**} - 2\varepsilon - 2e^{-t})\}$$

(with the convention $\sup\emptyset = C_7$). Note that t_* is finite since, by the weak convergence of χ^t to φ on B_R as $t \to \infty$, there exists t such that $q_R(t,h) \geq c_7$ for all $h \in (\tilde{h}+2\varepsilon, h_{**}-2\varepsilon)$ (see (5.8)). Moreover, by the definition of t_*, $q_R(t,h) \geq c_7$ for all $t > t_*$ and $h \in (\tilde{h}+2\varepsilon+2e^{-t}, h_{**}-2\varepsilon-2e^{-t})$. Therefore, we can apply Lemma 5.1 in this region of the (t,h)-plane to the function $|x|\mathbf{1}_{|x|\geq e^{-t}}$ to obtain (recall the definition of $\alpha(t)$ from (5.10))

$$\sum_{x\in\mathbb{Z}^d} \mathbb{E}[|\psi_x^t|\mathbf{1}_{|\psi_x^t|\geq e^{-t}}; \mathrm{Piv}_x |\chi_x^t = h]p_t(h) \leq e^{t^{C_6}}\alpha(t)(-\partial_h\theta) + \exp[t^{C_6} - r^{c_8}e^{-t^{C_6}}]\alpha(t)p_t(h).$$

Combined with (5.13) as well as the bounds in (5.11) and (5.12), this leads to the following differential inequalities (one for each $n \in \mathbb{N}$), which are valid for $\{(t,h) : t > t_*, h \in (\tilde{h} + 2\varepsilon + 2e^{-t}, h_{**} - 2\varepsilon - 2e^{-t})\} \cap \mathbb{S}_n$:

(5.14) $$|\partial_t\theta(t,h,r,R)| \leq -2e^{-t}\partial_h\theta(t,h,r,R) + \exp[-e^{c_9(\log r)^{1/3}}]e^{-t}.$$

Integrating this family of inequalities along $\gamma_\pm : s \mapsto h \pm 2(e^{-s} - e^{-t})$ between (t,h) and $(\infty, h \mp 2e^{-t})$, see Fig. 3, where $t > t_*$ and $h \in (\tilde{h} + 2\varepsilon + 2e^{-t}, h_{**} - 2\varepsilon - 2e^{-t})$ (chopping to this effect $\gamma_\pm$ into its pieces intercepted by each of the strips $\mathbb{S}_n$) yields that

(5.15)
$$\mathbb{P}[B_r \xleftrightarrow{\varphi\geq h-2e^{-t}} \partial B_R] \geq \mathbb{P}[B_r \xleftrightarrow{\chi^t\geq h} \partial B_R] - \exp[-e^{c_9(\log r)^{1/3}}]e^{-t},$$
$$\mathbb{P}[B_r \xleftrightarrow{\varphi\geq h+2e^{-t}} \partial B_R] \leq \mathbb{P}[B_r \xleftrightarrow{\chi^t\geq h} \partial B_R] + \exp[-e^{c_9(\log r)^{1/3}}]e^{-t},$$
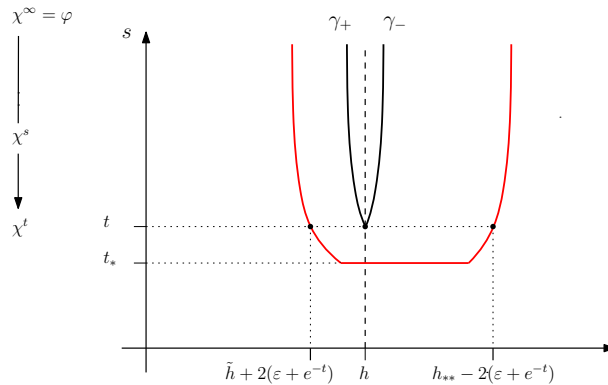
for all $t > t_*(R)$.

Figure 3 – The two interpolation curves used in (5.15). The red curve demarcates the boundary of the region in which the family of differential inequalities (5.14) hold.

Now let $t_{**} \geq C_7$ be the minimum number satisfying

(5.16) $$\sup\{a \geq 1 : a^d \exp[-e^{c_9(\log u(a))^{1/3}}]\}e^{-t_{**}} \leq \frac{c_7}{2},$$

which, in particular, is independent of R. We will now argue that $t_* = t_*(R) \leq t_{**}$ for all R. If this holds, then since $2\mathrm{e}^{-t} \leq \varepsilon$ by (5.11), (5.1) and (5.2) follow from (5.15) by choosing $L = L_{t_{**}}$ and confining h to the interval $(\tilde{h} + 3\varepsilon, h_{**} - 3\varepsilon)$.

Assume on the contrary, that $t_* = t_*(R) > t_{**}$ for some R. Let $q_N(h)$ denote the quantity defined below (5.8) with φ in place of χ^t and note that $q_N(h) \geq 2c_7$ for all $h \in (\tilde{h}+\varepsilon, h_{**}-\varepsilon) =: I$. Then (5.15) and (5.16) together imply that uniformly over $t > t_*(> t_{**})$ and $h \in (\tilde{h} + 2\varepsilon + 2\mathrm{e}^{-t}, h_{**} - 2\varepsilon - 2\mathrm{e}^{-t})$,

$$q_R(t, h) \geq \inf_{h' \in I} q_R(h') - R^d \exp[-\mathrm{e}^{c_9(\log u(R))^{1/3}}]\mathrm{e}^{-t_{**}} \geq \frac{3c_7}{2}.$$

On the other hand, from the definition of t_* it follows that

$$\inf\{q_R(t, h) : t < t_*, h \in (\tilde{h} + 2\varepsilon + 2\mathrm{e}^{-t}, \; h_{**} - 2\varepsilon - 2\mathrm{e}^{-t})\} \leq c_7.$$

However, the previous two displays violate the (joint) continuity of $q_R(\cdot, \cdot)$, cf. the discussion following (5.6). $\qquad\square$

Remark 5.3. The uniform bound on the error term in (5.16) yields a condition on the function $u(\cdot)$ entering the definition of $\tilde{h}$ (cf. (1.12)) not to grow too slowly. Another such condition will arise from the competing prefactors $\mathrm{e}^{C_9(\log L_n)^2}$ and $\mathrm{e}^{-c_{10}u^*(L_n)^\rho}$ in the estimate (5.19) below.

5.2   Proof of Lemma 5.1

Throughout this subsection, we fix all the parameters r, R, t, h and assume tacitly that $h \in (\tilde{h} + 2\varepsilon, h_{**} - 2\varepsilon)$, $t \geq 1$,

(5.17) $$C_5 := 10^2 \kappa L_0 \leq r \leq R/2$$

and (t, R) satisfy $q_R(t, h) \geq c_7$ as in Lemma 5.1, where L_0 will be given by 5.4 below. Although they depend on t, we will write χ and ψ instead of χ^t and ψ^t. We set T to be the smallest integer such that $u(L_T) \geq 20\kappa L_t$ and $\overline{T}$ the smallest integer such that $u(L_{\overline{T}}) \geq 20\kappa L_T$. Note that it follows directly from the definition of $u(\cdot)$ that $T \leq C(L_0)t^3$ and $\overline{T} \leq C(L_0)t^9$. We then define

(5.18) $$u^*(L_m) := \begin{cases} 0 & \text{if } m \leq \overline{T}, \\ u(L_m) & \text{if } m > \overline{T}. \end{cases}$$

The following lemma is a key step in proving Lemma 5.1. For $N \geq 1$, let $\mathrm{CoarsePiv}_x(N)$ denote the event that B_r and ∂B_R are connected in $\{\chi \geq h\} \cup B_N(x)$ but not in $\{\chi \geq h\}$.

Lemma 5.4. *For every $\varepsilon > 0$ and $L_0 \geq C_8(d, \varepsilon)$ there exist positive constants $C_9 = C_9(L_0, \varepsilon, d)$, $c_{10} = c_{10}(L_0, \varepsilon, d)$, $\rho = \rho(d)$ such that for all $x \in B_R \setminus B_{r-1}$,*

(5.19)
$$\mathbb{P}[\mathrm{CoarsePiv}_x(L_T)] \leq \mathrm{e}^{-c_{10}(|x|/L_T)^\rho} + \sum_{n \geq T} \mathrm{e}^{C_9(\log L_n)^2 - c_{10}u^*(L_n)^\rho} \left(\sum_{y \in B_{10\kappa L_n}(x)} \mathbb{P}[\mathrm{Piv}_y | \chi_y = h] \right).$$

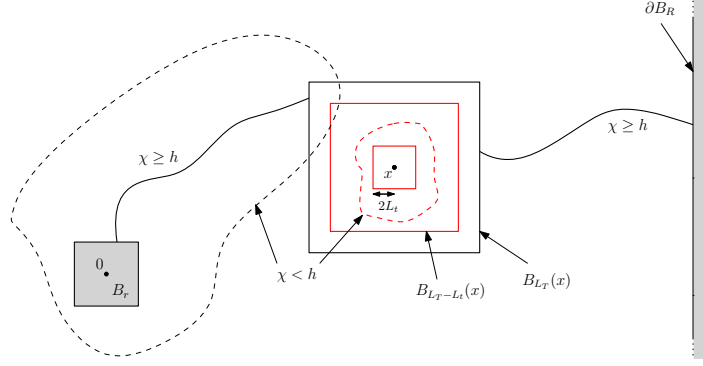We first assume Lemma 5.4 to hold and give the proof of Lemma 5.1.

Figure 4 – Decoupling in the proof of Lemma 5.1. By forcing the event E_x (in red), which is independent of $\mathbf{Z}(\Lambda_x)$ and not too costly since $h > \tilde{h}$, F_x and $f(\psi_x)$ decouple.

Proof of Lemma 5.1. Let $L_0 = C_8(d, \varepsilon)$ be given by Lemma 5.4. All constants C, c below may depend on L_0 (and therefore on ε and d). Assume first that $R \geq 8L_T$. Let us introduce the event E_x that $B_{2L_t}(x)$ and $\partial B_{L_T - L_t}(x)$ are not connected in $\{\chi \geq h\}$, and F_x the event that B_r and B_R are connected in $\{\chi \geq h\} \cup B_{L_T}(x)$ but not in $\{\chi \geq h\} \setminus \{x\}$, see Fig. 4.

Observe that, conditionally on $\mathbf{Z}(\Lambda_x)$ where $\Lambda_x := \{x\} \cup B_{L_T}(x)^c$ (recall the definition in (3.8)), the event F_x is decreasing in all the variables belonging to $\mathbf{Z}(\mathbb{Z}^d) \setminus \mathbf{Z}(\Lambda_x)$, and so is E_x. From the FKG inequality for independent random variables, we deduce that

$$
\begin{aligned}
\mathbb{E}[f(\psi_x); E_x \cap F_x \,|\, \mathbf{Z}(\Lambda_x)] &= f(\psi_x)\mathbb{P}[E_x \cap F_x \,|\, \mathbf{Z}(\Lambda_x)] \\
&\geq f(\psi_x)\mathbb{P}[E_x \,|\, \mathbf{Z}(\Lambda_x)]\,\mathbb{P}[F_x \,|\, \mathbf{Z}(\Lambda_x)] \\
&\geq c_7 L_T^{-d}\mathbb{E}[f(\psi_x); F_x \,|\, \mathbf{Z}(\Lambda_x)],
\end{aligned}
\tag{5.20}
$$

where in the final step we used the lower bound $q_R(t, h) \geq c_7$ from the hypothesis of Lemma 5.1 (note that $u(L_T - L_t) > 2L_t$ by definition of T) together with the fact that the event E_x is independent of $\mathbf{Z}(\Lambda_x)$ by (3.9). Since χ_x is measurable with respect to $\mathbf{Z}(\Lambda_x)$, integrating with respect to $\mathbf{Z}(\Lambda_x)$ gives

$$
\mathbb{E}[f(\psi_x); F_x|\chi_x = h] \leq c_7^{-1} L_T^d\, \mathbb{E}[f(\psi_x); E_x \cap F_x|\chi_x = h]
\tag{5.21}
$$

(to obtain (5.21), one first integrates (5.20) against a set of the form $\{h \leq \chi_x < h+\delta\}$, normalizes suitably and takes the limit $\delta \to 0$). Since $R - r \geq 4L_T$ (recall that $R \geq 8L_T$ by assumption) and consequently $B_{L_T}(x)$ cannot intersect both B_r and ∂B_R, the event $E_x \cap F_x$ is independent of (χ_x, ψ_x) as the range of χ is L_t. From this observation, we deduce that

$$
\begin{aligned}
\mathbb{E}[f(\psi_x); E_x \cap F_x|\chi_x = h] &= \mathbb{E}[f(\psi_x)|\chi_x = h]\mathbb{P}[E_x \cap F_x] \\
&\leq \mathbb{E}[f(\psi_x)|\chi_x = h]\mathbb{P}[\mathrm{CoarsePiv}_x(L_T)],
\end{aligned}
\tag{5.22}
$$

where in the second step we used the fact that $E_x \cap F_x \subset \mathrm{CoarsePiv}_x(L_T)$ when $R - r \geq 4L_T$. Now, Lemma 5.4 gives that

$$
\sum_{x \in B_R \setminus B_{r-1}} \mathbb{P}[\mathrm{CoarsePiv}_x(L_T)] \leq L_T^C e^{-c(r/L_T)^\rho} + e^{C(\log L_{\overline{T}})^2} \sum_{x \in \mathbb{Z}^d} \mathbb{P}[\mathrm{Piv}_x|\chi_x = h],
\tag{5.23}
$$

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where we used that (see (5.18) for $u^*(\cdot)$)

$$\sum_{n \geq T} |B_{10\kappa L_n}| e^{C_9 (\log L_n)^2 - c_{10} u^*(L_n)^\rho} \leq e^{C(\log L_{\overline{T}})^2},$$

$$\sum_{x \in B_R \setminus B_{r-1}} e^{-c_{10}(|x|/L_T)^\rho} \leq L_T^C e^{-c(r/L_T)^\rho},$$

which follow by considering separately the cases $T \leq n \leq \overline{T}$ and $n \geq \overline{T}$ in the first line and the cases $|x| \leq (L_T \vee r)$ and $|x| > (L_T \vee r)$ in the second. Lemma 5.1 now follows in the case $R \geq 8L_T$ from (5.21), (5.22) and (5.23) since $F_x \supset \text{Piv}_x$ and $T \leq C(L_0)t^3$, $\overline{T} \leq C(L_0)t^9$.

On the other hand if $R < 8L_T$, we simply bound

$$(5.24) \qquad \sum_{x \in \mathbb{Z}^d} \mathbb{E}[f(\psi_x); \text{Piv}_x \,|\, \chi_x = h] \leq \sum_{x \in B_R} \mathbb{E}[f(\psi_x)|\chi_x = h] \leq c' L_T^d \, \mathbb{E}[f(\psi_0)|\chi_0 = h].$$

The proof is thus concluded by noting that, since $r \leq 4L_T$ and $T \leq C(L_0)t^3$, one can find C_6 large enough (depending on d, ε, L_0 only) such that $c' L_T^d \leq \exp[C_6 t^3 - r^{c_8} e^{-C_6 t^3}]$. $\qquad\square$

We now turn to the proof of Lemma 5.4. Roughly speaking, we would like to show that conditionally on the event $\text{CoarsePiv}_x(L_T)$, some Piv_y occurs in the box $B_{L_T}(x)$ with not too small probability. A natural strategy consists in trying to create paths between the clusters of B_r and ∂B_R, which must necessarily intersect $B_{L_T}(x)$. However, the fact that the range of dependence of χ is L_t presents a potential barrier for constructing these paths, for instance by forcing the field to be quite large. In order to poke through this barrier, we will use a good bridge – in the sense mentioned below – connecting the clusters of B_r and ∂B_R in $\{\chi \geq h\} \cap B_R$ so we can apply a result akin to Lemma 3.5 (see Remark 3.7 and (5.31) below) to construct open paths.

To begin with, let

$$(5.25) \qquad \begin{aligned} F_{0,y} &:= \left\{ \varphi_z^{L_0} - \varphi_z^0 \geq -M + \varepsilon, \; \varphi_z^0 \geq -M, \;\; \forall z \in B_{L_0}(y) \right\}, \text{ for } y \in \mathbb{L}_0, \\ H_{n,y} &:= \left\{ \varphi_z^{L_n} - \varphi_z^{L_{n-1}} \geq -\frac{6\varepsilon}{(\pi n)^2}, \;\; \forall z \in B_{2L_n}(y) \right\}, \text{ for } n \geq 1 \text{ and } y \in \mathbb{L}_n, \end{aligned}$$

with $M = M(L_0)$ chosen large enough (eg. $M = \log L_0$) so that the bound in (**C3**) holds when $L_0 \geq C_8(d, \varepsilon)$. We call a bridge from Definition 2.1 *good* if it satisfies Definition 2.2, except that we only require **G2** to hold for all j satisfying $1 \vee m \leq j \leq n$ (which is a weaker condition). With a slight abuse of notation, we define $\mathcal{G}_{n,x} := \tau_x \mathcal{G}_n$ for $x \in \mathbb{Z}^d$, where $\mathcal{G}_n$ denotes the event from (2.2) corresponding to this weaker notion of good bridge, and with the choice $\Lambda_n = B_{10\kappa L_n} \setminus B_{\kappa L_n}$ in (2.1). For later reference, we also define $\mathcal{G}(S_1, S_2)$ as in (3.18) for any pair of admissible subsets S_1, S_2 of Λ_n.

In view of (5.25) and Theorem 2.3, for all $L_0 \geq C_8(d, \varepsilon)$ (which we will henceforth tacitly assume), there exist constants $c_{11} = c_{11}(L_0), \rho = \rho(d) > 0$ such that for every $n \geq 0$ and $x \in \mathbb{Z}^d$,

$$(5.26) \qquad \mathbb{P}[\mathcal{G}_{n,x}] \geq 1 - e^{-c_{11} L_n^\rho}.$$

We henceforth fix L_0 as above. All the constants C, c below may depend on L_0, ε and d.

By definition, $\mathcal{G}_{n,x}$ guarantees the existence of a good bridge between the clusters of B_r and ∂B_R in $\{\chi \geq h\} \cap B_R$ *provided* they are both admissible in Λ_n, i.e. they both intersect

$\partial B_{10\kappa L_n}(x)$ as well as $B_{8\kappa L_n}(x)$ – cf. above (2.2). But the latter condition is satisfied on the event $\mathrm{CoarsePiv}_x(8\kappa L_n)$ when $B_r \not\subset B_{10\kappa L_n}(x)$. Putting these two observations together and using (5.25), one ends up with the following lemma, whose proof is postponed for a few lines.

Lemma 5.5 (Creating pivotals from coarse pivotals). *For all $x \in B_R \setminus B_{r-1}$ and $n \geq T$ such that $B_r \not\subset B_{10\kappa L_n}(x)$, we have*

$$(5.27) \qquad \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_n), \mathcal{G}_{n,x}] \leq \mathrm{e}^{C(\log L_n)^2} \sum_{y \in B_{10\kappa L_n}(x)} \mathbb{P}[\mathrm{Piv}_y | \chi_y = h].$$

In order to prove Lemma 5.4, we then find the *first* scale L_n at which the event $\mathcal{G}_{n,x}$ occurs so that we can apply the previous lemma, which is the content of Lemma 5.6 below.

For $x \in B_R \setminus B_{r-1}$, let S_x denote the largest integer such that i) $B_r \not\subset B_{10\kappa L_{S_x}}(x)$, and ii) $B_{10\kappa L_{S_x}}(x)$ has empty intersection with at least one of B_r and ∂B_R. Recall that we used a condition similar to ii) to derive (5.22) (this was ensured by the assumption $R \geq 8L_T$ in the argument leading to (5.22)). The quantity S_x is well-defined, i.e. $S_x \geq 0$ by condition (5.17). Moreover, $L_{S_x} \geq c|x|$, as can be readily deduced from the following: if $d(x, B_r) > \frac{|x|}{4}$, the ball $B(x, \frac{|x|}{5})$ does not intersect B_r, whereas for $d(x, B_r) \leq \frac{|x|}{4}$, the ball $B(x, \frac{|x|}{4})$ does not intersect ∂B_R as $R \geq 2r$.

Lemma 5.6 (Finding the first good scale). *For all $x \in B_R \setminus B_r$ such that $S_x > T$, the following holds:*

$$\mathbb{P}[\mathrm{CoarsePiv}_x(L_T)] \leq \sum_{n=T}^{S_x} \mathrm{e}^{-cu^*(L_n)^\rho} \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_n), \mathcal{G}_{n,x}] + \mathbb{P}[\mathcal{G}^c_{S_x,x}].$$

Lemma 5.4 now follows readily by combining (5.26) with Lemmas 5.5 and 5.6 in case $S_x > T$ (this requires L_0 to be large enough, cf. above (5.26)), and simply bounding $\mathbb{P}[\mathrm{CoarsePiv}_x(L_T)]$ by 1 otherwise. The latter is accounted for by the first term on the right-hand side of (5.19) due to the factor $1/L_T$ appearing in the exponent and the fact that $L_{S_x} \geq c|x|$.

We now turn to the proofs of Lemmas 5.5 and 5.6.

Proof of Lemma 5.5. The proof is divided into two steps. In the first step, we prove an *unconditional* version of (5.27), namely

$$(5.28) \qquad \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_n), \mathcal{G}_{n,x}] \leq \mathrm{e}^{C(\log L_n)^2} \sum_{y \in \Lambda_n(x)} \mathbb{P}[\mathrm{Piv}_y, |\chi_y - \varphi^0_y| \leq M'],$$

where $M' := h + \varepsilon + M$ and $\Lambda_n(x) := \Lambda_n + x$. In the second step, we transform the unconditional probability into a conditional one:

$$(5.29) \qquad \mathbb{P}[\mathrm{Piv}_y, |\chi_y - \varphi^0_y| \leq M'] \leq C\,\mathbb{P}[\mathrm{Piv}_y | \chi_y = h].$$

It is clear that (5.27) follows from these two bounds as $\Lambda_n(x) \subset B_{10\kappa L_n}(x)$.

Let us first prove (5.28). To this end, consider any pair of disjoint subsets C_1 and C_2 of B_R such that $\mathcal{C}(C_1, C_2) := \{\mathcal{C}_{B_r} = C_1, \mathcal{C}_{\partial B_R} = C_2\} \subset \mathrm{CoarsePiv}_x(8\kappa L_n)$, where $\mathcal{C}_A$ denotes the cluster of A in $B_R \cap \{\chi \geq h\}$ (observe that $\mathrm{CoarsePiv}_x$ is measurable relative to the pair of

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random sets $\mathcal{C}_{B_r}$ and $\mathcal{C}_{\partial B_R}$). Taking the union over all possible choices of pairs (C_1, C_2) (call this collection of pairs $\mathscr{C}$) yields the decomposition

$$(5.30) \qquad \bigcup_{(C_1, C_2) \in \mathscr{C}} \mathcal{C}(C_1, C_2) = \mathrm{CoarsePiv}_x(8\kappa L_n).$$

By (5.30), the sets C_1 and C_2 are admissible in $\Lambda_n(x)$ for any pair $(C_1, C_2) \in \mathscr{C}$ – cf. below (5.26) – and so are $\overline{C}_1$ and $\overline{C}_2$, where $\overline{C} = (C \cup \partial_{\mathrm{out}} C) \cap \Lambda_n(x)$, for $C \subset \mathbb{Z}^d$. We will use a good bridge to create a (closed) pivotal point y in $(\partial_{\mathrm{out}} C_1 \cup \partial_{\mathrm{out}} C_2) \cap \Lambda_n(x)$, cf. Fig. 5.

Now, similarly to the bound (3.25) derived in Section 3, we can prove that

$$(5.31) \qquad \mathbb{P}\Big[\bigcup_{\substack{y \in \partial_{\mathrm{out}} C_1 \\ z \in \partial_{\mathrm{out}} C_2 \\ y, z \in \Lambda_n(x)}} \{\mathcal{N}(y) \xleftrightarrow[\Lambda_n(x) \setminus (\overline{C}_1 \cup \overline{C}_2)]{\chi \geq h} \mathcal{N}(z)\} \cap \mathcal{H}_y \cap \mathcal{H}_z \;\Big|\; \begin{array}{l} \mathcal{C}(C_1, C_2), \\ \mathcal{G}(\overline{C}_1, \overline{C}_2) \end{array} \Big] \geq e^{-C(\log L_n)^2},$$

where $\mathcal{H}_v = \{\chi_v - \varphi_v^0 \geq -M \text{ and } \varphi_v^0 \geq -M\}$. We now explain the small adjustments to the proof of Lemma 3.5 (or of (3.25)) needed in order to accommodate the different setup implicit in (5.31). First, property (3.20) is replaced by the following: for each box $B = B_{L_m}(y) \in \mathcal{B}$, where $\mathcal{B}$ is any good bridge in $\Lambda_n(x)$,

$$(5.32) \qquad \begin{aligned} \chi_z - \varphi_z^0 &\geq -M \text{ and } \varphi_z^0 \geq -M \quad \forall z \in B_{L_0}(y), \quad \text{when } m = 0, \\ \chi_z - \varphi_z^{L_m} &\geq -\varepsilon \quad \forall z \in \tilde{B} = B_{2L_m}(y), \quad \text{when } 1 \leq m \leq t, \end{aligned}$$

as follows from (5.25) and our (weaker) version of **G2** (see below (5.25)). For each $B = B_{L_m}(y) \in \mathcal{B}$, one then redefines the event A_B (see (3.21)) in the proof of Lemma 3.5 as follows:

$$A_B := \begin{cases} \{x_B \xleftrightarrow[B_{L_0}(y)]{\varphi^0 \geq h+M+\varepsilon} y_B\} & \text{if } m = 0, \\[2ex] \{x_B \xleftrightarrow[B_{2L_m}(y)]{\varphi^{L_m} \geq h+\varepsilon} y_B\} & \text{if } 1 \leq m \leq t, \\[2ex] \{x_B \xleftrightarrow[B_{2L_m}(y)]{\chi \geq h} y_B\} & \text{if } m > t, \end{cases}$$

and observes that, due to (5.32), x_B and y_B are connected in $\{\chi \geq h\}$ whenever $B \in \mathcal{B}$ and A_B occurs (the points x_B and y_B are chosen like in the paragraph above (3.21)). In view of the constraint $h \in (\tilde{h} + 2\varepsilon, h_{**} - 2\varepsilon)$ and the lower bound $q_R(t, h) \geq c_7$ from the hypothesis of Lemma 5.1, which are in force (see the beginning of this subsection), Lemmas 3.2 and 3.3 (see also Remark 3.4) together imply that $\mathbb{P}[A_B] \geq L_m^{-C'}$ for all $m \geq 1$ satisfying $L_m \leq R$. The rest of the proof of Lemma 3.5 then follows as before, yielding (5.31).

Rewriting (5.31) as an inequality involving the corresponding unconditional probabilities, using that $\mathcal{G}_{n,x} \subset \mathcal{G}(\overline{C}_1, \overline{C}_2)$ (see (2.2)), and subsequently summing over all possible choices of pairs $(C_1, C_2) \in \mathscr{C}$, we obtain

$$(5.33) \qquad \begin{aligned} &\mathbb{P}\Big[\bigcup_{y, z \in \Lambda_n(x)} E(y, z) \cap \mathcal{H}_y \cap \mathcal{H}_z \Big] \\ &\geq e^{-C(\log L_n)^2} \sum_{(C_1, C_2) \in \mathscr{C}} \mathbb{P}[\mathcal{G}_{n,x}, \mathcal{C}(C_1, C_2)] \overset{(5.30)}{=} e^{-C(\log L_n)^2} \mathbb{P}[\mathcal{G}_{n,x}, \mathrm{CoarsePiv}_x(8\kappa L_n)], \end{aligned}$$

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where

$$(5.34) \qquad E(y,z) := \{B_r \overset{\chi \geq h}{\longleftrightarrow} \mathcal{N}(y) \overset{\chi \geq h}{\longleftrightarrow} \mathcal{N}(z) \overset{\chi \geq h}{\longleftrightarrow} \partial B_R, \ B_r \overset{\chi \geq h}{\not\longleftrightarrow} \partial B_R, \ \chi_y < h\}.$$

Splitting into whether $z \in \partial_{\text{out}}\mathcal{C}_{B_r}$ or $z \notin \partial_{\text{out}}\mathcal{C}_{B_r}$, one can easily verify that $E(y,z) \subset E_1(z) \cup E_2(y,z)$, where $E_1(z) := \text{Piv}_z \cap \{\chi_z < h\}$ and

$$E_2(y,z) := \{y \text{ is pivotal in } \{\chi \geq h\} \cup \{z\}\} \cap \{\chi_y < h\}.$$

Also notice that

$$\{\chi_z < h\} \cap \mathcal{H}_z \subset \{\chi_z < h, \ \chi_z - \varphi_z^0 \geq -M, \ \varphi_z^0 \geq -M\} \subset \{|\chi_z - \varphi_z^0| \leq M'\}.$$

Altogether we have

$$E_1(z) \cap \mathcal{H}_z \subset \text{Piv}_z \cap \{|\chi_z - \varphi_z^0| \leq M'\}.$$

Therefore a simple union bound gives

$$(5.35) \qquad \begin{aligned} &\mathbb{P}\Big[\bigcup_{y,z \in \Lambda_n(x)} E(y,z) \cap \mathcal{H}_y \cap \mathcal{H}_z \}\Big] \\ &\leq |\Lambda_n| \sum_{z \in \Lambda_(x)} \mathbb{P}[\text{Piv}_z, |\chi_z - \varphi_z^0| \leq M'] + \sum_{y,z \in \Lambda_n(x)} \mathbb{P}[E_2(y,z) \cap \mathcal{H}_y \cap \mathcal{H}_z]. \end{aligned}$$

Now note that $E_2(y,z) \cap \mathcal{H}_y \cap \mathcal{H}_z$ is independent of φ_z^0, therefore

$$(5.36) \qquad \begin{aligned} \mathbb{P}[E_2(y,z) \cap \mathcal{H}_y \cap \mathcal{H}_z] &= C(L_0)\mathbb{P}[E_2(y,z) \cap \mathcal{H}_y \cap \mathcal{H}_z \cap \{\varphi_z^0 \geq M + h\}] \\ &\leq C(L_0)\mathbb{P}[\text{Piv}_y, |\chi_y - \varphi_y^0| \leq M'], \end{aligned}$$

where $C(L_0) := \mathbb{P}[\varphi_0^0 \geq M+h]^{-1}$. In the last inequality we used that $E_2(y,z) \cap \mathcal{H}_y \cap \mathcal{H}_z \cap \{\varphi_z^0 \geq M + h\} \subset \text{Piv}_y \cap \{|\chi_y - \varphi_y^0| \leq M'\}$, which can be easily verified. The desired inequality (5.28) then follows directly from (5.33), (5.35) and (5.36) together.

We now turn to the proof of (5.29). Below, for a stationary Gaussian process Φ indexed by $\mathbb{Z}^d$, we write p_Φ for the density of Φ_0. The key observation is that the pair $(\text{Piv}_y, \chi_y - \varphi_y^0)$ is independent of φ_y^0 (Piv_y is measurable with respect to $\chi_z, z \neq y$, which is independent of φ_y^0). Consequently,

$$\mathbb{P}[\text{Piv}_y | \chi_y - \varphi_y^0 = h_1, \varphi_y^0 = h_2] = \mathbb{P}[\text{Piv}_y | \chi_y - \varphi_y^0 = h_1]$$

for $(h_1, h_2) \in \mathbb{R}^2$, which leads to

$$(5.37) \qquad \mathbb{P}[\text{Piv}_y, |\chi_y - \varphi_y^0| \leq M'] = \int_{-M'}^{M'} \mathbb{P}[\text{Piv}_y | \chi_y - \varphi_y^0 = h_1] p_{\chi - \varphi^0}(h_1) dh_1 \,,$$

and also

$$(5.38)$$

$$p_\chi(h)\mathbb{P}[\text{Piv}_y, |\chi_y - \varphi_y^0| \leq M' \mid \chi_y = h] = \int_{-M'}^{M'} \mathbb{P}[\text{Piv}_y | \chi_y - \varphi_y^0 = h_1] p_{\chi - \varphi^0}(h_1) p_{\varphi^0}(h - h_1) dh_1.$$

Since φ^0 is a centered Gaussian variable, we have

$$\inf_{|h_1| \leq M'} p_{\varphi^0}(h - h_1) \geq c(L_0) > 0$$

and (5.29) now follows from the displays (5.37) and (5.38). This completes the proof. $\qquad \square$
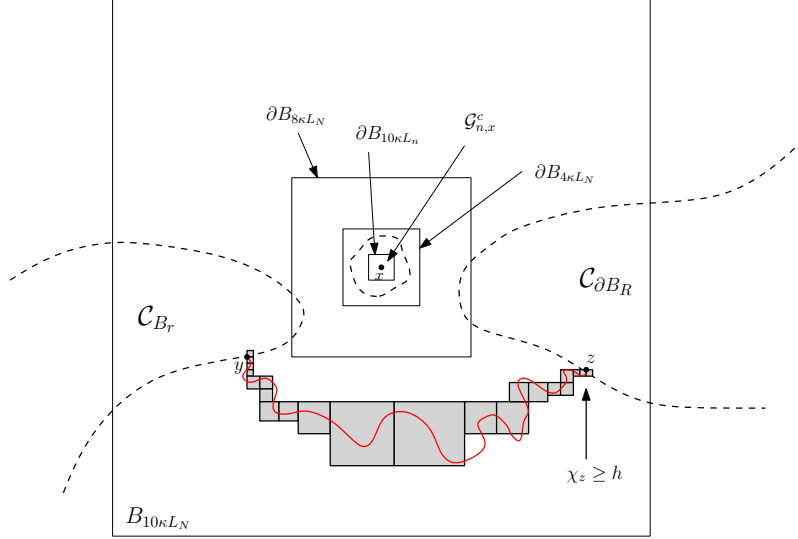
Figure 5 – Finding a good scale and reconstructing. On the event $\mathrm{CoarsePiv}_x(8\kappa L_N) \cap$ $\mathcal{G}_{N,x}$, one constructs a path (red) in $\{\chi \geq h\}$ connecting the boundaries (dotted) of the clusters of B_r and B_R. The point z is flipped to open and y becomes a pivotal point (Lemma 5.5). The occurrence of $\mathcal{G}_{n,x}^c$, decoupled by a dual surface, balances the reconstruction cost from the bridge (Lemma 5.6).

Proof of Lemma 5.6. In the proof below, we will consistently use the letters N and n as follows: n is the largest integer such that $u(L_N) \geq 20\kappa L_n$. Together with the definition of $\overline{T}$ (see above (5.18)), this implies $n > T$ whenever $N \geq \overline{T}$. Now, decomposing on the smallest good scale from T to S_x gives

$$1 = \sum_{N=T+1}^{S_x} \left(\mathbf{1}_{\mathcal{G}_{N,x}} \prod_{k=T+1}^{N-1} \mathbf{1}_{\mathcal{G}_{k,x}^c} \right) + \mathbf{1}_{\mathcal{G}_{S_x,x}^c} \leq \sum_{N=T+1}^{\overline{T}-1} \mathbf{1}_{\mathcal{G}_{N,x}} + \sum_{N=\overline{T}}^{S_x} \mathbf{1}_{\mathcal{G}_{N,x} \cap \mathcal{G}_{n,x}^c} + \mathbf{1}_{\mathcal{G}_{S_x,x}^c},$$

which in turn implies

(5.39)
$$\begin{aligned}
\mathbb{P}[\mathrm{CoarsePiv}_x(L_T)] &\leq \sum_{N=T}^{\overline{T}-1} \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), \mathcal{G}_{N,x}] \\
&\quad + \sum_{N=\overline{T}}^{S_x} \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), \mathcal{G}_{N,x}, \mathcal{G}_{n,x}^c] + \mathbb{P}[\mathcal{G}_{S_x,x}^c],
\end{aligned}$$

where we also used the monotonicity of $\mathrm{CoarsePiv}_x(L)$ in L. In view of (5.18), the first sum in the right-hand side of (5.39) is accounted for in the statement of Lemma 5.6. As for the second sum, we now decouple the events $\mathcal{G}_{n,x}^c$ using a similar technique as in the proof of Lemma 5.1, cf. also Fig. 4 and 5. The event $\mathcal{G}_{N,x} \cap \mathcal{G}_{n,x}^c$ is measurable relative to $\mathbf{Z}(\Lambda_x)$ where $\Lambda_x := B_{10\kappa L_n}(x) \cup B_{6\kappa L_N}(x)^c$; see the paragraph below (5.25) and (2.1)–(2.2). In particular,

$$\mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), \mathcal{G}_{N,x}, \mathcal{G}_{n,x}^c, B_{20\kappa L_n}(x) \overset{\chi \geq h}{\longleftrightarrow\!\!\!\!\!/} \partial B_{4\kappa L_N}(x) \,|\, \mathbf{Z}(\Lambda_x)]$$

$$= \mathbf{1}_{\mathcal{G}_{N,x} \cap \mathcal{G}_{n,x}^c} \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), B_{20\kappa L_n}(x) \overset{\chi \geq h}{\longleftrightarrow\!\!\!\!\!/} \partial B_{4\kappa L_N}(x) \,|\, \mathbf{Z}(\Lambda_x)].$$

Since our choice of (n, N) guarantees that $u(L_N) \geq 20\kappa L_n$ and $10\kappa L_n$ exceeds the range of χ, the standing assumption $q_R(t, h) \geq c_7$ implies a lower bound of the form $c_7 L_N^{-d}$ for the above disconnection probability under $\mathbb{P}[\cdot \mid \mathbf{Z}(\Lambda_x)]$ for all $N \leq S_x$. Now, applying the same argument as for (5.20) with E_x and F_x replaced by the above disconnection and coarse pivotality events, respectively, we deduce that

$$\mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N)|\mathbf{Z}(\Lambda_x)] \leq c_7^{-1} L_N^d \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), B_{20\kappa L_n}(x) \overset{\chi \geq h}{\longleftrightarrow\!\!\!/} \partial B_{4\kappa L_N}(x)|\mathbf{Z}(\Lambda_x)].$$

Plugging this inequality into the previous display and integrating with respect to $\mathbf{Z}(\Lambda_x)$ gives

$$\mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), \mathcal{G}_{N,x}, \mathcal{G}_{n,x}^c]$$
$$\leq c_7^{-1} L_N^d \, \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), B_{20\kappa L_n}(x) \overset{\chi \geq h}{\longleftrightarrow\!\!\!/} \partial B_{4\kappa L_N}(x), \mathcal{G}_{N,x}, \mathcal{G}_{n,x}^c].$$

However, since $N \leq S_x$ (recall its definition from the discussion preceding the statement of Lemma 5.6), $B_{10\kappa L_N}(x)$ has empty intersection with at least one of B_r or ∂B_R. We deduce from this fact that the event $\{\mathrm{CoarsePiv}_x(8\kappa L_N), B_{20\kappa L_n}(x) \longleftrightarrow\!\!\!/^{\chi \geq h} \partial B_{4\kappa L_N}(x), \mathcal{G}_{N,x}\}$ is measurable with respect to $\mathbf{Z}(B_{20\kappa L_n}(x)^c)$ and thus independent of $\mathcal{G}_{n,x}$ by (3.9) (our modification of property **G2**, cf. below (5.25), is geared towards this decoupling). Using this observation to factorize the right-hand side of the previous displayed inequality and subsequently bounding $\mathbb{P}[\mathcal{G}_{n,x}^c]$ by (5.26), we obtain for $\overline{T} \leq N \leq S_x$ that

$$\mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N) \cap \mathcal{G}_{N,x} \cap \mathcal{G}_{n,x}^c] \leq c_7^{-1} L_N^d \mathrm{e}^{-c_{11} L_n^\rho} \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), \mathcal{G}_{N,x}]$$
$$\leq \mathrm{e}^{-c L_n^\rho} \mathbb{P}[\mathrm{CoarsePiv}_x(8\kappa L_N), \mathcal{G}_{N,x}],$$

where in the final step we used the fact that $L_N \leq \exp[C(\log L_n)^3]$. Substituting this bound into (5.39) and using that $20\kappa \ell_0 L_n > u(L_N)$ completes the proof. $\qquad \square$

5.3   Adding the noise

We extend the bounds in (5.1) and (5.2) from $\delta = 0$ to some positive δ depending *only* on ε, with $L = L(\varepsilon)$ now fixed such that the conclusions of Proposition 1.4 hold for $\delta = 0$, see (5.1) and (5.2). It is enough to compare $\{\varphi^L \geq h\}$ to $T_\delta\{\varphi^L \geq h \pm \varepsilon\}$. To this end, consider some $\delta > 0$ and define for every $t \in [0, 1]$, the percolation process $\omega^{t,h} := T_{t\delta}\{\varphi^L \geq h\}$ (recall T_δ from below (1.13)) as well as $\theta(t, h) := \mathbb{P}[B_r \overset{\omega^{t,h}}{\longleftrightarrow} \partial B_R]$, for $r \leq R/2$. The analogue of (5.6) in this case reads

$$\partial_h \theta = -(1-\delta) \sum_{x \in \mathbb{Z}^d} \mathbb{P}[\mathrm{Piv}_x | \varphi_x^L = h] p(h) \quad \text{and} \quad \partial_t \theta = \frac{\delta}{2} \sum_{x \in \mathbb{Z}^d} \big(\mathbb{P}[\mathrm{Piv}_x, \omega_x^{t,h} = 0] - \mathbb{P}[\mathrm{Piv}_x, \omega_x^{t,h} = 1] \big),$$

where $p(\cdot)$ is the density of φ_0^L. Notice that

$$(5.40) \qquad\qquad |\partial_t \theta| \leq \frac{\delta}{2} \sum_{x \in \mathbb{Z}^d} \mathbb{P}[\mathrm{Piv}_x].$$

Define $\tilde{c}_7$ and $\tilde{q}_N(t, h)$ similarly to (5.8) and (5.9), but replacing φ by φ^L and $\{\chi^t \geq h\}$ by $\omega^{t,h}$, respectively. One then follows the proof of Lemma 5.1 – which is actually slightly simpler – to

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obtain that, under the condition that $\tilde{q}_R(t, h) \geq \tilde{c}_7$, the following inequality holds

$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}[\text{Piv}_x] \leq C(L)\Big(\sum_{x \in \mathbb{Z}^d} \mathbb{P}[\text{Piv}_x | \varphi_x^L = h] + \exp[-c(L)r^{c_8}] \Big).$$

Then, the proof follows similar lines of reasoning as the proof of (5.1) and (5.2) from Lemma 5.1 at the end of Section 5.1, choosing the prefactor δ appearing in (5.40) suitably small (recall that $L = L(\varepsilon)$ is fixed) to obtain an analogue of the differential inequality (5.14). We omit further details.

6   Proof of Proposition 1.3

Throughout this section, for a fixed $L \geq 0$ and $\delta \in (0, 1)$, we set

(6.1) $$\omega = \{\omega_h : h \in \mathbb{R}\}, \text{ where } \omega_h := T_\delta\{\varphi^L \geq h\}, \ h \in \mathbb{R}$$

(recall T_δ from the paragraph preceding the statement of Proposition 1.3). To be specific, we assume that ω is sampled in the following manner. There exists a collection of i.i.d. uniform random variables $\mathrm{U} = \{\mathrm{U}_x : x \in \mathbb{Z}^d\}$ independent of the process $\mathbf{Z}$ (recall the definition from Section 3.1) under $\mathbb{P}$ such that, for $h \in \mathbb{R}$,

(6.2) $$\omega_h(x) = \begin{cases} 0 & \text{if } \mathrm{U}_x \leq \delta/2, \\ \mathbf{1}_{\varphi_x^L \geq h} & \text{if } \mathrm{U}_x \in (\delta/2, 1 - \delta/2), \\ 1 & \text{if } \mathrm{U}_x \geq 1 - \delta/2. \end{cases}$$

We proceed to verify that ω satisfies the following properties:

(a) *Lattice symmetry.*   For all $h \in \mathbb{R}$, the law of ω_h is invariant with respect to translations of $\mathbb{Z}^d$, reflections with respect to hyperplanes and rotations by $\pi/2$.

(b) *Positive association.*   For all $h \in \mathbb{R}$, the law of ω_h is positively associated, i.e. any pair of increasing events satisfies the FKG-inequality.

(c) *Finite-energy.*   There exists $c_{\text{FE}} \in (0, 1)$ such that for any $h \in \mathbb{R}$,

$$\mathbb{P}[\omega_h(x) = 0 \,|\, \omega_h(y) \text{ for all } y \neq x] \in (c_{\text{FE}}, 1 - c_{\text{FE}}).$$

(d) *Bounded-range i.i.d. encoding.*   Let $\{\mathrm{V}(x) : x \in \mathbb{Z}^d\}$ denote a family of i.i.d. random variables (e.g. uniform in $[0, 1]$). Then, for every $h \in \mathbb{R}$, there exists a (measurable) function $g = g_h : \mathbb{R}^{\mathbb{Z}^d} \to \{0, 1\}^{\mathbb{Z}^d}$ such that the law of $g((\mathrm{V}(x))_{x \in \mathbb{Z}^d})$ is the same as that of ω_h and, for any $x \in \mathbb{Z}^d$, $g_x((v_y)_{y \in \mathbb{Z}^d})$ depends only on $\{v_y : y \in B_L(x)\}$. Thus, in particular, ω_h is an L-dependent process.

Property (a) is inherited from corresponding symmetries of the laws of U and φ^L. In view of (6.2), (3.2) and (3.7), ω_h is an increasing function of the independent collection $(\mathrm{U}, \mathbf{Z})$, and Property (b) follows by the FKG-inequality for independent random variables. Still by (6.2), (3.2) and (3.7), Properties (c) and (d) hold: for the former, take $c_{\text{FE}} = \delta/2$; for the latter one can use $\mathrm{V}(x)$ to generate the independent random variables U_x and $\mathbf{Z}_\ell(\tilde{z})$, for all $0 \leq \ell \leq L$ and $\tilde{z} \in \{x, x + \frac{1}{2}e_i, 1 \leq i \leq d\}$. We will use another property of ω_h, whose proof is more involved. We therefore state it as a separate lemma.

Lemma 6.1. *The field ω satisfies the following:*

(e) *Sprinkling property.* For every pair $h < h' \in \mathbb{R}$, there exists $\varepsilon = \varepsilon(h, h') > 0$ such that ω_h *stochastically dominates* $\omega_{h'} \vee \eta_\varepsilon$, where η_ε is a Bernoulli percolation with density ε independent of $\omega_{h'}$. Henceforth we will denote this domination by $\omega_h \succ \omega_{h'} \vee \eta_\varepsilon$.

Proof. We assume by suitably extending the underlying probability space that there exists $\eta = \{\eta_\varepsilon : \varepsilon \in (0, 1)\}$ with η independent of ω and η_ε distributed as i.i.d. Bernoulli variables with density ε. We will progressively replace the threshold h with h' in a finite number of steps. To this end we claim that for any $\kappa > 0$ there exists $\varepsilon > 0$ such that for any $\mathsf{h} \in [h, h']$ and $\mathbb{L} \in \mathcal{L}$, where $\mathcal{L}$ comprises all the (finitely many) translates of the sub-lattice $10L\,\mathbb{Z}^d$, we have

$$(6.3) \qquad\qquad \omega_{\mathsf{h}} \succ \omega_{\mathsf{h}+\kappa} \vee \eta_\varepsilon^{\mathbb{L}},$$

where $\eta_\varepsilon^{\mathbb{L}}(x) = \eta_\varepsilon(x)$ if $x \in \mathbb{L}$ and $\eta_\varepsilon^{\mathbb{L}}(x) = 0$ otherwise. Let us first explain how to derive property (e) from (6.3): choosing $\kappa := \frac{h'-h}{|\mathcal{L}|}$ gives $\omega_h \succ \omega_{h+\kappa} \vee \eta_\varepsilon^{\mathbb{L}}$, for suitable $\varepsilon = \varepsilon(h, h', L, d) > 0$ and any choice $\mathbb{L}$, so that iterating this over the lattices in $\mathcal{L}$ gives

$$\omega_h \succ \omega_{h+|\mathcal{L}|\kappa} \vee \bigvee_{\mathbb{L} \in \mathcal{L}} \eta_\varepsilon^{\mathbb{L}} = \omega_{h'} \vee \eta_\varepsilon,$$

as desired.

By approximation, it suffices to verify (6.3) for all fields restricted to a finite set $\Lambda \subset \mathbb{Z}^d$. Let $\omega_h(S) := \{\omega_h(x) : x \in S\}$ for $S \subset \mathbb{Z}^d$ and define similarly $\eta_\varepsilon(S)$, $\eta_\varepsilon^{\mathbb{L}}(S)$. Notice that $\omega_{\mathsf{h}} = \omega_{\mathsf{h}+\kappa} \vee \omega_{\mathsf{h}}$ and $\omega_{\mathsf{h}+\kappa} \vee \eta_\varepsilon^{\mathbb{L}}$ are both increasing functions of $(\omega_{\mathsf{h}+\kappa}, \omega_{\mathsf{h}})$ and $(\omega_{\mathsf{h}+\kappa}, \eta_\varepsilon^{\mathbb{L}})$ respectively. Therefore it suffices to show that, conditionally on any realization of $\omega_{\mathsf{h}+\kappa}$, the field $\omega_{\mathsf{h}}(\Lambda)$ stochastically dominates $\eta_\varepsilon^{\mathbb{L}}(\Lambda)$. To this end we fix an ordering $\{x_1, x_2, \dots\}$ of the vertices in Λ such that all the vertices in $\mathbb{L} \cap \Lambda$ appear before all the vertices in $\mathbb{L}^c \cap \Lambda$. In view of [37, Lemma 1.1], it then suffices to show that for any $k \geq 1$, with $\Lambda_k = \{x_1, \dots, x_{k-1}\}$ ($\Lambda_0 = \emptyset$),

$$(6.4) \qquad\qquad \mathbb{P}\text{-a.s.}, \quad \mathbb{P}[\,\omega_{\mathsf{h}}(x_k) = 1 \,|\, A, A'\,] \geq \varepsilon \cdot \mathbf{1}_{x_k \in \mathbb{L}},$$

where $A := \{\omega_{\mathsf{h}}(x) = \sigma(x), x \in \Lambda_k\}$, $A' := \{\omega_{\mathsf{h}+\kappa}(x) = \sigma'(x), x \in \Lambda\}$, for arbitrary $\sigma, \sigma' \in \{0, 1\}^{\mathbb{Z}^d}$ with $\sigma \geq \sigma'$ (as $\omega_{\mathsf{h}} \geq \omega_{\mathsf{h}+\kappa}$). We now show (6.4) and assume that $x_k \in \mathbb{L}$ (the other cases are trivial). To this end let us first define, for any $x \in \mathbb{Z}^d$, the pair of events

$$(6.5) \qquad\qquad \mathcal{T}(x) := \{\mathrm{U}_x \notin (\delta/2, 1 - \delta/2)\} \quad \text{and} \quad \mathcal{U}(x) := \bigcap_{y \in B_L(x) \setminus \{x\}} \mathcal{T}(y).$$

Notice that, in view of (6.2), $\mathcal{T}(x)$ corresponds to the event that the noise at x is *triggered*. We then write, with $\mathcal{U} = \mathcal{U}(x_k)$,

$$(6.6) \qquad\qquad \mathbb{P}[\omega_{\mathsf{h}}(x_k) = 1 \,|\, A, A'] \geq \mathbb{P}[\omega_{\mathsf{h}}(x_k) = 1 \,|\, \mathcal{U}, A, A'] \cdot \mathbb{P}[\mathcal{U} \,|\, A, A'].$$

We will bound the two probabilities on the right-hand side separately from below. It follows from the definition of ω_h in (6.2) that $\omega_{\mathsf{h}+\kappa}(y) = \zeta(y)$ on the event $\mathcal{T}(y)$, where $\zeta(y) = \mathbf{1}_{\mathrm{U}_y \geq 1-\delta/2}$. Consequently, decomposing A', we obtain, with $A_1' := \{\omega_{\mathsf{h}+\kappa} = \sigma' \text{ on } \Lambda \cap B_L(x_k)^c\}$ and $A_2' := \{\zeta = \sigma' \text{ on } B_L(x_k) \setminus \{x_k\}\}$, that

$$p := \mathbb{P}[\omega_{\mathsf{h}}(x_k) = 1 \,|\, \mathcal{U}, A, A'] = \mathbb{P}[\omega_{\mathsf{h}}(x_k) = 1 \,|\, \mathcal{U}, A, \omega_{\mathsf{h}+\kappa}(x_k) = \sigma'(x_k), A_1', A_2'].$$

However, in view of definition (6.2), (6.5) and the fact that $\mathbb{L}$ is $10L$-separated, it follows that both $\{\omega_{\mathsf{h}}(x_k) = 1, \omega_{\mathsf{h}+\kappa}(x_k) = \sigma'(x_k)\}$ and $\{\omega_{\mathsf{h}+\kappa}(x_k) = \sigma'(x_k)\}$ are independent of $\mathcal{U} \cap A \cap A_1' \cap A_2'$, leading to

$$p = \mathbb{P}[\omega_{\mathsf{h}}(x_k) = 1 \,|\, \omega_{\mathsf{h}+\kappa}(x_k) = \sigma'(x_k)].$$

Now, $p = 1$ when $\sigma'(x_k) = 1$. On the other hand, when $\sigma'(x_k) = 0$, we have, using (6.2), (6.5),

$$p \geq \mathbb{P}[\omega_{\mathsf{h}}(x_k) = 1 \,|\, \mathcal{T}(x_k)^c, \omega_{\mathsf{h}+\kappa}(x_k) = 0]\,\mathbb{P}[\mathcal{T}(x_k)^c \,|\, \omega_{\mathsf{h}+\kappa}(x_k) = 0]$$
$$= \mathbb{P}[\varphi_{x_k}^L \geq \mathsf{h} \,|\, \varphi_{x_k}^L < \mathsf{h} + \kappa]\,\mathbb{P}[\mathcal{T}(x_k)^c]\,\mathbb{P}[\varphi_{x_k}^L < \mathsf{h} + \kappa]/\mathbb{P}[\omega_{\mathsf{h}+\kappa}(x_k) = 0] \geq \varepsilon',$$

where $\varepsilon' > 0$ depends only on δ, h, h', L and d. As to bounding the second term on the right of (6.6), we write, with $A''(\rho) := \{\mathbf{1}_{\varphi_y^L \geq \mathsf{h}+\kappa} = \rho(y), y \in B_L(x_k) \setminus \{x_k\}\}$,

$$\mathbb{P}[\mathcal{U} \,|\, A, \, A'] \geq \inf_\rho \mathbb{P}[\mathcal{U} \,|\, A, \, A', \, A''(\rho)]$$

$$= \inf_\rho \prod_{y \in B_L(x_k) \setminus \{x_k\}} \mathbb{P}[\,\mathcal{T}(y) \,|\, \mathbf{1}_{\varphi_y^L \geq \mathsf{h}+\kappa} = \rho(y), \omega_{\mathsf{h}+\kappa}(y) = \sigma'(y)] \geq (\delta/2)^{|B_L|}$$

where the equality in the second line follows by (6.2) and (6.5) upon conditioning on $\omega_{\mathsf{h}}(x)$, $x \in \Lambda_k$, $\omega_{\mathsf{h}+\kappa}(y)$, $y \in (\Lambda \setminus B_L(x_k)) \cup \{x_k\}$ and $\mathbf{1}_{\varphi_y^L \geq \mathsf{h}+\kappa}$, $y \in B_L(x_k) \setminus \{x_k\}$, and the final lower bound follows by distinguishing whether $\rho(y) \neq \sigma'(y)$ (in which case the given conditional probability equals 1) or not, and using (6.2). Overall, the right-hand side of (6.6) is thus bounded from below by $\varepsilon := \varepsilon'(\delta/2)^{|B_L|}$, which implies (6.4) and completes the verification of (e). $\qquad\square$

The rest of this section is devoted to deriving Proposition 1.3 from Properties (a)–(e). We prove in two parts that $\tilde{h}(\delta, L) \geq h_*(\delta, L)$ and $h_*(\delta, L) = h_{**}(\delta, L) \geq \tilde{h}(\delta, L)$ which together imply $\tilde{h}(\delta, L) = h_*(\delta, L) = h_{**}(\delta, L)$, as asserted.

We first argue that $\tilde{h}(\delta, L) \geq h_*(\delta, L)$. As a consequence of Properties (a)–(c) and (e), one can adapt the argument in [33] in the context of Bernoulli percolation — see also [32, Chap. 7.2] — to deduce that ω_h percolates in "slabs", i.e. for every $h < h_*(\delta, L)$, there exists $M \in \mathbb{N}$ such that

$$(6.7) \qquad \mathbb{P}\big[0 \xleftarrow[\mathbb{Z}^2 \times \{0,\ldots,M\}^{d-2}]{\omega_h} \infty\big] > 0.$$

Now, fix $h < h_*(\delta, L)$ and M such that the above holds. Set $\mathbb{S} := \mathbb{Z}^2 \times \{0, \ldots, M\}^{d-2}$ and $x_i := (0, \ldots, 0, (M+L)i)$, and observe that since the range of ω_h is L, it follows that for every $R \geq C(h)$,

$$\mathbb{P}[B_{u(R)} \xleftrightarrow{\omega_h} B_R] \leq \prod_{i < u(R)/(M+L)} \mathbb{P}[x_i \xleftrightarrow[x_i + \mathbb{S}]{\omega_h} \infty] \overset{(6.7)}{\leq} \mathbb{P}[0 \xleftrightarrow[\mathbb{S}]{\omega_h} \infty]^{u(R)/(M+L)}.$$

But this implies $h \leq \tilde{h}(\delta, L)$, as desired.

The proof of $h_*(\delta, L) = h_{**}(\delta, L) \geq \tilde{h}(\delta, L)$ on the other hand follows directly from the exponential decay of ω_h in the subcritical regime. More precisely, we claim that for every $h > h_*(\delta, L)$, there exists $c = c(h) > 0$ such that

$$(6.8) \qquad \mathbb{P}[0 \xleftrightarrow{\omega_h} \partial B_R] \leq \exp(-cR), \quad \text{for every } R \geq 0,$$

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which implies that $h_*(\delta, L) = h_{**}(\delta, L)$, cf. (1.4) (recall that $h_{**}(\delta, L) \geq \tilde{h}(\delta, L)$, as explained below (1.12) for $\delta = 0$ and $L = \infty$). We therefore focus on the proof of (6.8). For each $h \in \mathbb{R}$, consider the family of processes $\gamma_\varepsilon := \omega_h \vee \eta_\varepsilon$ indexed by $\varepsilon \in [0, 1]$ where η_ε is a Bernoulli percolation with density ε independent of ω_h. Let $\varepsilon_c = \varepsilon_c(h) \in [0, 1]$ denote the critical parameter of the family of percolation processes $\{\gamma_\varepsilon : \varepsilon \in [0, 1]\}$ and suppose for a moment that for every $0 \leq \varepsilon < \varepsilon_c(h)$, there exists $c = c(h, \varepsilon) > 0$ such that for every $R \geq 1$,

$$(6.9) \qquad \theta_R(\varepsilon) := \mathbb{P}[0 \overset{\gamma_\varepsilon}{\longleftrightarrow} \partial B_R] \leq \exp(-cR).$$

Then (6.8) follows immediately by taking $\varepsilon = 0$ in case $\varepsilon_c(h) > 0$ for all $h > h_*(\delta, L)$. But the latter holds by the following reasoning. From Property (e), we see that whenever $h > h_*(\delta, L)$, choosing $h' = (h_*(\delta, L) + h)/2$, one has $\omega_{h'} \succ \omega_h \vee \eta_{\varepsilon'}$ for some $\varepsilon' > 0$, whence $\varepsilon' \leq \varepsilon_c(h)$ as $\omega_{h'}$ is subcritical. The remainder of this subsection is devoted to proving (6.9).

For this purpose, we use the strategy developed in the series of papers [29, 28, 27] to prove subcritical sharpness using decision trees. This strategy consists of two main parts. In the first part, one bounds the variance $\theta_R(1-\theta_R)$ using the OSSS inequality from [46] by a weighted sum of influences where the weights are given by revealment probabilities of a randomized algorithm (see (6.14) below). Then in a second part one relates these influence terms to the derivative of $\theta_R(\varepsilon)$ with respect to ε to deduce a system of differential inequalities of the following form:

$$(6.10) \qquad \theta'_R \geq \beta \frac{R}{\Sigma_R} \theta_R(1 - \theta_R),$$

where $\Sigma_R := \sum_{r=0}^{R-1} \theta_r$ and $\beta = \beta(\varepsilon) : (0, 1) \to (0, \infty)$ is a continuous function. Using purely analytical arguments, see [29, Lemma 3.1], one then obtains (6.9) for $\varepsilon < \varepsilon_c$.

In our particular context, we apply the OSSS inequality for product measures (see [29] for a more general version) and for that we use the encoding of ω_h in terms of $\{V(x) : x \in \mathbb{Z}^d\}$ provided by Property (d). In view of this, $\mathbf{1}_{\mathcal{E}_R}$ with $\mathcal{E}_R = \{0 \overset{\gamma_\varepsilon}{\longleftrightarrow} \partial B_R\}$ can now be written as a function of independent random variables $(V(x) : x \in B_{R+L})$ and $(\eta_\varepsilon(x) : x \in B_R)$. Using a randomized algorithm very similar to the one used in [27, Section 3.1] (see the discussion after the proof of Lemma 6.2 below for a more detailed exposition), we get

$$(6.11) \qquad \sum_{x \in B_{R+N}} \mathrm{Inf}_{V(x)} + \sum_{x \in B_R} \mathrm{Inf}_{\eta_\varepsilon(x)} \geq cL^{-(d-1)} \frac{R}{\Sigma_R} \theta_R(1 - \theta_R),$$

where $\mathrm{Inf}_{V(x)} := \mathbb{P}[\mathbf{1}_{\mathcal{E}_R}(V, \eta_\varepsilon) \neq \mathbf{1}_{\mathcal{E}_R}(\tilde{V}, \eta_\varepsilon)]$ with $\tilde{V}$ being the same as V for every vertex except at x where it is *resampled independently*, and $\mathrm{Inf}_{\eta_\varepsilon(x)}$ is defined similarly. In order to derive (6.10) from (6.11), we use the following lemma.

Lemma 6.2. *There exists a continuous function* $\alpha : (0, 1) \to (0, \infty)$ *such that for all* $R \geq 1$,

$$(6.12) \qquad \theta'_R(\varepsilon) \geq \alpha(\varepsilon) \left(\sum_{x \in B_{R+L}} \mathrm{Inf}_{V(x)} + \sum_{x \in B_R} \mathrm{Inf}_{\eta_\varepsilon(x)} \right).$$

Proof. Since the derivative is with respect to the parameter of the Bernoulli component of the process, it follows from standard computations that

$$\theta'_R = \mathbb{P}[\omega_h(0) = 0] \sum_{x \in B_R} \mathbb{P}[\mathrm{Piv}_x].$$

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where Piv_x is the same event as in Section 5 (see below (5.6)) with χ^t replaced by γ_ε. Now, on the one hand, $\mathrm{Inf}_{\eta_\varepsilon(x)} \le \mathbb{P}[\mathrm{Piv}_x]$. On the other hand, since $\mathrm{V}(x)$ affects the states of vertices only in $B_L(x)$ by Property (d), one immediately gets $\mathrm{Inf}_{\mathrm{V}(x)} \le \mathbb{P}[\mathrm{Piv}(B_L(x))]$, where $\mathrm{Piv}(B_L(x))$ – called the *pivotality of the box $B_L(x)$* – is the event that 0 is connected to ∂B_R in $\gamma_\varepsilon \cup B_L(x)$ where as it is not in $\gamma_\varepsilon \setminus B_L(x)$. In fact, due to finite-energy property of ω_h, we can write

$$\mathbb{P}[\mathrm{Piv}(B_L(x))] \le (c_{\mathrm{FE}}(1 - \varepsilon))^{-c' L^d} \mathbb{P}[\mathrm{Piv}(B_L(x)), \gamma_\varepsilon(y) = 0 \text{ for all } y \in B_L(x)].$$

If we start on the event on the right-hand side and then open the vertices inside $B_L(x)$ one by one in γ_ε until 0 gets connected to ∂B_R, which must happen by the definition of $\mathrm{Piv}(B_L(x))$. The last vertex to be opened is pivotal for the resulting configuration. This implies that

$$\mathbb{P}[\mathrm{Piv}(B_L(x)), \eta_\varepsilon(y) = 0 \text{ for all } y \in B_L(x)] \le \varepsilon^{-c' L^d} \sum_{y \in B_L(x)} \mathbb{P}[\mathrm{Piv}_y].$$

Combining all these displays yields (6.12). □

To conclude, we now give a full derivation of (6.11) for sake of completeness. To this end let us define a *randomized* algorithm T which takes $(\mathrm{V}(x) : x \in B_{R+L})$ and $(\eta_\varepsilon(x) : x \in B_R)$ as inputs, and determines the event $\mathcal{E}_R$ by revealing $\mathrm{V}(x), \eta_\varepsilon(x)$ one by one as follows:

Definition 6.3 (Algorithm T). Fix a deterministic ordering of the vertices in B_R and choose $j \in \{1, \ldots, R\}$ uniformly and independently of $(\mathrm{V}, \eta_\varepsilon)$. Now set x_0 to be the smallest $x \in \partial B_j$ in the ordering and reveal $\eta_\varepsilon(x_0)$ as well as $\mathrm{V}(y)$ for all $y \in B_L(x_0)$. Set $E_0 := \{x_0\}$ and $C_0 := \{x_0\}$ if $\gamma_\varepsilon(x_0) = 1$ and $C_0 := \emptyset$ otherwise. At each step $t \ge 1$, assume that E_{t-1} and C_{t-1} have been defined. Then,

- If the intersection of B_R with the set $(\partial_{\mathrm{out}} C_{t-1} \cup \partial B_j) \setminus E_{t-1}$ is non-empty, let x be the smallest vertex in this intersection and set $x_t := x$, $E_t := E_{t-1} \cup \{x_t\}$. Reveal $\eta_\varepsilon(x_t)$ as well as $\mathrm{V}(y)$ for all $y \in B_L(x_t)$ and set $C_t := C_{t-1} \cup \{x_t\}$ if $\gamma_\varepsilon(x_t) = 1$ and $C_t := C_{t-1}$ otherwise.

- If the intersection is empty, halt the algorithm.

In words, the algorithm T explores the clusters in γ_ε of the vertices in ∂B_j. By applying the OSSS inequality [46] to $\mathbf{1}_{\mathcal{E}_R}$ and the algorithm T, we now get

$$(6.13) \qquad \mathrm{Var}[\mathbf{1}_{\mathcal{E}_R}] = \theta_R (1 - \theta_R) \le \sum_{x \in B_{R+L}} \theta_{\mathrm{V}(x)}(\mathsf{T}) \, \mathrm{Inf}_{\mathrm{V}(x)} + \sum_{x \in B_R} \theta_{\eta_\varepsilon(x)}(\mathsf{T}) \, \mathrm{Inf}_{\eta_\varepsilon(x)},$$

where the function $\theta_\cdot(\mathsf{T})$, called the revealment of the respective variable, is defined as

$$(6.14) \qquad\qquad\qquad \theta_{\mathrm{V}(x)}(\mathsf{T}) := \pi[\mathsf{T} \text{ reveals the value of } \mathrm{V}(x)],$$

and $\theta_{\eta_\varepsilon(x)}(\mathsf{T})$ is defined in a similar way. Here π denotes the probability governing the extension of $(\mathrm{V}, \eta_\varepsilon)$ which accommodates the random choice of layer j, which is independent of $(\mathrm{V}, \eta_\varepsilon)$.

Let us now bound the revealments for $\mathrm{V}(x)$ and $\eta_\varepsilon(x)$. Notice that since $\mathrm{V}(x)$ affects the state of vertices only in $B_L(x)$ by Property (d), $\mathrm{V}(x)$ is revealed only if there is an explored

vertex $y \in B_L(x) \cap B_R$. The vertex y, on the other hand, is explored only if y is connected to ∂B_j in γ_ε. We deduce that

$$\theta_{\mathsf{V}(x)}(\mathsf{T}) \leq \frac{1}{R} \sum_{j=1}^{R} \mathbb{P}[(B_L(x) \cap B_R) \overset{\gamma_\varepsilon}{\longleftrightarrow} \partial B_j] \leq \frac{1}{R} \sum_{j=1}^{R} \sum_{y \in B_L(x) \cap B_R} \mathbb{P}[y \overset{\gamma_\varepsilon}{\longleftrightarrow} \partial B_j],$$

where in the last step we used a naive union bound. For $\eta_\varepsilon(x)$, it is even simpler:

$$\theta_{\eta_\varepsilon(x)}(\mathsf{T}) \leq \frac{1}{R} \sum_{j=1}^{R} \mathbb{P}[x \overset{\gamma_\varepsilon}{\longleftrightarrow} \partial B_j].$$

Now (6.11) follows by plugging the previous two displays into (6.13) combined with the translation invariance of γ_ε implied by Property (a) and the triangle inequality.

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