Practical Problems for "Introduction to GFD" Lectures

Warm-Up Problems

Problem 1: *Flow kinematics.*

Consider a 2D velocity field that is purely rotational:

$$u(t,x,y) = -\frac{\partial \psi}{\partial y}\,, \qquad v(t,x,y) = \frac{\partial \psi}{\partial x}\,,$$

where $\psi(t,x,y)$ is the evolving velocity streamfunction.

Note. To check whether a flow has a rotational component, you can put a small object in the flow and let the flow carry it. If the small object spins, the flow is rotational; if the object doesnt spin, the flow is irrotational. Some flows that you may think are rotational are actually irrotational. For example, away from the center, a vortex with tangential velocity component inversely proportional to radius is actually an irrotational flow; on the other hand, a plane shear flow is rotational.

(a) Show that an isoline of ψ at any fixed time is a streamline (i.e., line everywhere tangent to $\mathbf{u}$).

(b) Consider the flow field that is a combination of the uniform shear and propagating plane wave:

$$\psi(t,x,y) = -Uy + A\sin[k(x-ct)]\,,$$

and assume that $U,\ A,\ k,$ and c are positive constants. Sketch the streamlines at $t = 0$.

(c) Find the equation for the trajectory that passes through the origin at $t = 0$.

(d) Sketch the trajectories for $c = -U,\ 0,\ U,$ and $3U$; summarize in words the dependence on c.

(e) Prove that the flow is incompressible and find its vorticity.

Problem 2: *D'Alembert solution of wave equation.*

Solve one-dimensional wave equation with the given initial conditions, assuming that c is a constant:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \quad u_t(x,0) = 0\,.$$

Problem 3: *Point vortices.*

Flow fields corresponding to a point (i.e., δ-function) vortex located at $\mathbf{x} = \mathbf{x}_0$ are:

$$\zeta = C\,\delta(\mathbf{x}-\mathbf{x}_0)\,, \qquad v_\theta = V = \frac{C}{2\pi r}\,, \qquad \psi = c_0 + \frac{C}{2\pi}\,\ln r\,.$$

where C is the vortex circulation, $r = \sqrt{x^2+y^2}$ is the distance away from $\mathbf{x}_0$, and V is the tangential velocity component.

(a) Calculate the trajectories of three equal-circulation point vortices initially located at the vertices of an equilateral triangle.

(b) Calculate using point vortices the evolution of a tripole vortex with a total circulation of zero. Its initial configuration consists of one vortex between two others that are on opposite sides, each with half the strength and of the opposite sign to the central vortex.

Problem 4: *Material conservation.*

(a) Show that the area inside a closed material curve (i.e., timeline) in 2D incompressible flow is conserved with time. Does this result depend upon whether or not the external force is zero?

(b) Show that the area inside a closed material curve is not conserved with time in a shallow-water model.

Problem 5: *Free particle motion in rotating frame.*

Consider a particle free of any force and, therefore, moving with no acceleration in the inertial frame of reference. Consider now the same particle, but moving on the rotating f-plane, that is, in the rotating frame of reference.

(a) Write down the corresponding f-plane momentum equations and solve them. Prove that the particle speed remains constant.

(b) Solve for particle trajectory and analyze its properties. On a rotating sphere, how would trajectories of such particles vary with latitude, from one pole to the other?

Problem 6: *Acoustic wave equation.*

Start from the linearized momentum, continuity and thermodynamics equations and derive the single equation for pressure.

Main Problems

Problem 7: *Deep-water gravity waves.*

Derive a solution for an oceanic surface gravity wave with a surface elevation of the form

$$h(t, x, y) = h_0 \sin[kx - \omega t]\,, \qquad h_0,\ k,\ \omega > 0\,,$$

assuming the following:

(i) inviscid and constant-density (incompressible) fluid,

(ii) constant pressure at the free surface,

(iii) small-amplitude waves (so that the kinematic boundary condition at the free surface can be linearized about $z = 0$ in a Taylor series expansion, and that the dynamical equations can be linearized about the state of rest),

(iv) irrotational motion (hence, there is velocity potential ϕ),

(v) infinite water depth (hence, all motion vanishes at $z \to \infty$),

(vi) no Coriolis force: $f = 0$.

Explain what happens to this solution when the rigid-lid approximation is made.

Problem 8: *Integrals of motion.*

Show that for inviscid 2D flows the following quantities are *integral invariants* (i.e., they are conserved with time). Assume that $f(y) = f_0 + \beta(y - y_0)$, and the flow occupies the whole plane, but $\psi,\ \mathbf{u},\ \zeta \to 0$ as $|\mathbf{x}| \to \infty$.

(a) Kinetic energy: $\frac{1}{2}\int\!\!\int (u^2 + v^2)\, dx\, dy$.

Hint: multiply the 2D vorticity equation by $-\psi$ and integrate over the domain; integrate by parts.

(b) Circulation: $\int\!\!\int \zeta\, dx\, dy$,

(c) Enstrophy: $\int\!\!\int \zeta^2\, dx\, dy$,

(d) Potential enstrophy: $\int\!\!\int q^2\, dx\, dy$.

Problem 9: *Geostrophy and thermal wind.*

What are the vertical vorticity and horizontal divergence for a geostrophic velocity for $f = f_0$ and $f = f_0 + \beta(y - y_0)$? Explain why $f = f(y)$ is geophysically relevant. From the geostrophic and hydrostatic relations, derive the thermal-wind balance:

$$\frac{\partial u}{\partial z} = \frac{g}{\rho_0 f_0}\frac{\partial \rho}{\partial y}\,, \qquad \frac{\partial v}{\partial z} = -\frac{g}{\rho_0 f_0}\frac{\partial \rho}{\partial x}$$

Problem 10: *Fractional depth of fluid columns.*

Consider shallow-water model with bottom topography and prove that a particle half way up on a water column always stays half way up, that is, that the *fractional height* of particles is materially conserved quantity.

Problem 11: *Shallow-water PV conservation.*

Assume the shallow-water approximation for a single layer of constant-density fluid with depth h, flat bottom, density ρ and Coriolis parameter f. Consider material conservation law for the potential vorticity (PV) q:

$$\frac{D_h}{Dt}\, q = 0\,, \qquad q \equiv \frac{\zeta + f}{h}\,.$$

(i) Consider the rigid-lid approximation and β-plane. Assume that some external forcing instantaneously generates relative vorticity ζ_0 at y_0. How far can the fluid element with initial ζ_0 move meridionally?

(ii) Consider β-plane. Assume that some external forcing instantaneously generates relative vorticity ζ_0 at y_0. Assume also that the corresponding fluid element (column) moves meridionally over some distance y and stops. Given the measurement showing that at the final destination the fluid element was stretched by 10%, find its travel distance y.

Problem 12: *Topographic effects.*

Consider the conservative (i.e., inviscid and unforced) shallow-water dynamics with topography $B(x, y)$ on the bottom of the layer with the resting-flow depth H, and Coriolis parameter f. In this case it can be shown that potential vorticity is

$$q = \frac{f + \zeta}{H + \eta - B}\,,$$

where $\eta(t, x, y)$ is deformation of the free surface, and $\zeta(t, x, y)$ is flow vorticity.

(a) Show that stationary solutions are ones in which $q = F(\Psi)$, where F is any function, and Ψ is a *transport streamfunction*, such that $h\mathbf{u} = (-\Psi_y, \Psi_x)$.

(b) Show that (a) implies flow going along contours of

$$\frac{f}{h} = \frac{f_0 + \beta y}{H - B}\,,$$

if flow is weak enough, so that ζ and η are negligible compared to δf and δh.

(c) Sketch streamlines for an incident, steady and uniform eastward flow across a meridional mid-ocean ridge with $B = B(x) > 0$, or mid-ocean trench with $B = B(y) > 0$. What about the case with the same topographies, but with a steady and uniform meridional flow?

Problem 13: *Gradient-wind balance.*

Consider 2D inviscid dynamics of rotating fluid and derive the *divergence equation* in terms of pressure p and velocity streamfunction ψ. Prove, that for small Rossby numbers $p \sim \psi$.

Problem 14: *Ekman bottom boundary layer.*

Consider an f-plane with a 10 km deep layer of constant-density rotating fluid (atmosphere) with constant geostrophic wind $\mathbf{u}_g = (u_g, v_g)$ in the interior of the layer. Assume a flat bottom underneath the layer and the no-slip boundary condition on it. Assume that turbulence in the boundary layer redistributes horizontal momentum vertically in a such way that this process can be described by vertical viscosity with constant coefficient $\nu > 0$.

(i) Sketch velocity profile from the interior to the bottom. Write down horizontal momentum equations in the interior and in the boundary layer. Explain meanings of the terms and main balances. Write down boundary conditions on both sides of the boundary layer.

(ii) Assume that $\nu = 10\ \mathrm{m}^2\ \mathrm{s}^{-1}$ and Coriolis parameter is $f_0 = 10^{-5}\ \mathrm{s}^{-1}$, and estimate the boundary layer depth.

(iii) Solve for the full flow in the Ekman boundary layer, write down the explicit solution consisting of the geostrophic and Ekman parts.

For the other questions, assume that the geostrophic wind is zonal: $v_g = 0$.

(iv) Prove that very close to the bottom the full flow velocity is not aligned with the wind velocity and find the corresponding angle.

(v) Find vertically integrated additional transport due to the Ekman velocity. You may use:

$$\int_0^\infty e^{-x} \cos x \, dx = \int_0^\infty e^{-x} \sin x \, dx = \frac{1}{2}$$

Problem 15: *Exoplanetary waves.*

A group of fluid dynamicists working on atmospheric circulation of some exoplanet came to the conclusion that the midlatitude atmosphere can be approximated as a two-layer fluid, in which the bottom and top layers are of equal depths and can be described in terms of materially conserved quantities Π_1 and Π_2. In locally Cartesian coordinates:

$$\Pi_1 = \nabla^2(\psi_1 - \psi_2) + \frac{\partial^2 \psi_1}{\partial x \partial y} + S\,(\psi_1 - \psi_2) + \beta y + \alpha x\,,$$

$$\Pi_2 = \nabla^2(\psi_2 - \psi_1) + \frac{\partial^2 \psi_2}{\partial x \partial y} + S\,(\psi_2 - \psi_1) + \beta y + \alpha x\,,$$

where ψ_1 and ψ_2 are the top- and bottom-layer velocity streamfunction, respectively; x is zonal (i.e., eastward) coordinate, and y is meridional (i.e., northward) coordinate. Parameters of the problem are given as $S > 0,\ \beta > 0,\ \alpha > 0$. In this problem you have to help the fluid dynamicists to understand the linear wave properties of the exoplanet.

(i) What are physical dimensions of $\psi_1,\ S,\ \beta,\ \alpha,\ \Pi_1$?

(ii) In the Eulerian framework, write down the laws governing the evolution of Π_1 and Π_2, and explain the meaning of each term.

(iii) Linearize the laws obtained in (ii) around the state of rest, write them down and, thus, obtain the governing equations.

(iv) Prove that the governing equations can be decoupled by into vertically uniform (barotropic) and nonuniform (baroclinic) components, and write them down.

(v) Obtain the dispersion relation $\omega = \omega(k,l)$ for the barotropic waves, where ω is the frequency, and (k,l) is the wavevector. Sketch the dispersion curves, first, for fixed l, then, for fixed k.

(vi) Obtain the dispersion relation $\omega = \omega(k,l)$ for the baroclinic waves. Consider meridionally uniform waves, that is, $l = 0$ and find out whether these waves can propagate energy (hence, information) to the east.

Problem 16: *Reflecting Rossby waves.*

Consider QG shallow-water Rossby wave governed by the dispersion relation

$$\omega = \frac{-\beta k}{k^2 + l^2 + R_D^{-2}}$$

and its reflection from the meridional western boundary.

(a) Prove, that the incident and reflected wavevectors lie on a circle in the wavenumber (k,l) plane. Find and draw this circle, indicate on it both wavevectors.
Hint: Prove, that incident and reflected waves are characterized by (ω, k_i, l) and (ω, k_r, l), respectively, and use this result; consider $0 < k_i < k_r$.

(b) Find *energy flux* vector $\mathbf{S}$ for a plane Rossby wave (packet), as well as the energy E of this wave. What is the time-mean (i.e., averaged over one period) energy $\langle E \rangle$ of the wave? What is the time-mean energy flux $\langle \mathbf{S} \rangle$?
Hint: The energy equation is

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\psi)^2 + R_D^{-2}\psi^2}{2}\right] + \nabla\cdot\left[-\psi\,\nabla\frac{\partial\psi}{\partial t} - \hat{\mathbf{x}}\,\beta\,\frac{\psi^2}{2}\right] = 0\,.$$

(c) Prove that $\langle \mathbf{S} \rangle = \mathbf{C_g}\langle \mathbf{E} \rangle$, where $\mathbf{C}_g$ is the group velocity, thus, prove that energy of the wave packet propagates with the group velocity.

(d) Prove that $\langle \mathbf{S} \rangle$ is directed from the centre of the circle found in (a) to the tip of the corresponding wavevector $\mathbf{k}$.

(e) Find how much speed of the incident wave packet (i.e., magnitude of the group velocity) has changed upon reflection. What happened with the length of the wave packet?

(f) Show qualitatively what happens with wavevector upon Rossby wave reflection from the *northern* boundary. Find the dispersion circle as in (a).

Problem 17: *Baroclinic dynamics with localized periodic forcing.*

Consider the following forced, linear, two-layer QG PV dynamics on the β-plane:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi_1 - S_1\left(\psi_1-\psi_2\right)\right) + \beta\,\frac{\partial\psi_1}{\partial x} = f(x,y)\,e^{-i\omega t}\,,$$

$$\frac{\partial}{\partial t}\left(\nabla^2\psi_2 - S_2\left(\psi_2-\psi_1\right)\right) + \beta\,\frac{\partial\psi_2}{\partial x} = 0\,,$$

assuming that $f(x,y)$ is spatially compact function with spatial Fourier transform

$$\hat{f}(k,l) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-ikx}\,e^{-ily}\,dx\,dy\,.$$

Find the forced flow solution by spatial Fourier transform of the governing equations.

Problem 18: *Linear instability of sheared flow.*

Consider 1.5-layer QG PV model configured in a zonal channel $(-\infty < x < \infty;\ \ 0 \le y \le 1)$. Assume there is a rigid lid on the top of the active layer, a rotating β-plane, and that the fluid is inviscid. Let the stratification parameter be $S = R^{-2}$, where R is the deformation radius. Assume that the channel contains background flow given by zonal velocity with meridional profile $U(y) = U_0\,y\,(1-y)$, such that $U_0 > 0$ (eastward flow).

(i) Write down expressions for the background velocity streamfunction $\Psi(y)$ and potential vorticity $\Pi(y)$, and briefly explain the meaning of each term in Π. Write down expression for the meridional gradient of $\Pi(y)$.

(ii) Write down the expression for the potential vorticity (PV) anomaly that corresponds to fluctuations in the background flow, and explain the meaning of each term. Write down the governing equation describing material conservation law for the full PV, and explain the meaning of each term. Linearize the governing equation around the background flow.

(iii) What is the necessary condition for linear instability of a zonal background flow? Is the background flow $U(y)$ considered in this problem stable or unstable?

Problem 19: *Howard's semicircle theorem.*

Consider horizontal shear instability of 2D inviscid flow on the f-plane and in a zonal channel $0 < y < L$. The flow velocity is purely zonal $U(y)$ and characterized by the minimum and maximum values, U_{min} and U_{max}, respectively. Start from the governing equation for the amplitude of the wave solution for the perturbation velocity streamfunction $\psi(x,y,t) = \phi(y)\,\exp[i(kx-\omega t)]$:

$$\phi_{yy} - k^2\phi + \frac{-U_{yy}}{U-c}\,\phi = 0\,, \qquad c = c_r + ic_i = \frac{\omega}{k}\,.$$

Prove that growing perturbations must have bounded $c = c_r + ic_i$, so that it lies inside the upper half of the complex-plane circle centered at $c = ([U_{min}+U_{max}]/2, 0)$, and with the diameter $D = U_{max} - U_{min}$.

Problem 20: *Baroclinic instability.*

Explore linear stability of the two-layer *Phillips model*, which is the simplest β-plane model for an unstable, horizontally uniform, zonal current with an arbitrary vertical shear. Assume that both layers have equal thicknesses H, and background zonal currents in the upper and lower layers have velocities U_1 and U_2, respectively.

(a) Find background PV gradients Π_1 and Π_2, and write down linearized potential vorticity equations for both layers.

(b) Look for solutions in the wave form $\psi_{1,2} = A_{1,2}\exp(...)$ and derive the *necessary condition for instability*: meridional PV gradients in the layers have to be of the opposite sign. Find the critical shear $U_{crit} = U_1 - U_2$. Does increasing β stabilize or destabilize the background flow?
Hint: Multiply the first and second equations by $A_1^*/(U_1 - c)$ and $A_2^*/(U_2 - c)$, respectively; add them up and look at the resulting imaginary part.

(c) Find the underlying dispersion relation as the solvability condition for the pair of the governing linear equations. What determines instability of the solution? Is it unstable in the short-wave and long-wave limits?

Problem 21: *Kelvin-Helmholtz instability.*

Consider two infinite-depth layers of different densities and separated by interface, so that the lighter fluid is on the top. Let U_1 and ρ_1 be the velocity and density of the basic state in the upper layer, and U_2 and ρ_2 be those in the bottom layer. Assume that flow remains two-dimensional, incompressible and irrotational, and that flow perturbations die out far away from the interface. Carry out linear stability analysis of the basic state. Consider also the limiting situations: $U_1 = U_2 = 0$ and $\rho_1 = \rho_2$.

Problem 22: *Internal inertia-gravity waves.*

Assume conservative 3D Boussinesq approximation (with a simple equation of state) in an unbounded domain. Assume also uniform rotation (i.e., f-plane) $f = f_0$ and uniform (i.e., linear) static stratification:

$$\overline{\rho}(z) = \rho_0 - \frac{\rho_0}{g}\,N_0^2\,z\,.$$

Consider small-amplitude fluctuations around the state of rest.

(a) Derive the inertia-gravity wave dispersion relation.

(b) Prove, that wave frequency ω depends only on the direction of the wavevector, that is, not on its magnitude.

(c) Prove, that N_0 and f_0 are the largest and smallest, respectively, frequencies allowed for the inertia-gravity waves.

(d) Prove, that the phase and group velocities are orthogonal to each other and have opposite-signed vertical components.

Problem 23: *Poincare and Kelvin waves in a channel.*

Find solutions of the shallow-water free-surface evolution equation

$$\frac{\partial^2 \eta}{\partial t^2} + f^2 \eta = c^2 \, \nabla^2 \eta$$

in a zonal channel of width L, and with no-flow-through boundary conditions on the zonal walls. Recall that the corresponding momentum equation is

$$\left(\frac{\partial^2}{\partial t^2} + f^2 \right) \mathbf{u} = -g \, \nabla \, \frac{\partial \eta}{\partial t} + f g \left(- \frac{\partial \eta}{\partial y}, \, \frac{\partial \eta}{\partial x} \right)$$

Problem 24: *Energetics of geostrophic adjustment.*

Consider linearized shallow-water dynamics

$$\frac{\partial u}{\partial t} - fv = -g\,\frac{\partial \eta}{\partial x}\,, \qquad \frac{\partial v}{\partial t} + fu = -g\,\frac{\partial \eta}{\partial y}\,, \qquad \frac{\partial \eta}{\partial t} + H\,\nabla \mathbf{u} = 0\,,$$

and derive the corresponding energy equation. Next, recall the geostrophic adjustment problem for the simple discontinuity in the fluid height field,

$$\eta(x, t = 0) = +\eta_0 \quad if \quad x < 0\,, \qquad \eta(x, t = 0) = -\eta_0 \quad if \quad x > 0\,,$$

and its solution:

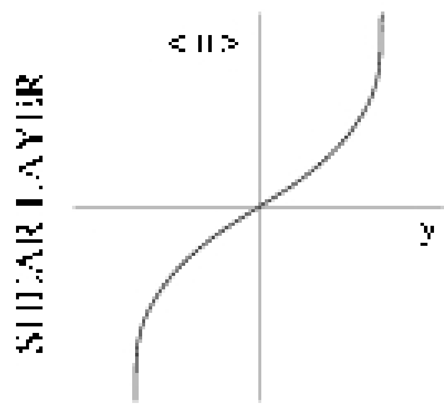
$$\eta = -\eta_0\,(1 - e^{-x/R_d}) \quad if \quad x > 0\,, \qquad \eta = +\eta_0\,(1 - e^{+x/R_d}) \quad if \quad x < 0$$

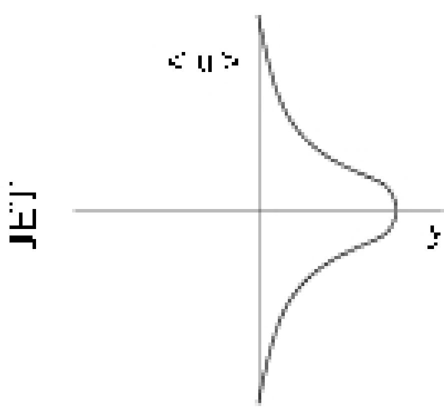
$$u = 0\,, \qquad v = -\frac{g\eta_0}{fR_d}\,e^{-|x|/R_d}$$

Estimate how much energy is radiated away by inertial-gravity waves during the geostrophic adjustment process.

Problem 25: *Eddy-mean interactions for zonal flows.*

Consider incompressible 2D dynamics and a zonal shear layer and an eastward jet localized at some latitudes (see Figure). Assume that these flows are the time-mean component of statistically equilibrated but highly turbulent flow, that decays to zero at $|y| \to \infty$.





(a) Derive eddy-mean interaction equations for enstrophy and potential enstrophy.

(b) Assume eddy viscosity (i.e., flux-gradient) relationship between the time-mean eddy Reynolds stress and vorticity for each given flow. Plot and compare time-mean Reynolds stress, vorticity and velocity profiles.

Extra Problems

Problem 26: *Eliassen-Palm flux*.

Consider some turbulent QG flow characterized by purely zonal time-mean component.
(a) Decompose the flow into zonally averaged and fluctuation components. Show that zonally averaged QG PV equation is

$$\frac{\partial}{\partial t}\langle q\rangle + \frac{\partial}{\partial y}\langle v'q'\rangle = 0\,,$$

where primes denote fluctuations and angular brackets denote zonal mean.

(b) Consider continuous stratification, so that

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left[\frac{f_0^2}{N^2}\,\frac{\partial\psi}{\partial z}\right],$$

and show that $\langle v'q'\rangle$ can be written as 2D (vertical-plane) divergence

$$\nabla_{yz} = \left(\mathbf{j}\,\frac{\partial}{\partial y},\ \mathbf{k}\,\frac{\partial}{\partial z}\right)$$

acting on the corresponding 2D vector field $\mathbf{E}$ and find this field.