WEIGHTED TANGO BUNDLES ON $\mathbb{P}^n$ AND THEIR MODULI SPACES

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Abstract. We define a new class of algebraic $(n-1)$-bundles on $\mathbb{P}^n$, that contains the bundles introduced by Tango [14] and their stable generalized pull-backs; we show that these bundles are invariant under small deformations and that they correspond to smooth points of moduli spaces.

It is a very difficult problem to find examples of non-splitting algebraic vector bundles on the complex projective space $\mathbb{P}^n$ whose rank is less than n. In particular for $n \geq 6$ the only known examples are essentially the mathematical instantons [3] (for odd n) and the bundles introduced by Tango [14]: all of them have rank $n-1$. Of course, pulling back the Tango bundles by a finite morphism $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ gives other examples of rank $n-1$ bundles.

In [9], Horrocks introduced a new technique of constructing new bundles from old ones, which generalizes the pull-back. This method, that we can call *generalized pull-back*, has been extensively studied in [1] and [2] and it applies only to bundles whose symmetry group contains a copy of $\mathbb{C}^*$.

In this paper we show that, for any $n \geq 3$, there exists a Tango bundle that is SL(2)-invariant: hence the generalized pull-back allows us to define a new class of $(n-1)$-bundles on $\mathbb{P}^n$.

More precisely, let α, γ be integer numbers such that $\gamma > n\alpha \geq 0$ and let $Q_{\alpha,\gamma}$ be the bundles on $\mathbb{P}^n$ described by the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-\gamma) \to \bigoplus_{k=0}^{n} \mathcal{O}_{\mathbb{P}^n}((n-2k)\alpha) \to Q_{\alpha,\gamma} \to 0.$$

$Q_{\alpha,\gamma}$ can also be defined as the generalized pull-back of the quotient bundle on $\mathbb{P}^n$ and, in particular, $Q_{0,1}$ is the quotient bundle. Let us define the rank $2n-1$ vector bundle:

$$\mathcal{V} = S^{2(n-1)}(\mathcal{O}_{\mathbb{P}^n}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^n}(-\alpha)) = \bigoplus_{k=0}^{2(n-1)} \mathcal{O}_{\mathbb{P}^n}((2n-1-2k)\alpha).$$

It will be proven that there exists an exact sequence of algebraic vector bundles over $\mathbb{P}^n$:

$$0 \to Q_{\alpha,\gamma}(-\gamma) \to \mathcal{V} \to F_{\alpha,\gamma}(\gamma) \to 0. \tag{1}$$

The $(n-1)$-bundles $F_{\alpha,\gamma}$ are called *weighted Tango bundles of weights* α *and* γ and they are stable if and only if $\gamma > 2(n-1)\alpha$. The bundles $F_{0,1}$ are the classical Tango

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bundles, moreover the generalized pull-backs of the Tango bundles are contained in the sequence (1). The main result of this paper is the following:

Theorem 0.1. *Let $F^o_{\alpha,\gamma}$ be a stable weighted Tango bundle on $\mathbb{P}^n$ of weights α and γ and let c_i be the i-th Chern class of $F^o_{\alpha,\gamma}$ (in particular $c_1 = 0$). There exists a smooth neighborhood of the point of the moduli space $\mathcal{M}_{\mathbb{P}^n}(0, c_2, \dots, c_{n-1})$ corresponding to $F^o_{\alpha,\gamma}$ entirely consisting of weighted Tango bundles of weights α and γ.*

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1. **Introduction.**

Let V be a $(n+1)$-dimensional vector space over $\mathbb{C}$, and let $\mathbb{P}^n = \mathbb{P}(V)$: it is possible to show (cf. [11]) that a Tango bundle F on $\mathbb{P}^n$ is contained in the following exact sequence:

$$0 \to Q(-1) \to \frac{\wedge^2 V}{W} \otimes \mathcal{O}_{\mathbb{P}^n} \to F(1) \to 0;$$

here Q is the quotient bundle (cf. [13]) on $\mathbb{P}^n$ and $W \subseteq \wedge^2 V$ is a linear subspace such that:

$$\begin{cases} \dim_{\mathbb{C}} \mathbb{P}(W) = m - 1 \\ \mathbb{P}(W) \cap \mathbb{G}(1, n) = \emptyset \end{cases} \tag{2}$$

where $m = \frac{(n-2)\ (n-1)}{2}$ and $\mathbb{G}(1, n)$ is the Grassmannian of the lines in $\mathbb{P}^n = \mathbb{P}(V)$: hence W does not contain any decomposable bivectors. Moore [12] has shown that F is uniquely determined by the subspace $W \subseteq \wedge^2 V$ and so by a point of the variety $\mathbb{G}(m-1, N-1)$, with $N = \frac{n\ (n+1)}{2}$; furthermore if W is invariant under the action of a group $G \subseteq \mathbb{P}\,\mathrm{GL}(n+1)$ then the Tango bundle, associated to W, is G-invariant too, i.e. $G \subseteq \operatorname{Sym} F$.

2. **Action of** $\mathrm{SL}(2)$.

Let U be a 2-dimensional vector space over $\mathbb{C}$ and let us consider the complex projective space $\mathbb{P}^n = \mathbb{P}(S^n U)$: in this way, we have a natural action of $\mathrm{SL}(2) = \mathrm{SL}(U)$ over $\mathbb{P}^n$.

We want to find a subspace $W \subseteq \wedge^2 S^n U$, $\mathrm{SL}(2)$-invariant and that satisfies (2). For this purpose we prove the following:

Proposition 2.1. *The decomposition of $\wedge^2 S^n U$ into irreducible representations is given by $S^{2(n-1)}U \oplus S^{2(n-3)}U \oplus S^{2(n-5)}U \oplus \dots$; moreover if $W = S^{2(n-3)}U \oplus S^{2(n-5)}U \oplus \dots$, then W satisfies (2).*

This proposition immediately implies that for any $n \in \mathbb{N}$, such a subspace W defines a $\mathrm{SL}(2)$-invariant Tango bundle F on $\mathbb{P}^n$, which is described by the exact sequence:

$$0 \to Q(-1) \to S^{2(n-1)}U \otimes \mathcal{O}_{\mathbb{P}^n} \to F(1) \to 0.$$

Before proceeding with the proof of the proposition, we prove the following lemma:

Lemma 2.2. *Let $\{v_0, \dots, v_n\}$ be a basis of V and $\omega \in \mathbb{G}(1,n) \subseteq \wedge^2 V$ a non-vanishing decomposable bivector, then:*

$$\omega = x_{i_0,j_0}(v_{i_0} \wedge v_{j_0}) + \sum_{i+j>i_0+j_0} x_{i,j}(v_i \wedge v_j)$$

where $x_{i,j} \in \mathbb{C}$ and $x_{i_0,j_0} \neq 0$.

Remark. In order to simplify the notations, we will often write $v_{i,j}$ instead of $v_i \wedge v_j$.

Proof. We proceed by induction on n. For $n = 1$, there is nothing to prove.

Let us suppose now $n > 1$ and let $\omega = v \wedge v'$ where $v = \sum x_i v_i$ and $v' = \sum y_i v_i$.

Let $z_{i,j} = x_i y_j - x_j y_i$ then

$$\omega = \sum_{i+j \geq k_0}^{i<j} z_{i,j} v_{i,j},$$

where $k_0 = \min\{k | z_{i,j} = 0 \text{ if } i + j = k\}$.

If there exist $i_0, j_0 \neq 0$ such that $i_0 + j_0 = k$, and $z_{i_0,j_0} \neq 0$ then, since $z_{0,i_0} = z_{0,j_0} = 0$, it easily follows $x_0 = y_0 = 0$: thus the lemma is true by induction.

Otherwise, if such i_0, j_0 do not exist, then:

$$\omega = z_{0,k} v_{0,k} + \sum_{i+j>k_0}^{i<j} z_{i,j} v_{i,j}.$$

□

Proof of proposition 2.1.

Let $V = S^n U$ and let $\{x, y\}$ be a basis of V: if $v_0 = x^n, \dots, v_n = y^n$, then $\{v_0 \dots v_n\}$ is a basis of V. The weights of $S^n U$ are $\{n, n-2, \dots, -n\}$ (cf. [6], pag. 146–153) and since the weights of $\wedge^2 S^n U$ are given by the sums of couples of different weights of $S^n U$, it easily follows:

$$\wedge^2 S^n U = S^{2(n-1)} U \oplus S^{2(n-3)} U \oplus S^{2(n-5)} U \oplus \dots$$

Indeed if $W = S^{2(n-3)} U \oplus S^{2(n-5)} U \oplus \dots$, then $\dim_{\mathbb{C}} W = m$.

Let us prove now that W does not contain any decomposable bivector, as required. We suppose that there exists $\omega \in W \cap \mathbb{G}(1,n)$, such that $\omega \neq 0$; by the previous lemma, we get:

$$\omega = x_{i_0,j_0}\ v_{i_0,j_0} + \sum_{i+j>i_0+j_0} x_{i,j}\ v_{i,j}$$

where $x_{i_0,j_0} \neq 0$. We want to show that, in this case, there exists a vector of weight $2(n-1)$ in W: this contradicts with the fact that $S^{2(n-1)} U \cap W = \{0\}$.

Let $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{SL}(2)$, and let $\tilde{Y}, \tilde{H}$ be the corresponding endomorphisms of $\wedge^2 S^n U$. If we suppose $v_{n+1} = 0$, we have:

$$\tilde{Y}(v_{i,j}) = (n-i)\ v_{i+1,j} + (n-j)\ v_{i,j+1} \qquad \text{for any } i, j = 0, \dots, n.$$

Hence if $k = (2n-1) - i_0 - j_0$, then $x_{i_0,j_0}\ \tilde{Y}^{(k)}(v_{i_0} \wedge v_{j_0}) = \tilde{Y}^{(k)}(\omega) \in W$. On the other hand it results that $\tilde{Y}^{(k)}(v_{i_0,j_0}) = m\ v_{n,n-1}$, where m is a positive integer: this implies that $v_{n,n-1} \in W$ and since $\tilde{H}(v_{n,n-1}) = -2(n-1) v_{n,n-1}$, we see that W contains a vector of weight $2(n-1)$. □

Remark. Moore [12] has shown that the Tango bundles on $\mathbb{P}^4$ have all symmetry groups isomorphic to $\mathbb{P}\,\mathrm{O}(3)$ and that $\mathbb{P}\,\mathrm{GL}(5)$ acts transitively on the moduli space of the Tango bundles $\mathcal{M}_{\mathbb{P}^4}(0,2,2)$. In higher dimensions the situation is different: in fact, with the help of the software Macaulay 2 [7], it has been possible to prove that on $\mathbb{P}^5$ the generic Tango bundle has a discrete symmetry group and that there exist Tango bundles with the symmetry group isomorphic to $\mathbb{C}^*$ (for istance the one defined by $W = < v_{0,5} + 5v_{2,3}, v_{1,4} + 3v_{2,3}, v_{0,4} - 2v_{1,3}, v_{2,5} + v_{3,4}, v_{0,3} + 3v_{1,2}, 2v_{2,5} - 3v_{3,4} >$).

The algorithm needed to calculate the dimension of the orbit of a subspace $W_0 \subseteq \wedge^2 V$ (where $n = 5$) under the action of $\mathbb{P}\,\mathrm{GL}(6)$ was communicated to the author by G. Ottaviani. We describe the fundamental steps of it:

1. Let us choose m as a (6×15)-matrix whose rows represent the generators of the subspace $W_0 \subseteq \wedge^2 V$;
2. We denote by $g = \{g_{i,j}\}$ a generic (6×6)-matrix and let's define $m' = m * \wedge^2 g$:

m' represents the image gW_0 of the matrix $g \in \mathbb{P}\,\mathrm{GL}(6)$ by the map $\eta : \mathbb{P}\,\mathrm{GL}(6) \to \mathbb{G}(6, \wedge^2 V)$; By the Plucker embedding $\phi : \mathbb{G}(m, \wedge^2 V) \hookrightarrow \mathbb{P}^{5004}$, the dimension of the orbit of W_0 is equal to the dimension of the ideal generated by the minors 6×6 of m', but its calculation is, computationally, too difficult. Therefore in order to make the computation easier, we first calculate the derivative $d(\phi \circ \eta)$ at the identity matrix and then we compute the dimension of its image: this number is exactly the dimension of the orbit. We proceed as follows:

3. Let $v_1(g), \ldots, v_6(g)$ be the rows of m', and let $v_i(g)_{g_{i,j}} = \frac{\partial v_i(g)}{\partial g_{i,j}}$.

In order to compute the derivative $d(\phi \circ \eta)$, we remind that, for any $I \subseteq \{1, \ldots, 15\}$ such that $\#I = 6$, we have:

$$\frac{\partial}{\partial g_{i,j}} \det \begin{pmatrix} v_0^I(g) \\ \vdots \\ v_6^I(g) \end{pmatrix} = \det \begin{pmatrix} v_0^I(g)_{g_{i,j}} \\ v_1^I(g) \\ \vdots \\ v_6^I(g) \end{pmatrix} + \cdots + \det \begin{pmatrix} v_0^I(g) \\ \vdots \\ v_5^I(g) \\ v_6^I(g)_{g_{i,j}} \end{pmatrix}$$

where $v_i^I(g)$ denotes the vector composed by the components of $v_i(g)$ with index in I.

4. Let's define $M_{i,j}^k = \begin{pmatrix} v_1(\mathrm{Id}_6) \\ \vdots \\ v_k(\mathrm{Id}_6)_{g_{i,j}} \\ \vdots \\ v_6(\mathrm{Id}_6) \end{pmatrix}$;
5. let $p_{i,j}$ be the sum of the vectors in $\mathbb{P}^{5004}$ defined by the minors of $M_{i,j}^k$ with $k = 1, \ldots, 6$;
6. The rank of the matrix $\begin{pmatrix} p_{1,1} \\ p_{1,2} \\ \vdots \\ p_{6,6} \end{pmatrix}$ is the dimension of the orbit of W_0.

3. **Weighted Tango Bundles.**

We have shown that for any n, there exists a Tango bundle F on $\mathbb{P}(S^nU)$ that is invariant under the $\mathbb{C}^*$-action defined by:

$$\begin{pmatrix} t^n & & & \\ & t^{n-2} & & \\ & & \ddots & \\ & & & t^{-n} \end{pmatrix} \in \mathbb{P}\,\mathrm{GL}(n+1) \qquad \text{for any } t \in \mathbb{C}^*$$

This map induces an embedding of $\mathbb{C}^*$ in $\mathrm{Sym}\, F$ and so it is possible to study the pull-backs over $\mathbb{C}^{n+1} \setminus 0$ of such bundles (cf. [1, 2]).

Let us fix $\alpha, \gamma \in \mathbb{N}$ such that $\gamma > n\alpha$ and let $f_0, \ldots, f_n \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous polynomial of degree:

$$\deg f_k = \gamma + (n-2k)\,\alpha \qquad \text{for each } k = 0, \ldots, n$$

and without common roots.

Let $\phi = (f_0, \ldots, f_n)$ and let us take into account the following diagram:

$$\begin{array}{ccc} \mathbb{C}^{n+1} \setminus 0 & \xrightarrow{\phi} & S^nU \setminus 0 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}^n & & \mathbb{P}^n \end{array}$$

According to [1, 9], there exists an algebraic vector bundle $F_{\alpha,\gamma}$ on $\mathbb{P}^n$ such that $\pi_1^* F_{\alpha,\gamma} = \phi^* \pi_2^* F$. Furthermore, since Q is an homogeneous bundle [13], there exists $Q_{\alpha,\gamma}$ such that $\pi_1^* Q_{\alpha,\gamma} = \phi^* \pi_2^* Q$. Such a bundle is contained in the weighted Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-\gamma) \to S^n\mathcal{U} \to Q_{\alpha,\gamma} \to 0 \tag{3}$$

where $\mathcal{U} = \mathcal{O}_{\mathbb{P}^n}(-\alpha) \oplus \mathcal{O}_{\mathbb{P}^n}(\alpha)$. In general, we will call *weighted quotient bundle of weights* α *and* γ any bundles $Q_{\alpha,\gamma}$ contained in a sequence (3).

On the other hand $F_{\alpha,\gamma}$ is contained in the exact sequence:

$$0 \to Q_{\alpha,\gamma}(-\gamma) \to \mathcal{V} \to F_{\alpha,\gamma}(\gamma) \to 0 \tag{4}$$

where $\mathcal{V} = S^{2(n-1)}\mathcal{U}$ and $Q_{\alpha,\gamma}$ is the pull-back over $\mathbb{C}^{n+1} \setminus 0$ of the quotient bundle Q defined by the map ϕ. Also in this case, we will call *weighted Tango bundle of weights* α *and* γ any bundles $F_{\alpha,\gamma}$ contained in the sequence (4), where $Q_{\alpha,\gamma}$ is any weighted quotient bundle of weights α and γ.

By these sequences, it immediately follows that $c_1(F_{\alpha,\gamma}) = 0$ and that $c_i(F_{\alpha,\gamma}) = c_i(\alpha, \gamma)$ for any $i = 2, \ldots, n-1$ (i.e. the Chern classes do not depend on the map ϕ).

Proposition 3.1. *A weighted Tango bundle* $F_{\alpha,\gamma}$ *is stable if and only if* $\gamma > 2(n-1)\alpha$.

Proof. Let $\gamma > 2(n-1)\alpha$. By the Hoppe criterion [8], it suffices to show that $\mathrm{H}^0(\wedge^q F_{\alpha,\gamma}) = 0$ for any $q = 1, \ldots, n-2$. By the sequence:

$$0 \to S^{k-1}S^n\mathcal{U}(-\gamma) \to S^kS^n\mathcal{U} \to S^kQ_{\alpha,\gamma} \to 0$$

obtained raising the sequence (3) to the k-th symmetric power, it results:

$$\mathrm{H}^i(S^kQ_{\alpha,\gamma}(t)) = 0 \qquad \text{for any } i = 1, \ldots, n-2 \text{ and } t \in \mathbb{Z}.$$

On the other hand by (4), we have the long exact sequence:

$$0 \to S^q Q_{\alpha,\gamma}(-q\gamma) \to \dots \to S^k Q_{\alpha,\gamma}(-k\gamma) \otimes \wedge^{q-k}\mathcal{V} \to \dots$$

$$\dots \to Q_{\alpha,\gamma}(-\gamma) \otimes \wedge^{q-1}\mathcal{V} \to \wedge^q \mathcal{V} \to \wedge^q F_{\alpha,\gamma}(q\gamma) \to 0$$

This sequence immediately implies that $\mathrm{H}^0(\wedge^q F_{\alpha,\gamma}) \subseteq \mathrm{H}^0(\wedge^q \mathcal{V}(-q\gamma))$, and since

$$\max\{t \in \mathbb{Z} | \mathcal{O}_{\mathbb{P}^n}(t) \subseteq \wedge^q \mathcal{V}(-q\gamma)\} = q((2n-q-1)\alpha - \gamma) < 0$$

we have that $\mathrm{H}^0(\wedge^q F_{\alpha,\gamma}) = 0$ for any $q = 1, \dots, n-2$, and so $F_{\alpha,\gamma}$ is stable.

Let us prove now that the condition is necessary. By the sequences:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-3\gamma) \to S^n\mathcal{U}(-2\gamma) \to Q_{\alpha,\gamma}(-2\gamma) \to 0$$

$$0 \to Q_{\alpha,\gamma}(-2\gamma) \to \mathcal{V}(-\gamma) \to F_{\alpha,\gamma} \to 0$$

it follows that if $\gamma \leq 2(n-1)\alpha$, then $\mathrm{H}^0(F_{\alpha,\gamma}) \neq 0$ and so $F_{\alpha,\gamma}$ cannot be stable. □

4. **Small deformations of $F_{\alpha,\gamma}$.**

Let E be a vector bundle on $\mathbb{P}^n$: we will indicate with $(\mathrm{Kur}\, E, e)$ the Kuranishi space of E (cf. [5]), where $e \in \mathrm{Kur}\, E$ is the point corresponding to the bundle E.

We are finally ready to introduce the main result of this paper:

Proposition 4.1. *Let $F^o_{\alpha,\gamma}$ be a weighted Tango bundle of weights α and γ. Every small deformation of $F^o_{\alpha,\gamma}$ is still a weighted Tango bundle and its Kuranishi space is smooth at the point corresponding to $F^o_{\alpha,\gamma}$.*

Before proceeding with the proof of the proposition, let us look at some preliminaries:

Lemma 4.2. *Let $Q^o_{\alpha,\gamma}$ be a weighted quotient bundle. Every small deformation of $Q^o_{\alpha,\gamma}$ is still a weighted quotient bundle and the Kuranishi space of $Q^o_{\alpha,\gamma}$ is smooth at the point corresponding to its isomorphism class.*

Proof. The proof of this lemma is very similar to the proof of prop. 3.1 of [1]. □

Lemma 4.3. *Let $F_{\alpha,\gamma}$ and $F'_{\alpha,\gamma}$ be two isomorphic weighted Tango bundles, defined by the sequences:*

$$0 \to Q_{\alpha,\gamma}(-\gamma) \to \mathcal{V} \to F_{\alpha,\gamma}(\gamma) \to 0$$

$$0 \to Q'_{\alpha,\gamma}(-\gamma) \to \mathcal{V} \to F'_{\alpha,\gamma}(\gamma) \to 0$$

where $Q_{\alpha,\gamma}$ and $Q'_{\alpha,\gamma}$ are weighted quotient bundles. Then $Q_{\alpha,\gamma}$ and $Q'_{\alpha,\gamma}$ are isomorphic.

Proof. By joining together the sequences (3) and (4), we get:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2\gamma) \xrightarrow{\phi} S^n\mathcal{U}(-\gamma) \to \mathcal{V} \to F_{\alpha,\gamma}(\gamma) \to 0.$$

By proposition 1.4 of [4] and by the fact that $-2\gamma < -\gamma - n\alpha$, the last sequence is the minimal resolution of $F_{\alpha,\gamma}(\gamma)$: hence $Q_{\alpha,\gamma}(-2\gamma) = \mathrm{Coker}\, \phi$ is directly defined by this resolution. □

Lemma 4.4. *Every isomorphism between two weighted Tango bundles* $F_{\alpha,\gamma} \to F'_{\alpha,\gamma}$ *is induced by an isomorphism of sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_{\alpha,\gamma}(-\gamma) & \longrightarrow & \mathcal{V} & \longrightarrow & F_{\alpha,\gamma}(\gamma) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q'_{\alpha,\gamma}(-\gamma) & \longrightarrow & \mathcal{V} & \longrightarrow & F'_{\alpha,\gamma}(\gamma) & \longrightarrow & 0 \end{array}$$

Proof. By the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2\gamma) \otimes \mathcal{V} \to S^n\mathcal{U}(-\gamma) \otimes \mathcal{V} \to Q_{\alpha,\gamma}(-\gamma) \otimes \mathcal{V} \to 0,$$

and since

$$h^1(S^n\mathcal{U}(-\gamma) \otimes \mathcal{V}) = h^2(\mathcal{O}_{\mathbb{P}^n}(-2\gamma) \otimes \mathcal{V}) = 0,$$

we get $h^1(Q_{\alpha,\gamma}(-\gamma) \otimes \mathcal{V}) = 0$; hence the lemma is proven. □

Lemma 4.5. *Two morphisms* $f, f' \in \mathrm{Hom}(Q_{\alpha,\gamma}(-\gamma), \mathcal{V})$ *give the same element of* $Quot_{\mathcal{V}|\mathbb{P}^n}$ *if and only if there exists an invertible* $h \in \mathrm{End}(Q_{\alpha,\gamma}(-\gamma))$ *such that*

$$f = f' \circ h.$$

Proof. It follows from the definition of $Quot_{\mathcal{V}|\mathbb{P}^n}$, (cf. [10]). □

Proof of proposition 4.1.

For brevity's sake, we will write $\widetilde{F}_o$ instead of $F^o_{\alpha,\gamma}$ and $\widetilde{Q}_o$ for $Q^o_{\alpha,\gamma}$. Let also $\sigma_0 \in \mathrm{Hom}(\widetilde{Q}_o(-\gamma), \mathcal{V})$ be such that $\widetilde{F}_o = \mathrm{Coker}\,\sigma_0$.

Let $\mathcal{Q}$ be the sub-variety of the irreducible component of $Quot_{\mathcal{V}|\mathbb{P}^n}$ composed by all the quotients of the maps $0 \to Q_{\alpha,\gamma}(-\gamma) \xrightarrow{\sigma} \mathcal{V}$ for some weighted bundle $Q_{\alpha,\gamma}$ and containing the point σ_0 corresponding to $\widetilde{F}_o$: the morphisms $\Phi : (\mathcal{Q}, \sigma_0) \longrightarrow (\mathrm{Kur}\,\widetilde{Q}_o, q_0)$ and $\Psi : (\mathcal{Q}, \sigma_0) \longrightarrow (\mathrm{Kur}\,\widetilde{F}_o, f_0)$ are canonically defined.

A generic fiber of Φ is given by all the cokernels of the morphisms $Q_{\alpha,\gamma}(-\gamma) \to \mathcal{V}$ with a fixed $Q_{\alpha,\gamma}$, and so, by lemma 4.5, its dimension is constantly equal (α and γ are fixed) to $h^0(Q^*_{\alpha,\gamma}(\gamma) \otimes \mathcal{V}) - h^0(\mathrm{End}\,Q_{\alpha,\gamma})$. Hence, since lemma 4.2 implies that $\dim_{q_0}(\mathrm{Kur}\,\widetilde{Q}_o) = h^1(\mathrm{End}\,\widetilde{Q}_o)$, we get:

$$\dim_{\sigma_0} \mathcal{Q} = h^0(\widetilde{Q}^*_o(\gamma) \otimes \mathcal{V}) - h^0(\mathrm{End}\,\widetilde{Q}_o) + h^1(\mathrm{End}\,\widetilde{Q}_o)$$

Let us study now the morphism $\Psi : \mathcal{Q} \longrightarrow \mathrm{Kur}\,\widetilde{F}_o$: if $\Sigma = \{\sigma \in Quot_{\mathcal{V}|\mathbb{P}^n} | F_\sigma \simeq \widetilde{F}_o\}$, then it results $\Psi^{-1}(f_0) \subseteq \Sigma$ and by lemma 4.3, 4.4 and 4.5, it follows:

$$\dim_{\sigma_o} \Sigma = h^0(\mathrm{End}\,\mathcal{V}) - \dim\{\varphi \in \mathrm{End}\,\mathcal{V} | \varphi \cdot \sigma_0 = \sigma_0\} - h^0(\mathrm{End}\,\widetilde{Q}_o).$$

By the sequence:

$$0 \to \widetilde{F}^*_o(-\gamma) \otimes \mathcal{V} \to \mathrm{End}\,\mathcal{V} \to \widetilde{Q}^*_o(\gamma) \otimes \mathcal{V} \to 0$$

obtained tensoring the dual sequence of (4) with $\mathcal{V}$, we have that:

$$\dim\{\varphi \in \mathrm{End}\,\mathcal{V} | \varphi \cdot \sigma_0 = \sigma_0\} = h^0(\widetilde{F}^*_o(-\gamma) \otimes \mathcal{V})$$

and so:

$$\dim_{\sigma_0} \Psi^{-1}(f_0) \le \dim_{\sigma_o} \Sigma = h^0(\mathrm{End}\,\mathcal{V}) - h^0(\widetilde{F}^*_o(-\gamma) \otimes \mathcal{V}) - h^0(\mathrm{End}\,\widetilde{Q}_o).$$

Hence:

$$h^1(\mathrm{End}\,\widetilde{F}_o) \ge \dim_{f_0}(\mathrm{Kur}\,\widetilde{F}_o) \ge h^1(\mathrm{End}\,\widetilde{Q}_o) + h^1(\widetilde{F}^*_o(-\gamma) \otimes \mathcal{V}).$$

To prove the proposition it suffices to show that

$$h^1(\mathrm{End}\,\widetilde{F}_o) \leq h^1(\mathrm{End}\,\widetilde{Q}_o) + h^1(\widetilde{F}_o^*(-\gamma)\otimes\mathcal{V}).$$

In fact this implies that $h^1(\mathrm{End}\,\widetilde{F}_o) = \dim_{f_0}(\mathrm{Kur}\,\widetilde{F}_o)$, i.e. $\mathrm{Kur}\,\widetilde{F}_o$ is smooth at the point f_0, and that $\dim_{f_0}(\mathrm{Kur}\,\widetilde{F}_o) = \dim_{\sigma_0}\mathcal{Q} - \dim\Psi^{-1}(f_0)$, i.e. Ψ is surjective.

By the exact sequence:

$$0 \to \widetilde{Q}_o(-2\gamma)\otimes\widetilde{F}_o^* \to \widetilde{Q}_o(-\gamma)\otimes\mathcal{V} \to \mathrm{End}\,\widetilde{Q}_o \to 0$$

and by the vanishing of $\mathrm{H}^1(\widetilde{Q}_o(-\gamma)\otimes\mathcal{V})$ and $\mathrm{H}^2(\widetilde{Q}_o(-\gamma)\otimes\mathcal{V})$, it results $\mathrm{H}^1(\mathrm{End}\,\widetilde{Q}_o) = \mathrm{H}^2(\widetilde{Q}_o(-2\gamma)\otimes\widetilde{F}_o^*)$. Hence by the sequence:

$$0 \to \widetilde{Q}_o(-2\gamma)\otimes\widetilde{F}_o^* \to \widetilde{F}_o^*(-\gamma)\otimes\mathcal{V} \to \mathrm{End}\,\widetilde{F}_o \to 0$$

and for what we have seen, we get the sequence of cohomology groups:

$$\cdots \to \mathrm{H}^1(\widetilde{F}_o^*(-\gamma)\otimes\mathcal{V}) \to \mathrm{H}^1(\mathrm{End}\,\widetilde{F}_o) \to \mathrm{H}^1(\mathrm{End}\,\widetilde{Q}_o) \to \dots$$

In particular $h^1(\mathrm{End}\,\widetilde{F}_o) \leq h^1(\mathrm{End}\,\widetilde{Q}_o) + h^1(\widetilde{F}_o^*(-\gamma)\otimes\mathcal{V})$, as required. □

Theorem 0.1 easily follows from the previous proposition. In fact if $\gamma \geq 2(n-1)\alpha$, we can consider the canonical algebraic map $\mathcal{Q} \longrightarrow \mathcal{M}(0, c_2, \dots, c_{n-1})$. The image of this map is a smooth quasi projective set composed uniquely by weigthed Tango bundles and it is an open neighborhood of $F^o_{\alpha,\gamma}$ in $\mathcal{M}(0, c_2, \dots, c_{n-1})$.

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