Solutions of Practical Problems

Problem 1: *Flow kinematics.*

(a) The relation $\psi(x,y) = C$ defines a curve in (x,y) space, such that $d\psi/ds = 0$ along the curve, where s is the along-curve coordinate. By the chain rule

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} \qquad \Rightarrow \qquad v\,\frac{dx}{ds} - u\,\frac{dy}{ds} = 0 \qquad \Rightarrow \qquad \frac{dx}{u} = \frac{dy}{v}\,, \qquad \frac{d\mathbf{x}}{ds} \times \mathbf{u} = 0.$$

Thus, the curve is a streamline.

(b) Let $\psi(0,x,y) = \psi_0$. The streamline at $t = 0$ is

$$y = \frac{A}{U}\,\sin[kx] - \frac{\psi_0}{U}\,.$$

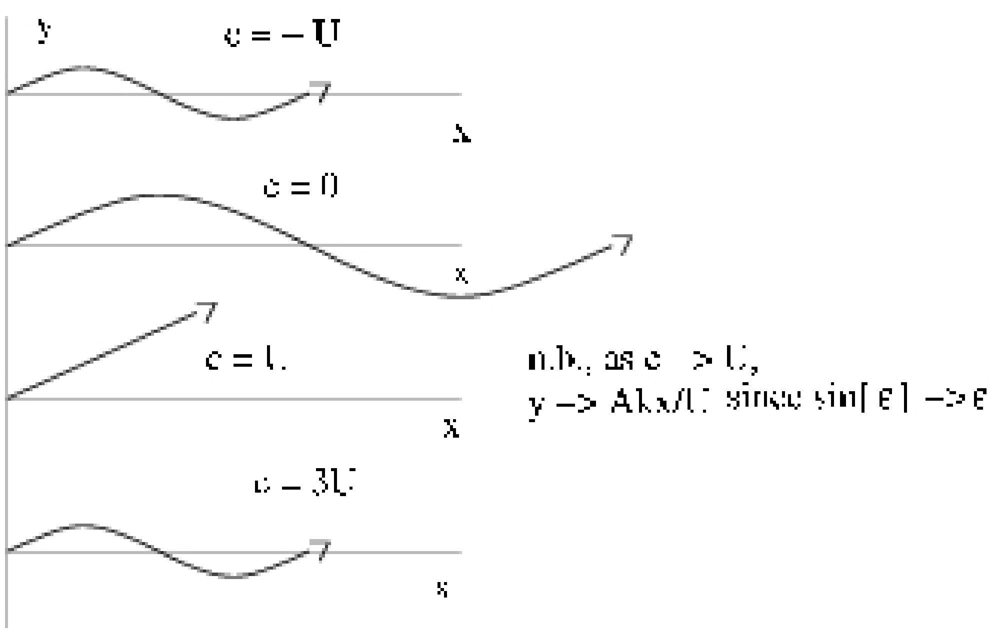
(c) The trajectory is defined by

$$\frac{dx}{dt} \equiv u = U\,, \qquad \frac{dy}{dt} \equiv v = Ak\cos[k(x - ct)]\,.$$

Integrating the first equation with $x(0) = 0$ gives $x = Ut$. Substituting this into the second equation and integrating with $y(0) = 0$ gives:

$$y(t) = \frac{A}{U - c}\,\sin[k(U - c)t]\,, \qquad y(x) = \frac{A}{U - c}\,\sin\left[k\,\frac{U - c}{U}\,x\right]$$

(d) Same period for $c = -U$ and $c = 3U$, longer period for $c = 0$.
Linear growth for $c = U$, because as $c \to U$: $y \to Akx/U$.



(e)

$$\zeta = \nabla\times\mathbf{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2\psi = -Ak^2\sin[k(x-ct)]\,,$$

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x\partial y} = 0$$

Problem 2: *D'Alembert solution of wave equation.*

The Fourier transform of the wave equation in space yields:

$$\hat{u}_{tt} + c^2k^2\,\hat{u} = 0 \qquad (*)$$

If we fix $k,$ so that $\hat{u}(k,t) = U(t),$ then the equation reduces to

$$U'' + c^2k^2\,U = 0\,.$$

Hence, $U(t) = a\,e^{-ickt} + b\,e^{ickt},$ and the general solution of $(*)$ is

$$\hat{u}(k,t) = \hat{F}(k)\,e^{-ickt} + \hat{G}(k)\,e^{ickt} \qquad \rightarrow \qquad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}\,e^{ikx}\,dk = ...$$

$$... = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(k)\,e^{ik\,(x-ct)}\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{G}(k)\,e^{ik\,(x+ct)}\,dk = F(x-ct) + G(x+ct),$$

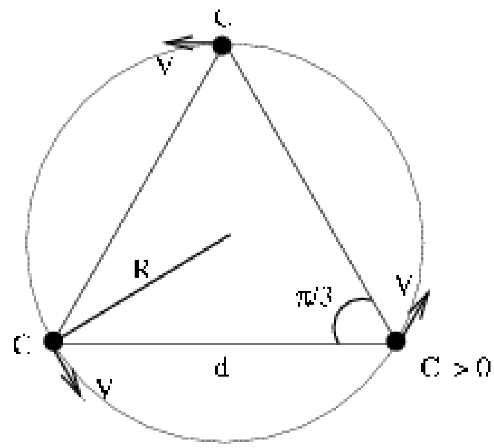
which is the D'Alembert form of the solution of the wave equation.
Now, let's apply the initial conditions:

$$u(x,0) = f(x) = F(x) + G(x), \qquad u_t(x,0) = 0 = -c\,F'(x) + c\,G'(x)$$

$$\rightarrow \qquad F'(x) = G'(x) \qquad \rightarrow \qquad F(x) = G(x) + C \qquad \rightarrow \qquad \text{let's take :}\quad F(x) = G(x) = \frac{1}{2}\,f(x).$$

$$\rightarrow \qquad u(x,t) = \frac{1}{2}\,f(x-ct) + \frac{1}{2}\,f(x+ct)$$

Problem 3: *Point vortices.*

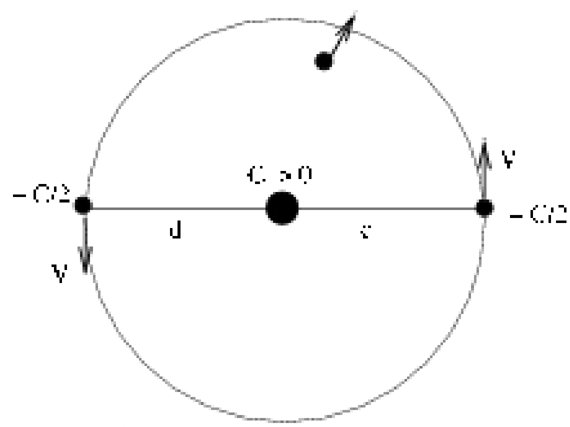


(a) Each vortex moves counterclockwise in a circle of radius $R = d/\sqrt{3}$ about the centroid of the triangle at a speed V, that is the sum of the far-field velocities of the other two vortices:

$$V = 2\,\frac{C}{2\pi d}\,\sin\frac{\pi}{3} = 2\,\frac{C}{2\pi d}\,\frac{\sqrt{3}}{2} = \frac{\sqrt{3}\,C}{2\pi d}$$

The period of rotation of vortices around the circle is

$$P = \frac{2\pi R}{V} = \frac{4\pi^2 d^2}{3C}$$



(b) The center vortex is stationary because of cancelling velocities from the two peripheral vortices. Each peripheral vortex moves counterclockwise in a circle with the combined speed of the other two vortices:

$$V = \frac{C}{2\pi d} + \frac{-C/2}{2\pi(2d)} = \frac{3C}{8\pi d}\,.$$

The period of rotation is

$$P = \frac{2\pi d}{V} = \frac{16\pi^2 d^2}{3C}\,.$$

Problem 4: *Material conservation.*

(a) The area A within a material curve (i.e., timeline) changes by the net effect of displacements in the normal to the curve direction by all the points on the curve C :

$$\frac{dA}{dt} = \oint_C \mathbf{u}\cdot\mathbf{n}\,ds = \iint_A \nabla\cdot\mathbf{u}\;dx\,dy = 0\,.$$

Above, we used Green's theorem in 2D and incompressibility (i.e., nondivergence) of the flow. This result does not depend upon whether external force is zero or not; it also does not depend upon the momentum balance that determines motion of the curve. It depends only on incompressibility.

(b) As in (a), except that

$$\nabla\cdot\mathbf{u} = -\frac{1}{h}\,\frac{Dh}{Dt} \neq 0 \qquad \Longrightarrow \qquad \frac{dA}{dt} \neq 0\,.$$

Problem 5: *Free particle motion in rotating frame*.

(a) Newton's law for particle on the f-plane gets additional Coriolis (pseudo)force:

$$\frac{du}{dt} - fv = 0\,, \qquad \frac{dv}{dt} + fu = 0\,, \qquad f = 2\Omega\sin\theta$$

hence, the general solution is

$$u = A\,\sin(ft+\phi)\,, \qquad v = A\,\cos(ft+\phi) \qquad \rightarrow \qquad \sqrt{u^2+v^2} = A\,,$$

and the speed is preserved and equal to A, which comes from the initial condition.

(b) Particle trajectories are found by integration:

$$\frac{dx}{dt} = u\,, \qquad \frac{dy}{dt} = v \qquad \rightarrow \qquad x = x_0 - \frac{A}{f}\,\cos(ft+\phi)\,, \qquad y = y_0 + \frac{A}{f}\,\sin(ft+\phi)$$

Note, that

$$(x-x_0)^2 + (y-y_0)^2 = \frac{A^2}{f^2}$$

is a circle, and its radius changes from finite values at the poles to infinity at the equator (note, that f, hence, direction of inertial rotation is of opposite sign in the northern and souther hemispheres). At the pole period of the particle motion (inertial period), $T_i = 2\pi/f = \pi/(\Omega\sin\theta) = \pi/\Omega$, around the circle is exactly one half of the planetary rotation period. Inertial period increases towards the equator, so that at latitude $\theta = 30°\ :\ f = 2\Omega\sin\theta = \Omega$, that is, inertial and planetary periods coincide.

Problem 6: *Acoustic wave equation*.

Linearized momentum, continuity and thermodynamics equations are

$$\frac{\partial\mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\,\nabla p\,, \qquad \frac{\partial\rho}{\partial t} = -\rho_0\nabla\mathbf{u}\,, \qquad \frac{\partial\rho}{\partial t} - \frac{1}{c_s^2}\,\frac{\partial p}{\partial t} = 0$$

By substituting the tendency from the continuity equation in the thermodynamics equation, we obtain

$$\frac{\partial p}{\partial t} + \rho_0\,c_s^2\,\nabla\mathbf{u} = 0\,.$$

By differentiating this equation with respect to time and substituting the tendency term from the momentum equation we obtain

$$\frac{\partial^2 p}{\partial t^2} - c_s^2\,\nabla^2 p = 0\,,$$

which is the 3D wave equation for pressure.

Main Problems

Problem 7: *Deep-water gravity waves.*

Irrotational velocity field implies velocity potential ϕ, such that $\mathbf{u}=\nabla\phi$; since flow is nondivergent: $\nabla^2\phi=0$. Let's look for solution in the form $\phi(t,x,z)=f(z)\,\cos[kx-\omega t]$, then:

$$\nabla^2\phi=0 \quad\rightarrow\quad f''-k^2f=0 \quad\rightarrow\quad \phi=Ce^{kz}\cos[kx-\omega t].$$

Linearized kinematic boundary condition is $w(t,x,h)=\partial h/\partial t$, but it still applies to $z=h$, whereas we would like to have it linearized at $z=0$.
Let's write down Taylor expansion of the vertical velocity around $z=0$:

$$w(t,x,h)=w(t,x,0)+h\,w'(t,x,0)+...$$

Use the above boundary condition, keep only the first term on the rhs, and express it through ϕ to obtain:

$$z=0: \qquad w=\frac{\partial\phi}{\partial z}=\frac{\partial h}{\partial t} \qquad\qquad (*)$$

Similar treatment of the pressure continuity boundary condition yields:

$$p(t,x,h)=p_a \quad\rightarrow\quad p_a=p(t,x,0)+hp'(t,x,0)+...$$

The second term on rhs is actually linear for the static part of pressure, and the nonlinear dynamic-pressure part of this term we neglect by the linearization. Next, let's take $p_a=0$, for simplicity, then the linearized pressure continuity boundary condition at $z=0$ becomes

$$p(t,x,0)=-h\,\frac{\partial p_s}{\partial z}(t,x,0)=g\rho_0 h\,, \qquad\qquad (**)$$

assuming hydrostaticity. The linearized momentum equations are:

$$(***) \qquad \frac{\partial u}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}\,, \qquad v=0\,, \qquad \frac{\partial w}{\partial t}=-\frac{1}{\rho_0}\frac{\partial(p_s+p)}{\partial z}-g \quad\rightarrow\quad \frac{\partial w}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

The final solution for ϕ satisfying $(*)$ is

$$\phi=-\frac{h_0\omega}{k}\,e^{kz}\cos[kx-\omega t]\,,$$

and the flow velocity components can be found as

$$u=\frac{\partial\phi}{\partial x}=h_0\omega\,e^{kz}\sin[kx-\omega t]\,, \qquad v=\frac{\partial\phi}{\partial y}=0\,, \qquad w=\frac{\partial\phi}{\partial z}=-h_0\omega\,e^{kz}\cos[kx-\omega t]$$

Dynamic pressure can be found from either momentum equation:

$$(***) \quad\rightarrow\quad p=\frac{\rho_0h_0\omega^2}{k}\,e^{kz}\sin[kx-\omega t]\,,$$

and notice that it satisfies the linearized pressure continuity boundary condition $(**)$ only for (solvability condition) $\omega^2=gk$, which is the classical dispersion relation for dispersive "deep-water" surface gravity waves.

Rigid-lid approximation sets $h_0=0$ and, thus, completely removes the surface waves.

Problem 8: *Integrals of motion.*

All of the results are obtained by manipulation of the 2D vorticity equation

$$\frac{\partial \zeta}{\partial t} + J[\psi,\ \zeta + f(y)] = 0\,, \qquad (*)$$

where

$$J[A,B] \equiv \frac{\partial A}{\partial x}\,\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\,\frac{\partial B}{\partial x}\,,$$

with integration over the area, integration by parts, and use of the boundary conditions in the far field. A key step is that Jacobian can be written as curl of some vector field F, and, therefore, by *Green's (Stokes) Theorem* the 2D surface integral can be written as the boundary contour integral of F :

$$\iint J[A,B]\,dx\,dy = \iint \Big(\frac{\partial}{\partial x}\Big[A\,\frac{\partial B}{\partial y}\Big] - \frac{\partial}{\partial y}\Big[A\,\frac{\partial B}{\partial x}\Big]\Big)\,dx\,dy = \iint \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dA = \oint_C \mathbf{F}\cdot d\mathbf{r}\,, \qquad (**)$$

$$\mathbf{F} \equiv \Big(A\,\frac{\partial B}{\partial x},\ A\,\frac{\partial B}{\partial y},\ 0\Big)\,,$$

where the last integral in $(**)$ is zero, if the contour of integration C is expanded to the far field, and $\mathbf{F}$ decays to zero in the far field. Thus, area integral of the Jacobian is zero.
Note, that the same argument can be made from the *Divergence Theorem* by considering divergence of slightly different vector field.

The other useful identity can be obtained in a similar way:

$$A\,J[A,B] = \frac{1}{2}\,J[A^2,B]$$

(a) Multiply $(*)$ by $-\psi$ and use: $-\psi\nabla^2\psi_t = -\nabla\cdot(\psi\nabla\psi_t) + \nabla\psi\cdot\nabla\psi_t$. The first term on the rhs is divergence of vector field, and its area integral is equal to the vector flux through the boundary (Divergence Theorem). Since the corresponding vector vanishes on the far-field boundary, this integral is zero. Thus, the tendency term becomes

$$-\iint \psi\nabla^2\psi_t\,dx\,dy = \frac{d}{dt}\iint \frac{1}{2}\,(\nabla\psi)^2\,dx\,dy = \frac{d}{dt}\iint \frac{1}{2}\,\mathbf{u}^2\,dx\,dy\,.$$

Integration of the divergence term yields zero, because $\psi \to 0$ in the far field, thus:

$$\frac{d}{dt}\iint \frac{1}{2}\,\mathbf{u}^2\,dx\,dy = 0\,.$$

(b) Integrate $(*)$ over the domain and use $(**)$ to obtain

$$\frac{d}{dt}\iint \zeta\,dx\,dy = 0$$

(c) Multiply $(*)$ by ζ and integrate. There is additional term with β, but the corresponding function can be also written as curl of vector field that vanishes at infinity:

$$\beta\iint \zeta\,\frac{\partial \psi}{\partial x}\,dx\,dy = \beta\iint \psi_x\,(\psi_{xx}+\psi_{yy})\,dx\,dy = \frac{\beta}{2}\iint \Big(\frac{\partial}{\partial x}[\psi_x^2 - \psi_y^2] + \frac{\partial}{\partial y}[\psi_x\,\psi_y]\Big)\,dx\,dy = 0$$

$$\Longrightarrow \qquad \frac{d}{dt}\iint \frac{1}{2}\zeta^2\,dx\,dy = 0$$

(d) Multiply $(*)$ by $q = \zeta + f$ and integrate to get

$$\frac{d}{dt}\iint \frac{1}{2}q^2\,dx\,dy = 0\,.$$

Problem 9: *Geostrophy and thermal wind.*

The geostrophic and hydrostatic relations are:

$$u = -\frac{1}{f\rho_0}\frac{\partial p}{\partial y}\,, \qquad v = +\frac{1}{f\rho_0}\frac{\partial p}{\partial x}\,, \qquad \frac{\partial p}{\partial z} = -g\rho \qquad (*)$$

$$f = f_0: \qquad \nabla\times\mathbf{u} = \frac{1}{f\rho_0}\nabla^2 p\,, \qquad \nabla\cdot\mathbf{u} = 0\,;$$

$$f = f_0 + \beta(y-y_0): \qquad \nabla\times\mathbf{u} = \frac{1}{f\rho_0}\nabla^2 p - \frac{\beta}{f^2\rho_0}\frac{\partial p}{\partial y}\,, \qquad \nabla\cdot\mathbf{u} = -\frac{\beta}{f^2\rho_0}\frac{\partial p}{\partial x}\,.$$

Geophysical relevance of $f = f(y)$ is due to the fact that angle between the vertical coordinate axis (aligned with gravity force) and Earth's rotation vector varies with latitude.

Differentiate $(*)$ to obtain:

$$\frac{\partial u}{\partial z} = \frac{g}{\rho_0 f_0}\frac{\partial \rho}{\partial y}\,, \qquad \frac{\partial v}{\partial z} = -\frac{g}{\rho_0 f_0}\frac{\partial \rho}{\partial x}\,.$$

Thermal wind corresponding to these balances is a vertical shear in the geostrophic wind caused by a horizontal temperature gradient or vice versa.
Midlatitude jet stream phenomenon:

$$\frac{\partial \rho}{\partial y} > 0\,, \quad u(0) = 0 \qquad \rightarrow \qquad \frac{\partial u}{\partial z} > 0\,.$$

Problem 10: *Fractional depth of fluid columns*.

The continuity equation is

$$\frac{Dh}{Dt} + h\nabla\mathbf{u} = 0\,.$$

Also, note that at any level

$$\frac{Dz}{Dt} = w\,,$$

and at the bottom $(z = h_b)$:

$$\frac{Dz}{Dt} = \frac{Dh_b}{Dt}\,.$$

Let's now integrate the continuity equation:

$$\int_{h_b}^{z} \frac{\partial w}{\partial z}\,dz = -\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)\int_{h_b}^{z} dz \quad \rightarrow \quad \Big[w\Big]_{h_b}^{z} = -\nabla\mathbf{u}\,(z-h_b) \quad \rightarrow \quad \frac{Dz}{Dt}-\frac{Dh_b}{Dt} = -(z-h_b)\,\nabla\mathbf{u}$$

With the expression for velocity divergence from the continuity equation, this becomes

$$\frac{1}{z-h_b}\,\frac{D}{Dt}\,(z-h_b)-\frac{1}{h}\,\frac{Dh}{Dt} = 0 \quad \rightarrow \quad h\,\frac{D}{Dt}\,(z-h_b)-(z-h_b)\,\frac{Dh}{Dt} = 0 \quad \rightarrow \quad h^2\,\frac{D}{Dt}\Big[\frac{z-h_b}{h}\Big] = 0\,.$$

Thus, fractional height $(z-h_b)/h$ is conserved on fluid particles; that is, a particle half way up on a water column always stays half way up, but the column itself can be stretched up or squashed.

Problem 11: *Shallow-water PV conservation*.

(i) Rigid-lid approximation implies that fluid depth is constant.
Initial PV is $(\zeta_0 + \beta\, y_0)/H$ and must be equal to the final PV $\beta\,(y + y_0)/H$. This yields $y = \zeta_0/\beta$.

(ii) Initial PV must be equal to the final PV:

$$\frac{\zeta_0 + \beta y_0}{h} = \frac{\beta\,(y_0 + y)}{h + 0.1h} \qquad \rightarrow \qquad (1 + 0.1)\,\zeta_0 + 0.1\beta y_0 = \beta y \qquad \rightarrow \qquad y = 0.1 y_0 + 1.1\,\frac{\zeta_0}{\beta}$$

Problem 12: *Topographic effects.*

(a) Stationary and conservative solutions of the shallow-water equations must satisfy

$$\mathbf{u}\cdot\nabla q = 0\,, \qquad\qquad \nabla\cdot(h\mathbf{u}) = 0\,.$$

The second relation is satisfied by any scalar function $\Psi(\mathbf{x})$, such that $h\mathbf{u} = (-\Psi_y, \Psi_x)$. This implies that the first relation multiplied by h can be rewritten as

$$J[\Psi, q] = 0\,.$$

Therefore, all solutions must satisfy the functional relation $q = F(\Psi)$, for an arbitrary functional F.

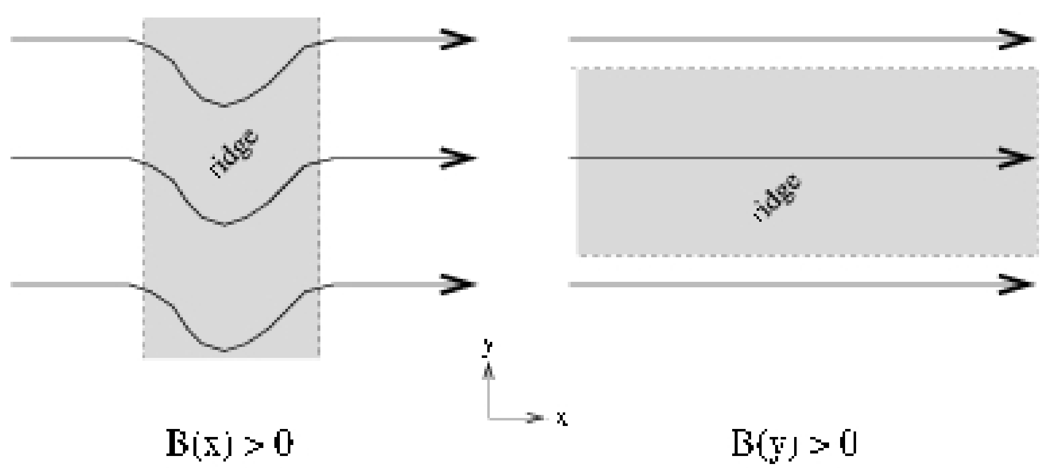
(b) If we assume very weak flow and neglect ζ and η, then

$$q \approx \frac{f_0 + \beta y}{H - B(x, y)}\,, \qquad\qquad (*)$$

and $q = F(\Psi)$ implies that isolines of Ψ and q must be parallel, and the flow is oriented along isolines given by $(*)$.

(c) In order to conserve q, incoming steady zonal flow over a *meridional* ridge $(B(x) > 0)$ has to deflect the streamlines southwards (left Figure), that is, y must go negative to compensate for the ridge. In contrast, zonal flow over a *zonal* ridge $(B(y) > 0)$ retains parallel streamlines, because q does not vary zonally.

Steady uniform meridional flow cannot exist over a spatially confined ridge, because it does not conserve q. If we assume such a streamline, then q should remain constant along it in accord with $(*)$, but this does not happen even outside the ridge $(q(y) \sim y)$, thus, contradicting $(*)$.



Problem 13: *Gradient-wind balance.*

Start from the momentum and continuity equations,

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,,$$

take x-derivative of the u-momentum equation and sum it up with y-derivative of the v-momentum equation. Note, that the tendency term vanishes, because flow is nondivergent, and divergences of the nonlinear and Coriolis terms are:

$$\frac{\partial}{\partial x}\Big(u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}\Big) = -2\,\Big(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\Big) = -2\,J\Big(\frac{\partial \psi}{\partial x},\ \frac{\partial \psi}{\partial y}\Big)\,,$$

$$\frac{\partial}{\partial x}(-fv) + \frac{\partial}{\partial y}(fu) = -\nabla(f\nabla\psi)\,,$$

The resulting *gradient-wind balance* equation is

$$\frac{1}{\rho}\,\nabla^2 p = \nabla(f\nabla\psi) + 2\,J\Big(\frac{\partial \psi}{\partial x},\ \frac{\partial \psi}{\partial y}\Big)\,.$$

This relation between p and ψ is different from the linear geostrophic relation; without the last term, this is the (linear) geostrophic balance. The last term comes from the additional flow acceleration along curved streamlines (i.e., this term is divergence of such acceleration) — *centripetal force*. Flow along a curved trajectory must be accelerated due to an imbalance between the pressure gradient and Coriolis forces. It can be shown that for cyclonic flow Coriolis force is weaker than pressure gradient force, hence actual wind is weaker than geostrophic wind; for anticyclonic flow the situation is opposite and actual wind is stronger. Hence, for the same pressure gradient magnitude, anticyclonic winds are stronger than cyclonic ones.

Let Rossby number be small: $\epsilon = V/(f_0 L) \ll 1$ and apply usual scalings,

$$V = \epsilon f_0 L\,, \qquad \Psi = \epsilon f_0 L^2\,, \qquad P = \rho f_0 V L\,, \qquad [\beta y] = \epsilon f_0\,.$$

The lhs term and the first rhs term corresponding to f_0 are of the ϵ-order, whereas the β-term and the nonlinear term are of the ϵ^2-order, therefore, at the leading order: ensure linear relation between pressure and velocity streamfunction:

$$\nabla^2 p = \rho f_0 \nabla^2 \psi \qquad \rightarrow \qquad p = \rho f_0 \psi$$

Effect of centripetal force. Consider (counter-clockwise) cyclonic motion around negative pressure anomaly. Disbalance between the larger inward-pointing pressure gradient force F_{pg} and the smaller outward-pointing Coriolis force F_c is needed to keep particle on circular path. This disbalance is called *centripetal force*. Since F_c is smaller, the corresponding flow velocity is *subgeostrophic*. By similar arguments, in the case of positive pressure anomaly, the corresponding (clockwise) anticyclonic motion is characterized by inward-pointing F_c, such that $|F_c! > |F_{pg}|$, hence, the corresponding flow velocity is *supergeostrophic*. The QG dynamics doesn't have this asymmetry!

Stability of cyclones. Being subgeostrophic stabilizes, because velocity is weaker, therefore, cyclones are generally more stable. A cyclone also sucks in the air from the bottom Ekman layer (it has total transport to the left from the direction of geostrophic flow); this air updrafts, makes clouds and, thus, releases latent heat, which makes the pressure anomaly even more negative — this reinforces the cyclone.

Problem 14: *Ekman bottom boundary layer.*

(i) The interior geostrophic balance is

$$-f_0 v_g = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial x}, \qquad f_0 u_g = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y}.$$

The full velocity in the boundary layer is $\mathbf{u}=(u_g+u_E,\ v_g+v_E)$, and the boundary layer balance is

$$-f_0(v_g+v_E) = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial x} + \nu\frac{\partial^2 u_E}{\partial z^2}, \qquad f_0(u_g+u_E) = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y} + \nu\frac{\partial^2 v_E}{\partial z^2}.$$

The Ekman flow balance is combination of the above balances:

$$-f_0 v_E = \nu\frac{\partial^2 u_E}{\partial z^2}, \qquad f_0 u_E = \nu\frac{\partial^2 v_E}{\partial z^2}, \qquad (*)$$

and the boundary conditions at the bottom (no-slip velocity) and top (vanishing velocity) of the boundary layer are, respectively:

$$z=0: \quad u_E=-u_g, \quad v_E=-v_g \qquad\qquad z=\infty: \quad u_E=0, \quad v_E=0$$

(ii) The boundary layer thickness can be defined from the main balance:

$$f_0\,U = \nu\,\frac{U}{H^2} \qquad \to \qquad H=\sqrt{\frac{\nu}{f_0}} \sim 1km$$

(iii) Let's introduce complex velocity $W=u_E+iv_E$, combine $(*)$ into the single equation

$$f_0\,W = -i\nu\frac{\partial^2 W}{\partial z^2},$$

and look for the exponential solution

$$W \sim e^{\lambda z} \qquad \to \qquad \lambda^2 = i\,\frac{f_0}{\nu} = e^{i(\pi/2+2\pi k)}\,\frac{f_0}{\nu} \qquad \to \qquad \lambda = \sqrt{\frac{f_0}{\nu}}\,e^{i(\pi/4+\pi k)} \qquad \to \qquad \lambda = \pm(1+i)\sqrt{\frac{f_0}{2\nu}}.$$

Only vertically decaying solution is physical, therefore, introduce $d=\sqrt{2\nu/f_0}$ and obtain

$$W = A\,e^{-z/d}\,e^{-iz/d} = A\,e^{-z/d}\,(\cos\frac{z}{d} - i\sin\frac{z}{d}) \qquad (**)$$

The bottom boundary condition yields $A=(-u_g,-iv_g)$ and the real and imaginary parts of $(**)$ yield

$$u_e = -e^{-z/d}\,(u_g\cos\frac{z}{d} + v_g\sin\frac{z}{d}), \qquad v_e = e^{-z/d}\,(u_g\sin\frac{z}{d} - v_g\cos\frac{z}{d})$$

The full velocity is

$$u = u_g - e^{-z/d}\,(u_g\cos\frac{z}{d} + v_g\sin\frac{z}{d}), \qquad v = v_g + e^{-z/d}\,(u_g\sin\frac{z}{d} - v_g\cos\frac{z}{d}) \qquad (***)$$

(iv) Near the very bottom, and given that $v_g = 0$, $(***)$ can be written as

$$\frac{u}{u_g} = 1 - e^{-z/d} \cos\frac{z}{d} \approx 1 - (1 - \frac{z}{d}) \approx \frac{z}{d}, \qquad \frac{v}{u_g} = e^{-z/d} \sin\frac{z}{d} \approx \frac{z}{d},$$

hence the flow is turned by $+45°$ to the left from the interior flow.

(v)

$$U = -u_g \int_0^\infty e^{-z/d} \cos\frac{z}{d}\, dz = -u_g\, d \int_0^\infty e^{-x} \cos x\, dx = -\frac{u_g d}{2}$$

$$V = u_g \int_0^\infty e^{-z/d} \sin\frac{z}{d}\, dz = \frac{u_g d}{2}$$

Problem 15: *Exoplanetary waves.*

(i) Dimensions are

$$[\psi] = L^2T^{-1}\,, \qquad [S] = L^{-2}\,, \qquad [\beta] = [\alpha] = L^{-1}T^{-1}\,, \qquad [\Pi] = T^{-1}$$

(ii)

$$\frac{D\Pi_1}{Dt} = 0\,, \qquad \frac{D\Pi_2}{Dt} = 0\,, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y}$$

(iii)

$$\frac{\partial}{\partial t}\left[\nabla^2(\psi_1 - \psi_2) + \frac{\partial^2\psi_1}{\partial x\partial y} + S\,(\psi_1 - \psi_2)\right] + \beta\,\frac{\partial\psi_1}{\partial x} - \alpha\,\frac{\partial\psi_1}{\partial y} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla^2(\psi_2 - \psi_1) + \frac{\partial^2\psi_2}{\partial x\partial y} + S\,(\psi_2 - \psi_1)\right] + \beta\,\frac{\partial\psi_2}{\partial x} - \alpha\,\frac{\partial\psi_2}{\partial y} = 0$$

(iv) Define $\phi_1 = \psi_1 + \psi_2$ (barotropic; here, ψ_1 and ψ_2 have equal weights, because fluid layer have equal depths, $H_1 = H_2 = H$) and $\phi_2 = \psi_1 - \psi_2$ (baroclinic) velocity streamfunctions. By summing up and subtracting the governing equations obtain:

$$\frac{\partial}{\partial t}\left[\frac{\partial^2\phi_1}{\partial x\partial y}\right] + \beta\,\frac{\partial\phi_1}{\partial x} - \alpha\,\frac{\partial\phi_1}{\partial y} = 0\,,$$

$$\frac{\partial}{\partial t}\left[2\nabla^2\phi_2 + \frac{\partial^2\phi_2}{\partial x\partial y} + 2S\phi_2\right] + \beta\,\frac{\partial\phi_2}{\partial x} - \alpha\,\frac{\partial\phi_2}{\partial y} = 0\,.$$

(v) Look for wave solutions in the form $\exp[i(kx + ly - \omega t)]$ and obtain:

$$\omega kl + k\beta - l\alpha = 0 \qquad \rightarrow \qquad \omega = -\frac{\beta}{l} + \frac{\alpha}{k}$$

Dispersion curves corresponding to fixed l are hyperbolas confined to the left half of (ω, k) plane and displaced vertically by $const = -\beta/l$.

(vi) The dispersion relation is obtained similarly:

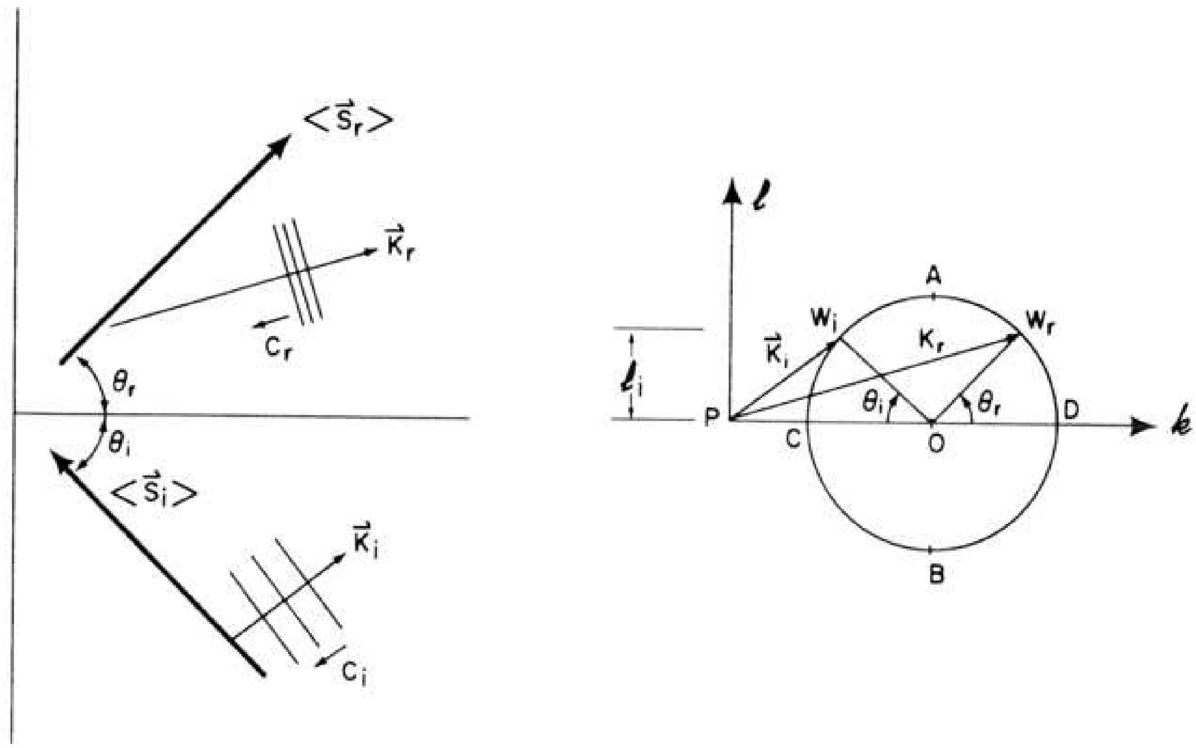
$$-i\omega\left[-2\,(k^2 + l^2) - kl + 2S\right] + ik\beta - il\alpha = 0 \qquad \rightarrow \qquad \omega = \frac{-k\beta + l\alpha}{2\,(k^2 + l^2) + kl - 2S}$$

For meridionally uniform wave $(l = 0)$:

$$\omega = \frac{-k\beta}{2k^2 - 2S} \qquad \rightarrow \qquad \frac{\partial\omega}{\partial k} = \frac{-\beta\,(2k^2 - 2S) + 4k^2\beta}{(2k^2 - 2S)^2}$$

Zonal group velocity is positive (eastward) if $-k^2 + S + 2k^2 > 0$, which is always the case.

Problem 16: *Reflecting Rossby waves.*



(a) Without loss of generality, let's consider $k > 0$, then: $\omega < 0$. Total flow field (e.g., expressed by streamfunction) consists of the incident and reflected plane waves:

$$\psi = \psi_i + \psi_r = A_i\, e^{i(k_i x + l_i y - \omega_i t)} + A_r\, e^{i(k_r x + l_r y - \omega_r t)}\,.$$

No-flow-through boundary condition on the western boundary $(x = 0)$ requires:

$$-u = \frac{\partial \psi}{\partial y} = 0 \qquad \rightarrow \qquad l_i\, A_i\, e^{i(l_i y - \omega_i t)} + l_r\, A_r\, e^{i(l_r y - \omega_r t)} = 0\,,$$

which can be true only if: $\omega_i = \omega_r = \omega, \quad l_i = l_r = l, \quad A_i = -A_r = A.$

Next, notice that for a given ω, wavevector $\mathbf{k}$ must lie in the wavenumber plane on the circle (see Figure) given by dispersion relation:

$$\Big[k - \frac{\beta}{-2\omega}\Big]^2 + l^2 = \frac{\beta^2}{4\omega^2} - R_D^{-2}\,, \qquad (*)$$

hence, the center of this circle is $(k_0, l_0) = (\beta/(-2\omega),\ 0)$, and its radius is $r = \sqrt{\beta^2/4\omega^2 - R_D^{-2}}$.

In (b) it is shown, that energy of the incident wave is directed to the western boundary (otherwise, it wouldn't be incident on it!), if $0 < k_i < k_r$, hence, it is clear which root of $(*)$ describes incident or reflected wave:

$$k_i = \frac{\beta}{-2\omega} - \Big[\frac{\beta^2}{4\omega^2} - (R_D^{-2} + l^2)\Big]^{1/2}, \qquad k_r = \frac{\beta}{-2\omega} + \Big[\frac{\beta^2}{4\omega^2} - (R_D^{-2} + l^2)\Big]^{1/2}.$$

(b) The energy equation can be written in terms of the usual conservation law and energy flux vector $\mathbf{S}$:

$$\boxed{\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{S} = 0}, \qquad \mathbf{S} = -\psi\,\nabla\frac{\partial \psi}{\partial t} - \mathbf{i}\,\beta\,\frac{\psi^2}{2}\,, \qquad E = \frac{1}{2}\,(\nabla\psi)^2 + \frac{R_D^{-2}}{2}\,\psi^2\,.$$

Mind that $\mathbf{S}$ is more fundamental descriptor of energy propagation than group velocity $\mathbf{C}_g$, and in dissipative fluids it is even different from the latter.
The energy density is readily obtained for each wave:

$$\psi = A\cos(kx+ly-\omega t) \qquad \rightarrow \qquad E = \frac{1}{2}\,(k^2+l^2)\,A^2\sin^2(kx+ly-\omega t) + \frac{1}{2}\,R_D^{-2}\,A^2\cos^2(kx+ly-\omega t)$$

The energy flux vector is obtained from ψ as

$$\mathbf{S} = -\mathbf{k}\,A^2\,\omega\,\cos^2(kx+ly-\omega t) - \mathbf{i}\,\beta\,\frac{A^2}{2}\,\cos^2(kx+ly-\omega t)\,, \qquad \mathbf{k} = (\mathbf{i}\,k,\ \mathbf{j}\,l)$$

The appropriate definition of the *wave energy* involves averaging (denoted by angular brackets) of E over space and time periods:

$$\langle E\rangle = (k^2 + l^2 + R_D^{-2})\,\frac{A^2}{4}\,.$$

(c) First, let's find the group velocity, which at the leading order is velocity of a nearly monochromatic wave packet:

$$\mathbf{C}_g \equiv \Big(\frac{\partial\omega}{\partial k},\ \frac{\partial\omega}{\partial l}\Big) = \Big(\beta\,\frac{k^2 - l^2 - R_D^{-2}}{(k^2+l^2+R_D^{-2})^2},\ \frac{2\beta kl}{(k^2+l^2+R_D^{-2})^2}\Big)$$

Second, let's find the average energy flux:

$$\langle\mathbf{S}\rangle = \frac{A^2}{2}\Big[-\mathbf{k}\,\omega - \mathbf{i}\,\frac{\beta}{2}\Big] = \frac{A^2}{2}\Big[\mathbf{i}\,k\,\frac{\beta k}{k^2+l^2+R_D^{-2}} + \mathbf{j}\,l\,\frac{\beta k}{k^2+l^2+R_D^{-2}} - \mathbf{i}\,\frac{\beta}{2}\Big] =$$

$$= \frac{A^2}{4}\Big[\mathbf{i}\,\beta\,\frac{k^2 - l^2 - R_D^{-2}}{k^2+l^2+R_D^{-2}} + \mathbf{j}\,\frac{2\beta kl}{k^2+l^2+R_D^{-2}}\Big] = \mathbf{C}_g\langle E\rangle\,,$$

given expression for $\mathbf{C}_g$.
Note, that $\mathbf{C}_g$ is spatially homogeneous (i.e., a constant) in the problem considered, therefore, the wave energy equation can be written as

$$\boxed{\frac{\partial\langle E\rangle}{\partial t} + \mathbf{C}_g\cdot\nabla\langle E\rangle = 0}$$

Thus, divergence of energy flux can be written as the advection operator.

(d) The average energy flux vector can be further manipulated as

$$\langle\mathbf{S}\rangle = \frac{A^2}{2}\,\frac{\beta}{k^2+l^2+R_D^{-2}}\,k\,\Big[\mathbf{i}\,\frac{k^2-(l^2+R_D^{-2})}{2k} + \mathbf{j}\,l\Big] = \frac{A^2}{2}\,(-\omega)\,\Big[\mathbf{i}\,\Big(k - \frac{k^2+l^2+R_D^{-2}}{2k}\Big) + \mathbf{j}\,l\Big]$$

$$= \frac{A^2}{2}(-\omega)\left[\mathbf{i}\left(k - \left[\frac{\beta}{-2\omega}\right]\right) + \mathbf{j}\,l\right].$$

Note, that vector contained in the square brackets (above) connects the centre of the dispersion circle with its circumference (see Figure) and the factor in front of it depends only on ω. Thus, we conclude that $|\langle \mathbf{S} \rangle|$ is preserved upon reflection, because ω is preserved.

(e) Wave energy increases upon the reflection:

$$\frac{\langle E_r \rangle}{\langle E_i \rangle} = \frac{k_r^2 + l^2 + R_D^{-2}}{k_i^2 + l^2 + R_D^{-2}} > 1\,.$$

Group speed decreases upon the reflection:

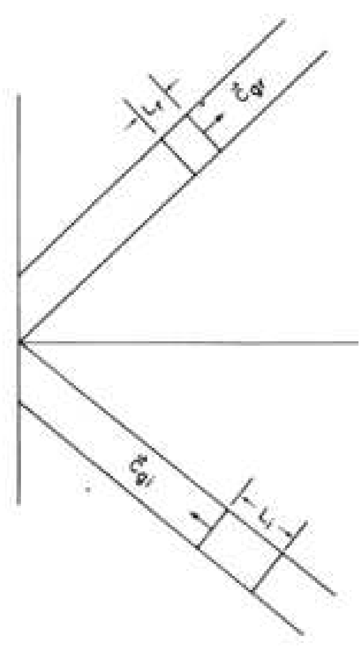
$$|\mathbf{C}_g| = \frac{2(-\omega)}{k^2 + l^2 + R_D^{-2}}\left[\frac{\beta^2}{4\omega^2} - R_D^{-2}\right] \quad \Longrightarrow \quad \frac{|\mathbf{C}_g(reflected)|}{|\mathbf{C}_g(incident)|} = \frac{k_i^2 + l^2 + R_D^{-2}}{k_r^2 + l^2 + R_D^{-2}} < 1\,,$$

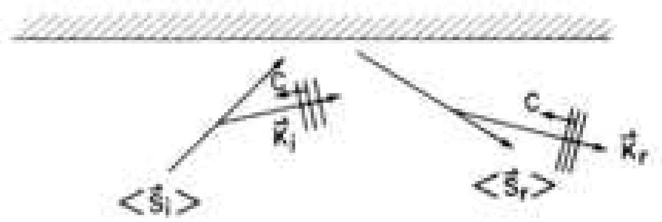
since frequency (and period) does not change upon reflection, length of the wave packet must change:

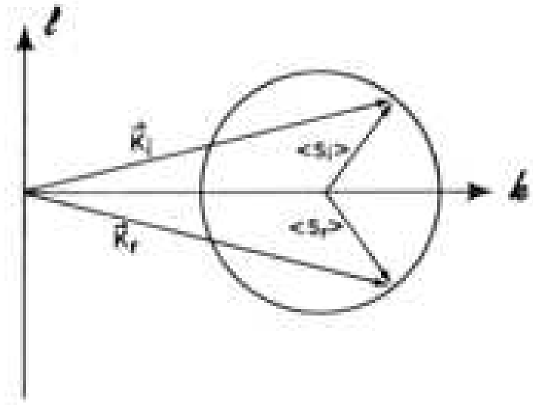
$$\Longrightarrow \quad \frac{L_r}{L_i} = \frac{|\mathbf{C}_g(reflected)|}{|\mathbf{C}_g(incident)|} < 1\,.$$

NOTE: Incident Rossby wave has relatively low energy density but fast group speed, so that its total energy flux is equal to the energy flux of the reflected wave, which has relatively high energy density but slow group speed. However, the reflected wave is characterized by shorter lengthscale, hence, it is more affected by dissipation. Thus, energy flux away from the western boundary must be less than the incoming energy flux, and this should lead to an *energy build-up on western boundaries*.

(f) Reflection from the northern boundary is illustrated by Figure. Upon reflection only l experiences sign change; both wavelength and group speed of the wave are preserved.







Problem 17: *Baroclinic dynamics with localized periodic forcing*.

Look for $\psi_n = \phi_n(x,y)\exp(-i\omega t)$ to get

$$-i\omega(\nabla^2\phi_1 - S_1(\phi_1-\phi_2)) + \beta\frac{\partial\phi_1}{\partial x} = f(x,y)\,,$$

$$-i\omega(\nabla^2\phi_2 - S_2(\phi_2-\phi_1)) + \beta\frac{\partial\phi_2}{\partial x} = 0\,,$$

and Fourier transform these equations to obtain:

$$\left[-\omega\left(k^2+l^2+S_1\right)-\beta k\right]\hat{\phi}_1 \qquad\qquad + \omega S_1\hat{\phi}_2 = i\hat{f}$$

$$\omega S_2\hat{\phi}_1 + \left[-\omega\left(k^2+l^2+S_2\right)-\beta k\right]\hat{\phi}_2 = 0\,.$$

These equations can be written in the matrix form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} if_1 \\ 0 \end{pmatrix}$$

and solved:

$$\hat{\phi}_1(k,l) = i\,\frac{d\hat{f}}{ad-bc}\,, \qquad\qquad \hat{\phi}_2(k,l) = -i\,\frac{c\hat{f}}{ad-bc}\,.$$

Inverse Fourier transform yields the solution:

$$\psi_n(t,x,y)) = e^{-i\omega t}\,\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\phi}_n(k,l)\,e^{ikx}\,e^{ily}\,dk\,dl\,.$$

Problem 18: *Linear instability of sheared flow.*

(i) Background velocity streamfunction is

$$\Psi(y) = \int -U(y)\,dy = \frac{U_0}{3}\,y^3 - \frac{U_0}{2}\,y^2\,.$$

Background PV consists of the planetary vorticity, background velocity curl and background isopycnal deformation:

$$\Pi(y) = \beta y - \frac{dU}{dy} - S\,\Psi = \beta y + U_0\,(2y-1) - S\left(\frac{U_0}{3}\,y^3 - \frac{U_0}{2}\,y^2\right).$$

The meridional PV gradient is

$$\frac{d\Pi}{dy} = \beta + 2U_0 - SU_0\,(y^2 - y)\,.$$

(ii) Potential vorticity anomaly is

$$q = \nabla^2\psi - S\,\psi\,,$$

where the first term is relative vorticity anomaly, and the second term is isopycnal deformation. Material conservation law for PV involves tendency and advective terms:

$$\frac{D}{Dt}(\Pi + q) = \left[\frac{\partial}{\partial t} + \left(U(y) - \frac{\partial\psi}{\partial y}\right)\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\,\frac{\partial}{\partial y}\right](\Pi + q) = 0$$

The linearized material conservation law for PV can be obtained as

$$\left(\frac{\partial}{\partial t} + U(y)\,\frac{\partial}{\partial x}\right)\left[\nabla^2\psi - S\,\psi\right] + \frac{\partial\psi}{\partial x}\,\frac{d\Pi}{dy} = 0\,,$$

because both the nonlinear terms and x-derivatives of Π drop out.

(iii) The necessary condition for instability to occur, that is, for existence of exponentially growing solutions, is that $d\Pi/dy$ changes sign somewhere in the channel (i.e., for some $0 \le Y \le 1$):

$$\frac{d\Pi}{dy} = \beta + 2U_0 - SU_0\,(Y^2 - Y) = 0 \qquad \rightarrow \qquad Y = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4(\beta + 2U_0)}{SU_0}}$$

Only one of the roots is positive, and it has to be ≤ 1 :

$$\sqrt{1 + \frac{4(\beta + 2U_0)}{SU_0}} \le 1 \qquad \rightarrow \qquad \frac{4(\beta + 2U_0)}{SU_0} \le 0\,,$$

and this never happens for $U_0 > 0,$ therefore, the flow is STABLE.

Problem 19: *Howard's semicircle theorem.*

Let's look for solution in the form

$$\phi = (U - c)\, a \,,$$

note that

$$\phi' = (U-c)a' + U'a \,, \qquad \phi'' = (U-c)a'' + U''a + U'a' \,,$$

and substitute ϕ in the governing equation:

$$(U-c)\,\frac{d^2\phi}{dy^2} - k^2\,(U-c)\,\phi - \frac{d^2U}{dy^2}\,\phi = 0 \qquad \rightarrow$$

$$\boxed{2\,(U-c)\,U'\,a' + (U-c)^2\,a'' - k^2\,(U-c)^2\,a = 0}\,. \qquad (*)$$

Let's multiply $(*)$ by the complex-conjugate a^*, inegrate with respect to y across the channel, and apply one integration by parts, assuming that $a(0) = a(L) = 0$. Let's focus on the second-term integral,

$$\int_0^L (U-c)^2\,a''\,a^*\,dy = a'\,a^*\,(U-c)^2\Big|_0^L - \int_0^L a'\,a^{*\prime}\,(U-c)^2\,dy - \int_0^L a'\,a^*\,2(U-c)\,U'\,dy \,,$$

and note that on the rhs the first term is zero due to the boundary conditions, and the last term will cancel out with the first-term integral coming from $(*)$. The outcome is

$$\int_0^L (U-c)^2\,(|a'|^2 + k^2|a|^2)\,dy = 0 \,.$$

Next, separate real and imaginary parts:

$$(U-c)^2 = (U - c_r - i\,c_i)^2 = (U-c_r)^2 - c_i^2 - i\,2\,(U-c_r)\,c_i \,.$$

The resulting integral equation must have both real part,

$$\int_0^L [(U-c_r)^2 - c_i^2]\,P\,dy = 0 \,, \qquad (*)$$

and imaginary part,

$$\int_0^L (U-c_r)\,P\,dy = 0 \,, \qquad (**)$$

equal to zero, where $P = |a'|^2 + k^2|a|^2$ is positively defined function.
From $(**)$ it follows that $U - c_r$ must change sign somewhere in the domain, therefore:

$$U_{min} < c_r < U_{max} \,,$$

which provides the bounds on c_r. The location where phase speed of the growing perturbations is equal to the flow velocity is called a *critical level*.

Let's now find bounds on c_i and start by writing the obvious inequality:

$$\int_0^L (U - U_{min})\,(U_{max} - U)\,P\,dy \geq 0\,.$$

Let's add this inequality to $(*)$:

$$\int_0^L \Big[(U - c_r)^2 - c_i^2 + (U - U_{min})\,(U_{max} - U)\Big]\,P\,dy \geq 0$$

and notice that we can get rid of U by subtracting from the above expression equation $(**)$ premultiplied by $U_{min} + U_{max} - 2c_r$:

$$\int_0^L \Big[(U - c_r)^2 - c_i^2 + (U - U_{min})\,(U_{max} - U) - (U_{min} + U_{max} - 2c_r)\,(U - c_r)\Big]\,P\,dy \geq 0$$

$$\rightarrow \qquad \int_0^L \Big[-c_r^2 - c_i^2 + 2\,\frac{U_{min} + U_{max}}{2}\,c_r - U_{min}U_{max}\Big]\,P\,dy \geq 0$$

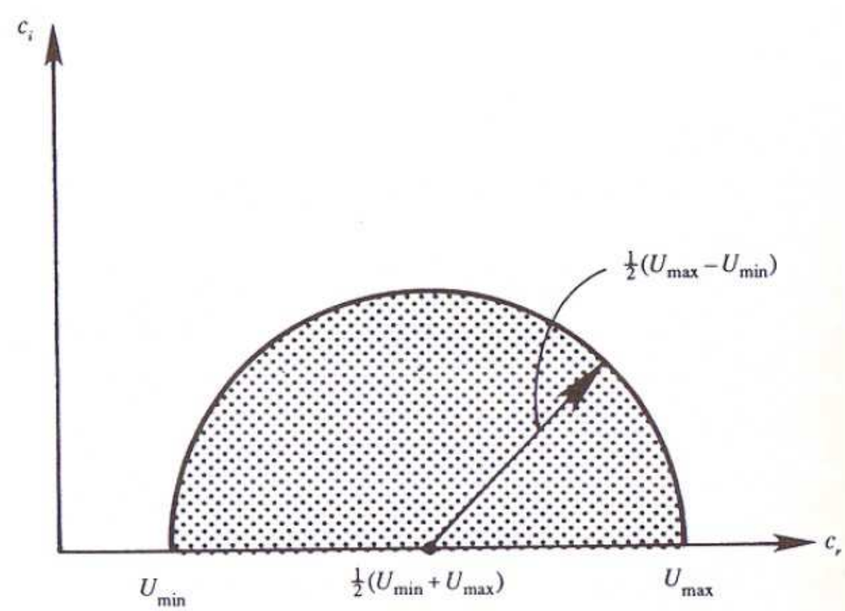
$$\rightarrow \qquad \int_0^L \Big[\Big(c_r - \frac{U_{min} + U_{max}}{2}\Big)^2 + c_i^2 - \Big(\frac{U_{min} + U_{max}}{2}\Big)^2 + U_{min}U_{max}\Big]\,P\,dy \leq 0$$

$$\rightarrow \qquad \int_0^L \Big[\Big(c_r - \frac{U_{min} + U_{max}}{2}\Big)^2 + c_i^2 - \Big(\frac{U_{max} - U_{min}}{2}\Big)^2\Big]\,P\,dy \leq 0$$

The expression in square brackets is just a constant, which, given the inequality, must be negative, therefore:

$$\boxed{\Big(c_r - \frac{U_{min} + U_{max}}{2}\Big)^2 + c_i^2 \leq \Big(\frac{U_{max} - U_{min}}{2}\Big)^2}\,.$$

Hence, the theorem is proven (see Figure).



Problem 20: *Baroclinic instability.*

(a) The mean PV gradients are

$$\frac{d\Pi_1}{dy} = \beta + \frac{1}{R_D^2}\,(U_1 - U_2)\,, \qquad \frac{d\Pi_2}{dy} = \beta + \frac{1}{R_D^2}\,(U_2 - U_1)\,, \qquad R_D^2 = \frac{g'H}{f_0^2}\,,$$

and the linearized PV equations are

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} + U_1\,\frac{\partial}{\partial x}\Big)\,\Big(\nabla^2\psi_1 - \frac{1}{R_D^2}\,(\psi_1-\psi_2)\Big) + \frac{\partial\psi_1}{\partial x}\,\frac{d\Pi_1}{dy} \;&=\; 0\,,\\
\Big(\frac{\partial}{\partial t} + U_2\,\frac{\partial}{\partial x}\Big)\,\Big(\nabla^2\psi_2 - \frac{1}{R_D^2}\,(\psi_2-\psi_1)\Big) + \frac{\partial\psi_2}{\partial x}\,\frac{d\Pi_2}{dy} \;&=\; 0\,.
\end{aligned}$$

(b) Let's look for solutions in the wave form,

$$\psi_{1,2} = A_{1,2}\exp[i(k(x-ct)+ly)]\,, \qquad c = c_r + i\frac{\omega_i}{k}\,, \qquad K^2 = k^2 + l^2\,,$$

and obtain the following linear system for the amplitudes:

$$\begin{aligned}
(U_1 - c)\,\Big(K^2A_1 + \frac{1}{R_D^2}\,(A_1 - A_2)\Big) - A_1\,\frac{d\Pi_1}{dy} \;&=\; 0\,,\\
(U_2 - c)\,\Big(K^2A_2 + \frac{1}{R_D^2}\,(A_2 - A_1)\Big) - A_2\,\frac{d\Pi_2}{dy} \;&=\; 0\,.
\end{aligned}$$

Let's now multiply the first equation by $A_1^*/(U_1-c)$ and the second equation by $A_2^*/(U_2-c)$, and add the results together:

$$\Big(|A_1|^2 + |A_2|^2\Big)\,K^2 + \frac{1}{R_D^2}\,\Big(|A_1|^2 + |A_2|^2 - 2\,Re[A_1A_2^*]\Big) - \frac{|A_1|^2}{U_1-c}\,\frac{d\Pi_1}{dy} - \frac{|A_2|^2}{U_2-c}\,\frac{d\Pi_2}{dy} = 0\,.$$

The imaginary part of this equation must be zero:

$$\frac{\omega_i}{k}\,\Big(-\frac{|A_1|^2}{|U_1-c|^2}\,\frac{d\Pi_1}{dy} - \frac{|A_2|^2}{|U_2-c|^2}\,\frac{d\Pi_2}{dy}\Big) = 0\,,$$

hence, growing solutions $(\omega_i > 0)$ can exist only if expression in the brackets is zero. This is possible only if $d\Pi_1/dy$ and $d\Pi_2/dy$ have different signs, which implies

$$\boxed{|U_1 - U_2| > \beta\,R_D^2}$$

Thus, βR_D^2 is the *critical shear*, and both β and R_D have stabilizing effects on the background flow.

(c) Let's now write the linearized problem in the matrix form:

$$\begin{aligned}
A_1\Big[(c-U_1)\,\Big(K^2 + \frac{1}{R_D^2}\Big) + \beta + \frac{U_1-U_2}{R_D^2}\Big] \qquad + A_2\Big[-(c-U_1)\,\frac{1}{R_D^2}\Big] \qquad &= 0\\
A_1\Big[-(c-U_2)\,\frac{1}{R_D^2}\Big] \qquad + A_2\Big[(c-U_2)\,\Big(K^2 + \frac{1}{R_D^2}\Big) + \beta - \frac{U_1-U_2}{R_D^2}\Big] &= 0
\end{aligned}$$

For existence of nontrivial solutions, the determinant must be zero (i.e., solvability condition). This leads to quadratic equation for the phase speed, and the corresponding roots are

$$c = \frac{U_1 + U_2}{2} - \frac{\beta\,(K^2 + R_D^{-2})}{K^2(K^2 + 2R_D^{-2})} \quad \pm \frac{\left[4\beta^2 R_D^{-4} - K^4 (U_1 - U_2)^2\,(4R_D^{-4} - K^4)\right]^{1/2}}{2K^2(K^2 + 2R_D^{-2})}\,.$$

Unstable solutions correspond to the negative values inside the square root, which implies

$$K^4 \left(\frac{4}{R_D^4} - K^4\right) \,>\, \frac{4\beta^2}{(U_1 - U_2)^2\, R_D^4}\,.$$

The lhs of this inequality is a hyperparabola facing down and passing through both $K = 0$ and some positive K. The rhs sets some positive threshold, hence solutions are unstable within the range of K between some short-wave and long-wave cutoffs.

Problem 21: *Kelvin-Helmholtz instability.*

Since this 2D flow is incompressible and irrotational, it is potential in each layer: $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$. The boundary conditions are: $y \to \infty: \ \nabla\phi_1 = U_1; \quad y \to -\infty: \ \nabla\phi_2 = U_2$.
The deformed interface is described by the kinematic boundary condition:

$$\frac{D\eta(x,t)}{Dt} = v \qquad at \qquad y = \eta(x,t) \qquad \to \qquad \frac{\partial\eta}{\partial t} + (U_1 + u_1)\,\frac{\partial\eta}{\partial x} = v_1 \qquad at \qquad y = \eta(x,t)$$

Similar statement can be made for the deep layer, and vertical velocity can be expressed via potential:

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial\eta}{\partial t} + (U_1 + u_1)\,\frac{\partial\eta}{\partial x}\,, \qquad \frac{\partial\phi_2}{\partial y} = \frac{\partial\eta}{\partial t} + (U_2 + u_2)\,\frac{\partial\eta}{\partial x} \qquad at \qquad y = \eta(x,t)$$

The dynamic boundary condition can be obtained from the unsteady Bernoulli equation:

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\,(\nabla\phi)^2 + gy = C(t)\,,$$

which is just an outcome of integrating the momentum equation written in terms of ϕ. The Bernoulli equation holds on both sides of the interface:

$$\frac{\partial\phi_1}{\partial t} + \frac{p_1}{\rho_1} + \frac{1}{2}\,(\nabla\phi_1)^2 + gy = C_1(t)\,, \qquad \frac{\partial\phi_2}{\partial t} + \frac{p_2}{\rho_2} + \frac{1}{2}\,(\nabla\phi_2)^2 + gy = C_2(t)\,,$$

therefore, the dynamic boundary condition of $p_1 = p_2$ at the interface yields

$$y = \eta\ : \qquad \rho_1\left(\frac{\partial\phi_1}{\partial t} + \frac{(\nabla\phi_1)^2}{2} + g\eta - C_1\right) = \rho_2\left(\frac{\partial\phi_2}{\partial t} + \frac{(\nabla\phi_2)^2}{2} + g\eta - C_2\right). \qquad (*)$$

The dynamic boundary condition is also satisfied for the basic state, hence

$$\rho_1\left(\frac{U_1^2}{2} - C_1\right) = \rho_2\left(\frac{U_2^2}{2} - C_2\right). \qquad (**)$$

Now, let's consider small perturbations around the basic state, assuming that they are potential flows governed by Laplace equations:

$$\phi_1 = U_1 x + \phi_1'\,, \qquad \phi_2 = U_2 x + \phi_2' \qquad \to \qquad \boxed{\nabla^2\phi_1' = 0\,, \quad \nabla^2\phi_2' = 0},$$

In the far field, assume that the solutions decay:

$$\boxed{y \to \infty: \ \ \nabla\phi_1' = 0\,, \qquad y \to -\infty: \ \ \nabla\phi_2' = 0\,.}$$

Next, let's linearize the kinematic boundary condition and apply it at $y = 0$, so that finally we have

$$\boxed{y = 0\ : \qquad \frac{\partial\phi_1'}{\partial y} = \frac{\partial\eta}{\partial t} + U_1\,\frac{\partial\eta}{\partial x}\,, \qquad \frac{\partial\phi_2'}{\partial y} = \frac{\partial\eta}{\partial t} + U_2\,\frac{\partial\eta}{\partial x}\,.}$$

By substituting the flow decomposition into $(*)$ and cancelling the terms corresponding to $(**)$, we obtain

$$y = \eta\ : \qquad \rho_1\left(\frac{\partial\phi_1'}{\partial t} + U_1\,\nabla\phi_1' + \frac{(\nabla\phi_1')^2}{2} + g\eta - C_1\right) = \rho_2\left(\frac{\partial\phi_2'}{\partial t} + U_2\,\nabla\phi_2' + \frac{(\nabla\phi_2')^2}{2} + g\eta - C_2\right),$$

and the linearization yields the final dynamic boundary condition:

$$\boxed{y=0\ :\qquad \rho_1\Big(\frac{\partial\phi_1'}{\partial t}+U_1\,\nabla\phi_1'+g\eta\Big)=\rho_2\Big(\frac{\partial\phi_2'}{\partial t}+U_2\,\nabla\phi_2'+g\eta\Big)\,.}$$

Given the horizontal symmetry of the problem, let's look for the following wave solutions:

$$\eta=\tilde\eta\,e^{ik(x-ct)}\,,\qquad \phi_1'=\tilde\phi_1\,e^{ik(x-ct)}\,,\qquad \phi_2'=\tilde\phi_2\,e^{ik(x-ct)}\,,$$

where $c=c_r+ic_i$. The Laplace equations yield

$$\frac{d^2\tilde\phi_1}{dy^2}=k^2\,\tilde\phi_1\,,\qquad \frac{d^2\tilde\phi_2}{dy^2}=k^2\,\tilde\phi_2\qquad\rightarrow\qquad \tilde\phi_1=A\,e^{-ky}+C\,e^{ky}\,,\qquad \tilde\phi_2=D\,e^{-ky}+B\,e^{ky}$$

The far-field boundary conditions restrict potentials of the wave solution to

$$\phi_1'=A\,e^{-ky}\,e^{ik(x-ct)}\,,\qquad \phi_2'=B\,e^{ky}\,e^{ik(x-ct)}\,.$$

Next, we substitute the wave solution into the kinematic boundary condition:

$$-kA\,e^{-ky}\,e^{ik(x-ct)}=-ikc\,\tilde\eta\,e^{ik(x-ct)}+ikU_1\,\tilde\eta\,e^{ik(x-ct)}\,,$$
$$kB\,e^{ky}\,e^{ik(x-ct)}=-ikc\,\tilde\eta\,e^{ik(x-ct)}+ik\,U_2\,\tilde\eta\,e^{ik(x-ct)}$$
$$\rightarrow\qquad A=-i\,(U_1-c)\,\tilde\eta\,,\qquad B=i\,(U_2-c)\,\tilde\eta$$

The remaining unknown $\tilde\eta$ is found from the dynamic boundary condition at $y=0$:

$$\rho_1\,e^{ik(x-ct)}\Big(-ik\,c\,A\,e^{-ky}+ik\,U_1\,A\,e^{-ky}+g\tilde\eta\Big)=\rho_2\,e^{ik(x-ct)}\Big(-ik\,c\,B\,e^{ky}+ik\,U_2\,B\,e^{ky}+g\tilde\eta\Big)$$
$$\rightarrow\qquad \rho_1\Big(-ik\,A\,(U_1-c)+g\tilde\eta\Big)=\rho_2\Big(ik\,B\,(U_2-c)+g\tilde\eta\Big)$$
$$\rightarrow\qquad \rho_1\Big(k\,(U_1-c)^2\,\tilde\eta+g\tilde\eta\Big)=\rho_2\Big(-k\,(U_2-c)^2\,\tilde\eta+g\tilde\eta\Big)$$
$$\rightarrow\qquad \rho_1\,k\,(U_1-c)^2+\rho_2\,k\,(U_2-c)^2=g\,(\rho_2-\rho_1)\,.$$

This quadratic equation for c has the following solution:

$$c=c_r+ic_i=\frac{\rho_1\,U_1+\rho_2\,U_2}{\rho_1+\rho_2}\pm\Big[\frac{g}{k}\,\frac{\rho_2-\rho_1}{\rho_1+\rho_2}-\rho_1\,\rho_2\,\Big(\frac{U_1-U_2}{\rho_1+\rho_2}\Big)^2\Big]^{1/2}$$

The solution is unstable, if c is complex, and this happens when expression under the square root is negative, that is,

$$\frac{g}{k}\,\frac{\rho_2-\rho_1}{\rho_1+\rho_2}<\rho_1\,\rho_2\,\Big(\frac{U_1-U_2}{\rho_1+\rho_2}\Big)^2\qquad\rightarrow\qquad k>\frac{g}{\rho_1\,\rho_2}\,\frac{\rho_2^2-\rho_1^2}{(U_1-U_2)^2}$$

Thus, if $U_1\neq U_2$, then there are always unstable short waves.

If $U_1=U_2=0$, that is, there is no flow, then

$$c=\pm\Big[\frac{g}{k}\,\frac{\rho_2-\rho_1}{\rho_1+\rho_2}\Big]^{1/2}=\pm\Big[\frac{g}{k}\,At\Big]^{1/2}\qquad\rightarrow\qquad \omega^2=gk\,At\,,$$

where $At\equiv(\rho_2-\rho_1)/(\rho_1+\rho_2)$ is the Atwood number, and the dispersion relation is the one for "deep-water" internal gravity waves.

If there is velocity discontinuity (i.e., a vortex sheet), but there is no density jump, that is, $\rho_1=\rho_2=\rho$, then

$$c=\frac{U_1+U_2}{2}\pm i\,\frac{U_1-U_2}{2}\,.$$

This implies that the vortex sheet is always unstable, and for all wavenumbers; and the phase of the perturbations propagates with the average velocity of the basic flow.

Problem 22: *Internal inertia-gravity waves.*

Internal waves in the atmosphere are often visualized by regular cloud patterns; in the ocean they are difficult to observe (e.g., “dead water” phenomenon of wave crest phase-locked to the ship stern; solitary waves); also, tides and Lee waves. Internal wave breaking input into vertical mixing and radiation from balanced motions are both problems of outstanding importance.

(a) Consider the conservative Boussinesq approximation in terms of flow fluctuations linearized about a uniformly rotating and linearly stratified state of rest:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - f_0 v &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + f_0 u &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial b}{\partial t} + N^2 w &= 0\,.
\end{aligned}
$$

With a normal-mode solution form appropriate to an unbounded domain,

$$
u = u_0\, e^{i(kx+ly+mz-\omega t)}\,, \qquad v = v_0\, e^{i(kx+ly+mz-\omega t)}\,, \qquad w = w_0\, e^{i(kx+ly+mz-\omega t)}\,,
$$

$$
p/\rho_0 = p_0\, e^{i(kx+ly+mz-\omega t)}\,, \qquad b = b_0\, e^{i(kx+ly+mz-\omega t)}\,.
$$

we obtain the following set of linear equations:

$$
\begin{aligned}
\omega u_0 - i f_0 v_0 - k p_0 &= 0\,, \\
\omega v_0 + i f_0 u_0 - l p_0 &= 0\,, \\
\omega w_0 - m p_0 - i b_0 &= 0\,, \\
k u_0 + l v_0 + m w_0 &= 0\,, \\
\omega b_0 + i N^2 w_0 &= 0\,.
\end{aligned}
$$

In the matrix form, this can be written as

$$
\begin{pmatrix}
\omega & -if_0 & 0 & -k & 0 \\
if_0 & \omega & 0 & -l & 0 \\
0 & 0 & \omega & -m & -i \\
k & l & m & 0 & 0 \\
0 & 0 & iN^2 & 0 & \omega
\end{pmatrix}
\begin{pmatrix}
u_0 \\ v_0 \\ w_0 \\ p_0 \\ b_0
\end{pmatrix}
= 0\,,
$$

and *solvability condition* states that determinant of this matrix must be equal to zero, and this provides dispersion relation:

$$
\omega^2 K^2 = (k^2+l^2)\, N^2 + m^2 f_0^2 \qquad \rightarrow \qquad \omega = \pm\Big[\frac{(k^2+l^2)\, N^2 + m^2 f_0^2}{k^2+l^2+m^2}\Big]^{1/2}\,, \qquad \omega^2 K^2 = (k^2+l^2)\, N^2 + m^2 f_0^2
$$

(b) Let's express Cartesian wavevector $\mathbf{k} = (k, l, m)$ in spherical polar coordinates (λ is upward angle) as $\mathbf{k} = (K, \theta, \lambda)$, by the following transformation:

$$k = K \cos\theta \, \cos\lambda \,, \qquad l = K \cos\theta \, \sin\lambda \,, \qquad m = K \sin\theta \,.$$

By substituting these expressions in $(*)$, we obtain

$$\omega = \pm (N^2 \cos^2\theta + f_0^2 \sin^2\theta)^{1/2} \,.$$

Note, that ω depends on the angle θ between the wavector and horizontal plane, which is also the angle formed by lines of constant phase to the vertical, and it is independent of K and λ (horizontal symmetry).

(c) Let's assume that $f_0 < N$, which is quite realistic in the stably stratified ocean and atmosphere, then, the largest and smallest wave frequencies are found by searching over all θ in (b), and they correspond to $\theta{=}0$ and $\theta{=}\pi/2$, respectively. Thus, we find:

$$max\, |\omega| = N \quad for \quad m = 0 \,, \qquad min\, |\omega| = f_0 \quad for \quad k = l = 0 \,.$$

In other words, the maximum frequency N is achieved for vertically uniform wave, and the minimum frequency f_0 is achieved for horizontally uniform wave.

(d) Components of the fundamental phase velocity

$$\mathbf{C}_p = \Big(\frac{\omega k}{K^2}, \; \frac{\omega l}{K^2}, \; \frac{\omega m}{K^2} \Big) \sim (k, \; l, \; m)$$

and the group velocity

$$\mathbf{C}_g = \Big[\frac{k\,(N^2 - \omega^2)}{\omega K^2}, \; \frac{l\,(N^2 - \omega^2)}{\omega K^2}, \; \frac{m\,(f_0^2 - \omega^2)}{\omega K^2} \Big] \sim [k\,(N^2 - \omega^2), \; l\,(N^2 - \omega^2), \; m\,(f_0^2 - \omega^2)]$$

are calculated from the dispersion relation. By taking their scalar product, one can show that these velocity vectors are orthogonal (i.e., this is a *transverse wave*):

$$\mathbf{C}_p \cdot \mathbf{C}_g \sim k^2\,(N^2 - \omega^2) + l^2\,(N^2 - \omega^2) + m^2\,(f_0^2 - \omega^2) = (k^2 + l^2)\,N^2 - \omega^2\,K^2 + m^2\,f_0^2 = 0 \,,$$

where the last step used the first expression for $\omega^2 K^2$ from $(*)$. Hence, the wave energy propagates perpendicular to the wave vector and direction of phase propagation. Furthemore, the vertical components of the fundamental phase and group velocities have *opposite signs* (except when $\omega = f_0$), because:

$$sign\,[C_p^z] = sign\,[\omega]\,sign\,[m] \,, \qquad sign\,[C_g^z] = sign\,[\omega]\,sign\,[m]\,sign[f_0^2 - \omega^2] \,,$$

and $|\omega| \geq f_0$.

If background stratification is nonlinear, then a trapped wave can form with horizontal and parallel phase and group velocities. This is a vertically standing wave, which propagates horizontally in the waveguide.

Problem 23: *Poincare and Kelvin waves in a channel.*

Meridional momentum equation predicts that at $y = 0, L$:

$$f\,\frac{\partial\eta}{\partial x} - \frac{\partial^2\eta}{\partial t\partial y} = 0\,,$$

which are the boundary conditions for the free-surface evolution equation. Let's look for the wave solution of the form $\eta(x, y, t) = \phi(y)\,\exp[i\,(kx - \omega t)]$ and obtain the following ODE:

$$(-i\omega)^2\,\phi + f^2\,\phi = c^2\,(ik)^2\,\phi + c^2\,\phi_{yy} \qquad \rightarrow \qquad \phi_{yy} + \alpha^2\,\phi = 0\,, \qquad \alpha^2 = \frac{\omega^2 - f^2}{c^2} - k^2$$

The general solution of the ODE is

$$\phi = A\,\sin(\alpha y) + B\,\cos(\alpha y)\,.$$

The boundary conditions at $y = 0, L$ yield

$$A\,\alpha + \frac{kf}{\omega}\,B = 0\,, \qquad A\,\alpha\cos(\alpha L) - B\,\alpha\,\sin(\alpha L) + \frac{kf}{\omega}\Big(A\,\sin(\alpha L) + B\,\cos(\alpha L)\Big)$$

This is the system of 2 linear equations for A and $B,$ and the solvability condition is that the determinant of the system's matrix vanishes; that is, it is

$$\Big(\alpha^2 + \frac{k^2 f^2}{\omega^2}\Big)\,\sin(\alpha L) = 0\,.$$

There are now 2 cases to consider.
(i) The first situation corresponds to

$$\sin(\alpha L) = 0 \qquad \rightarrow \qquad \alpha L = n\pi\,, \qquad n = 0, \pm 1, \pm 2, ...$$

By recalling the definition of $\alpha,$ we obtain the Poincare waves dispersion relation

$$\omega^2 = c^2\,(k^2 + \alpha^2) + f^2 = c^2\,\Big(k^2 + \Big(\frac{n\pi}{L}\Big)^2\Big) + f^2$$

The final solution in this case is

$$\eta(x, y, t) = Re\Big[\Big(A\,\sin\Big[\frac{n\pi y}{L}\Big] + B\,\cos\Big[\frac{n\pi y}{L}\Big]\Big)\,e^{i\,(kx-\omega t)}\Big]\,.$$

Note, that $n = 0$ case corresponds to the trivial solution (all fields are zeros).
(ii) The second case corresponds to

$$\alpha^2 = -\frac{f^2\,k^2}{\omega^2} \qquad \rightarrow \qquad \alpha = \pm i\,\frac{fk}{\omega} \qquad \rightarrow \qquad \phi(y) = A\,e^{fky/\omega} + B\,e^{-fky/\omega}$$

Since the domain is bounded, the exponential solutions are valid. By recalling the definition of $\alpha,$ we obtain the Kelvin waves dispersion relation:

$$\omega^2 = c^2\,(k^2 + \alpha^2) + f^2 \qquad \rightarrow \qquad \omega^2 - f^2 = \frac{c^2\,k^2}{\omega^2}\,(\omega^2 - f^2)$$

$$\rightarrow \qquad \omega^2 = c^2 k^2 \qquad or \qquad \omega^2 = f^2 \,.$$

The second situation can be absorbed in the first case, and the first situation is nondispersive Kelvin wave, which does not depend on f.

Problem 24: *Energetics of geostrophic adjustment.*

Multiply the momentum equations by Hu and Hv, respectively, and the continuity equation by $g\eta$. Sum these equations up and collect the terms to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\left(H\,(u^2+v^2)+g\,\eta^2\right)+gH\nabla(\mathbf{u}\eta)=0 \qquad (*)$$

The total energy

$$E=\frac{1}{2}\int\int\left(H\,(u^2+v^2)+g\,\eta^2\right)dx\,dy\,, \qquad \frac{dE}{dt}=0\,,$$

is conserved, provided that the divergence term in $(*)$ vanishes upon integration over large area, given that the solution decays in the far field.

The initial potential energy density (per unit length along the discontinuity) was

$$P_{init}=\int_0^\infty g\,\eta_0^2\,dx\,,$$

and the final potential energy density is

$$P_{final}=\frac{1}{2}\,g\,\eta_0^2\Big[\int_0^\infty\Big(1-e^{-x/R_d}\Big)^2dx+\int_{-\infty}^0\Big(1-e^{x/R_d}\Big)^2dx\Big]\,,$$

and the difference is

$$P_{init}-P_{final}=g\,\eta_0^2\int_0^\infty\Big(2\,e^{-x/R_d}-e^{-2x/R_d}\Big)\,dx=\frac{3}{2}\,g\,\eta_0^2\,R_d\,.$$

The initial kinetic energy is zero, and the final one is

$$K_{final}=\frac{1}{2}\,H\int\mathbf{u}^2\,dx=H\left(\frac{g\,\eta_0}{f\,R_d}\right)^2\int_0^\infty e^{-2x/R_d}\,dx=\frac{1}{2}\,g\,\eta_0^2\,R_d$$

Thus, only one third of the released potential energy was converted into the kinetic energy of the generated flow, hence the energy released into the inertial-gravity waves and radiated away is

$$E_{waves}=g\,\eta_0^2\,R_d\,.$$

Problem 25: *Eddy-mean interactions for zonal flows.*

Assume that given zonally symmetric flow is also zonally periodic with period L and introduce zonal averaging (zonal-mean) operator:

$$\langle A\rangle \equiv \frac{1}{L}\int_L A\,dx \qquad \Longrightarrow \qquad \Big\langle\frac{\partial A}{\partial x}\Big\rangle = \frac{1}{L}\int \frac{\partial A}{\partial x}\,dx = 0 \qquad (*)$$

Let's write down all flow fields in terms of their zonal-mean and fluctuation components:

$$\begin{aligned}
\mathbf{u} &= \mathbf{i}\,\langle u\rangle(t,y) + \mathbf{u}'(t,x,y)\,, \qquad \langle \mathbf{u}'\rangle = 0 \\
\zeta &= \langle\zeta\rangle(t,y) + \zeta'(t,x,y)\,, \qquad \langle\zeta'\rangle = 0 \\
\langle u\rangle &= -\frac{\partial\langle\psi\rangle}{\partial y}\,, \qquad \langle v\rangle = 0\,, \qquad \langle\zeta\rangle = \frac{\partial^2\langle\psi\rangle}{\partial y^2} \\
\mathbf{u}' &= \Big(-\frac{\partial\psi'}{\partial y},\ \frac{\partial\psi'}{\partial x}\Big)\,, \qquad \zeta' = \nabla^2\psi'
\end{aligned}$$

Let's prepare for zonal averaging of the governing vorticity equation, which is

$$\boxed{\frac{\partial\zeta}{\partial t} = -J[\psi,\zeta] - \beta v + F} \qquad (**)$$

Its Jacobian part can be treated as the following:

$$-J[\psi,\zeta] = -J[\langle\psi\rangle,\langle\zeta\rangle] - J[\langle\psi\rangle,\zeta'] - J[\psi',\langle\zeta\rangle] - J[\psi',\zeta'] = ...$$

Both $\langle\psi\rangle$ and $\langle\zeta\rangle$ have no x-dependence, therefore, the first Jacobian is zero, and the other two Jacobians involve only y-derivatives of the zonally averaged fields:

$$... = \frac{\partial\langle\psi\rangle}{\partial y}\,\frac{\partial\zeta'}{\partial x} - \frac{\partial\psi'}{\partial x}\,\frac{\partial\langle\zeta\rangle}{\partial y} - J[\psi',\zeta'] = ...$$

Zonal averaging removes the first two terms because of $(*)$:

$$\Longrightarrow \qquad -\langle J[\psi,\zeta]\rangle = -\langle J[\psi',\zeta']\rangle = -\Big\langle u'\,\frac{\partial\zeta'}{\partial x} + v'\,\frac{\partial\zeta'}{\partial y}\Big\rangle = ...$$

Use nondivergence of the velocity field and write this in the flux divergence form:

$$... = \Big\langle -\frac{\partial(u'\zeta')}{\partial x} - \frac{\partial(v'\zeta')}{\partial y}\Big\rangle = -\frac{\partial\langle v'\zeta'\rangle}{\partial y}$$

Notice, that the first term vanished because of $(*)$.
The resulting dynamical equation for zonally averaged vorticity $\langle\zeta\rangle$ is

$$\boxed{\frac{\partial\langle\zeta\rangle}{\partial t} = -\frac{\partial\langle v'\zeta'\rangle}{\partial y} + \langle F\rangle} \qquad (***)$$

Equation for the vorticity fluctuation can be obtained by subtracting $(***)$ from $(**)$ and noting that x-derivatives of the zonally averaged fields are zeros:

$$\boxed{\frac{\partial\zeta'}{\partial t} = -\langle u\rangle\,\frac{\partial\zeta'}{\partial x} - v'\,\frac{\partial\langle\zeta\rangle}{\partial y} - \beta v' - J[\psi',\zeta'] + \frac{\partial\langle v'\zeta'\rangle}{\partial y} + F'} \qquad (****)$$

Before deriving integral equations for zonal-mean and fluctuation enstrophies, let's notice that

$$-\int \langle\zeta\rangle \, \frac{\partial\langle v'\zeta'\rangle}{\partial y} \, dy = \int \Big[-\frac{\partial}{\partial y}(\langle v'\zeta'\rangle\langle\zeta\rangle) + \langle v'\zeta'\rangle \, \frac{\partial\langle\zeta\rangle}{\partial y}\Big] \, dy = ...$$

The first integral is zero, because flow fluctuations decay at $|y| \to \infty$

$$... = \int \langle v'\zeta'\rangle \, \frac{\partial\langle\zeta\rangle}{\partial y} \, dy \qquad\qquad (\star)$$

With the above expression in mind, equation for zonal-mean enstrophy is obtained by multiplying $(***)$ with $\langle\zeta\rangle$ and integrating over y :

$$\boxed{\frac{d}{dt}\int \frac{1}{2}\,\langle\zeta\rangle^2 \, dy = \int \langle v'\zeta'\rangle \, \frac{\partial\langle\zeta\rangle}{\partial y} \, dy + \int \langle\zeta\rangle \, \langle F\rangle \, dy}$$

Equation for fluctuation (eddy) enstrophy is obtained by multiplying $(****)$ with ζ' and both averaging over x and integrating over y :

$$\boxed{\frac{d}{dt}\int \frac{1}{2}\,\langle\zeta'^2\rangle \, dy = -\int \langle v'\zeta'\rangle \, \frac{\partial\langle\zeta\rangle}{\partial y} \, dy + \int \langle\zeta' F'\rangle \, dy}$$

and by taking into account that the following integrals do not contribute for the reasons similar to $(\star)$ and exercise 8:

$$\int \langle u\rangle \, \Big\langle \zeta' \, \frac{\partial\zeta'}{\partial x} \Big\rangle \, dy = 0\,,$$

$$\int \beta\langle v'\zeta'\rangle \, dy = 0\,,$$

$$\int \langle\zeta' J[\psi',\zeta']\rangle \, dy = 0\,,$$

$$\int \Big\langle \zeta' \, \frac{\partial\langle v'\zeta'\rangle}{\partial y} \Big\rangle \, dy = 0\,.$$

Now, let's take a look at the conversion term between zonal-mean and fluctuation (eddy) enstrophy and denote it by C_{enstr} :

$$C_{enstr} = -\int \langle v'\zeta'\rangle \, \frac{\partial\langle\zeta\rangle}{\partial y} \, dy\,,$$

then the evolution equations can be written as

$$\begin{aligned} \frac{d}{dt}\int \frac{1}{2}\,\langle\zeta\rangle^2 \, dy &= -C_{enstr} + ... \\ \frac{d}{dt}\int \frac{1}{2}\,\langle\zeta'^2\rangle \, dy &= +C_{enstr} + ... \end{aligned}$$

Hypothetical flux-gradient (eddy viscosity) relationship for the eddy vorticity flux is

$$\boxed{-\langle v'\zeta'\rangle = \nu\,\frac{\partial\langle\zeta\rangle}{\partial y}}$$

It makes the eddy-mean enstrophy conversion term $C_{enstr} > 0$ (i.e., eddy enstrophy is generated from zonal-mean enstrophy, as it happens in a mean-flow instability process):

$$-\int\langle v'\zeta'\rangle\,\frac{\partial\langle\zeta\rangle}{\partial y}\,dy = +\int\nu\left[\frac{\partial\langle\zeta\rangle}{\partial y}\right]^2 dy\;\geq\;0 \qquad for \qquad \nu > 0\,.$$

Positive eddy viscosity corresponds to *down-gradient* mean transport by the eddies. In this closure representation, an equilibrium state is one where the zonal-mean enstrophy is maintained by the zonal-mean forcing term $(\langle\zeta\rangle\,\langle F\rangle > 0)$ and $C_{enstr} > 0$. The eddy enstrophy can be also dissipated or generated by the eddy/forcing correlations via term $\langle\zeta' F'\rangle < 0,$ if the external forcing fluctuates.

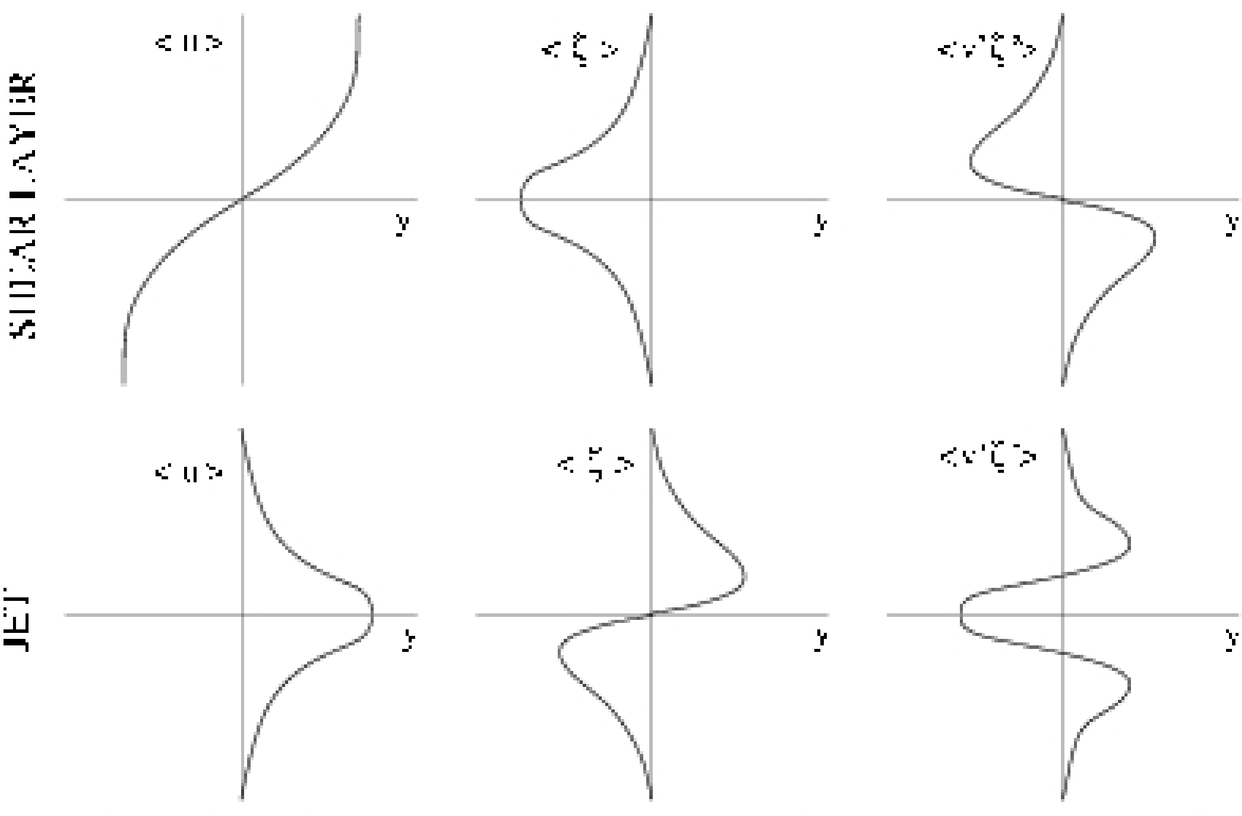
The potential enstrophy equations are closely related to those for the enstrophy, since

$$q = \zeta + \beta y\,, \qquad \langle q\rangle = \langle\zeta\rangle + \beta y\,, \qquad q' = \zeta'\,.$$

In particular, the conversion term is identical, because (this can be demonstrated by integration by parts)

$$\int\langle v'\zeta'\rangle\,\frac{\partial\langle\zeta\rangle}{\partial y}\,dy = \int\langle v'q'\rangle\,\frac{\partial\langle q\rangle}{\partial y}\,dy\,.$$

The figure below illustrates how the eddy flux profiles relate to the mean-flow profiles in a down-gradient transport regime for both a shear layer and a jet.
In both cases, the vorticity is meridionally fluxed by the eddies away from the vorticity extrema, thus, smearing and weakening the zonal-mean vorticity profile.


SHEAR LAYER
JET
< u >
< ζ >
< v'ζ' >
y

Problem 26: *Eliassen-Palm flux.*

(a) is trivial. For (b) let's consider separately the first and second terms of the meridional eddy PV flux

$$v'q' = \frac{\partial \psi'}{\partial x}\Big(\nabla^2 \psi' + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\Big)\,. \qquad (*)$$

The idea is to write $(*)$ in the flux divergence form. The first term in $(*)$ can be written as

$$\frac{\partial \psi'}{\partial x}\nabla^2 \psi' = \frac{\partial}{\partial y}\Big(\frac{\partial \psi'}{\partial x}\frac{\partial \psi'}{\partial y}\Big) + \frac{1}{2}\frac{\partial}{\partial x}\Big[\Big(\frac{\partial \psi'}{\partial x}\Big)^2 - \Big(\frac{\partial \psi'}{\partial y}\Big)^2\Big]\,,$$

and the second term in $(*)$ takes the form

$$\frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z} = \frac{\partial}{\partial z}\Big(\frac{\partial \psi'}{\partial x}\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\Big) - \frac{\partial^2 \psi'}{\partial x \partial z}\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z} = \frac{\partial}{\partial z}\Big(\frac{\partial \psi'}{\partial x}\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\Big) - \frac{1}{2}\frac{\partial}{\partial x}\frac{f_0^2}{N^2}\Big(\frac{\partial \psi'}{\partial z}\Big)^2\,.$$

Zonal averaging removes all x-derivatives, therefore:

$$\overline{v'q'} = \nabla_{yz}\cdot \mathbf{E} \equiv \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z}\,,$$

where

$$\mathbf{E} = \Big(0\,,\ \ \overline{\frac{\partial \psi'}{\partial x}\frac{\partial \psi'}{\partial y}}\,,\ \ \overline{\frac{\partial \psi'}{\partial x}\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}}\Big) = (0\,,\ \ -\overline{u'v'}\,,\ \ \frac{f_0^2}{N^2}\overline{v'b'})$$

is the *Eliassen-Palm flux*; its meridional component corresponds to the Reynolds stress effect, and its vertical component corresponds to the form stress effect.

Thus, zonally averaged momentum equation can be written as

$$\boxed{\frac{\partial}{\partial t}\langle q\rangle + \frac{\partial}{\partial y}\nabla_{yz}\cdot\mathbf{E} = 0}$$