Regularized transformation optics for transient heat transfer

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***Abstract* – We report on certain cloaking strategies for transient heat transfer. Regularized Kohn's transform is employed to design cylindrical cloaks and to prove a near-cloak result. Our main result says that, after the lapse of a certain threshold time, the temperature field outside the cylindrical cloak is close to that of the uniformly conducting medium irrespective of the conductivity enclosed in the cloaked region.**

I. Introduction

The by now classical transformation optics approach to cloaking employs singular change-of-variables such as the Pendry's transform [1] where one blows up a point to the cloaked region. Although physicists have made rapid progress in the fabrication and characterization of electromagnetic cloaks [2], mathematical analysis of such singular structures is highly non-trivial. Rather than working with the singular transform, we adapt the viewpoint of [3] where the singular scheme is replaced by a regularized one and the notion of "perfect cloak" is replaced by the the notion of "near-cloak".

More precisely, the regularized transformation in [3] blows up a ball of radius ε rather than a point. We consider an arbitrary spatial domain $\Omega \subset \mathbb{R}^2$ which contains B_2 (we denote by B_r the ball of radius r centred at the origin). Given a small parameter $\varepsilon > 0$, we consider the Lipschitz invertible map $\mathcal{F}_\varepsilon : \Omega \mapsto \Omega$ defined as $\mathcal{F}_\varepsilon(x) := x$ for $x \in \Omega \setminus B_2$, $\mathcal{F}_\varepsilon(x) := \left(\frac{2-2\varepsilon}{2-\varepsilon} + \frac{|x|}{2-\varepsilon}\right)\frac{x}{|x|}$ for $x \in B_2 \setminus B_\varepsilon$ and $\mathcal{F}_\varepsilon(x) := \frac{x}{\varepsilon}$ for $x \in B_\varepsilon$. Note that taking $\varepsilon = 0$ in the mapping $\mathcal{F}_\varepsilon$ yields the singular transform of Pendry. In what follows, we denote by $D\mathcal{F}_\varepsilon(x)$, the Jacobian matrix associated with the Lipschitz map $\mathcal{F}_\varepsilon : \Omega \mapsto \Omega$.

Change-of-variables based cloaking essentially exploits the coordinate invariance property of the underlying differential equation. We consider an evolution equation for a temperature field $u(t,x)$ in space-time variables:

$$\rho(x)\partial_t u(t,x) = \mathrm{div}\,(A(x)\nabla u(t,x)) + f(x)$$

where the coefficients $\rho(x)$ represents the volumetric heat capacity, $A(x)$ represents the thermal conductivity coefficient and $f(x)$ represents the heat source independent of time, ∂_t is the partial derivative with respect to the time variable and div, ∇ are the usual divergence and gradient operators with respect to space variable.

In this short note, we announce some of our recent results on "near-cloaking" strategies for the above parabolic problem. To the best of our knowledge, this is the first work to consider near-cloaking strategies to address time-dependent heat conduction problem. Here, we only state some of the main results of our work and give some numerical illustrations. We choose to give elaborate explanations, mathematical proofs and further analysis elsewhere (see [4]). We cite [5] which formally treats the thermal cloaking problem using Pendry's transform. Note that the evolution equation which we consider is a good model for [6], which designs and fabricates a microstructured thermal cloak that molds the heat flow around an object in a metal plate.

II. Main objectives and results

Given a positive bounded density $\eta(x)$ and a positive definite bounded conductivity $\beta(x)$ on B_1, our goal is to construct coefficients $\rho_{\mathrm{cl}}(x)$ and $A_{\mathrm{cl}}(x)$ such that $(\rho_{\mathrm{cl}}(x), A_{\mathrm{cl}}(x)) = (\eta(x), \beta(x))$ for $x \in B_1$ such that the solution to the following evolution

$$\rho_{\mathrm{cl}}(x)\partial_t u_{\mathrm{cl}}(t,x) = \mathrm{div}\,(A_{\mathrm{cl}}(x)\nabla u_{\mathrm{cl}}(t,x)) \tag{1}$$

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satisfies $u_{\rm cl}(t,x) \approx u_{\rm hom}(t,x)$ for $x \in \Omega \setminus B_2$, probably for time instants $t > 0$ after a threshold $T^* < \infty$, with $u_{\rm hom}(t,x)$ being solution to the homogeneous equation

$$\partial_t u_{\rm hom}(t,x) = \Delta u_{\rm hom}(t,x). \tag{2}$$

More importantly, the aforementioned approximation should hold irrespective of the choice of $(\eta(x), \beta(x))$ in B_1. Note further that the evolution equations (1) and (2) are to be supplemented by initial data in Ω and boundary data on $\mathbb{R}_+ \times \partial\Omega$. In fact, we treat them as initial boundary value problems with Neumann data. Furthermore, the chosen data are the same for both the evolutions (1)-(2).

Theorem 1 *Given $\varepsilon > 0$, there exists a threshold time $T^*(\varepsilon) < \infty$ such that for all $t \geq T^*(\varepsilon)$ we have*

$$\|u_{\rm hom}(t,\cdot) - u_{\rm cl}(t,\cdot)\|_{{\rm L}^2(\partial\Omega)} \leq C\,\varepsilon^2 \tag{3}$$

where the constant C is independent of ε. The temperature fields $u_{\rm cl}(t,x)$ and $u_{\rm hom}(t,x)$ in the above estimate are solutions to the evolution problems (1) and (2) respectively. Furthermore, the cloaking coefficients in (1) are

$$\rho_{\rm cl}(y),\, A_{\rm cl}(y) = \begin{cases} 1 & {\rm Id} & \textit{for } y \in \Omega \setminus B_2, \\ \mathcal{F}^*_\varepsilon 1(y) & \mathcal{F}^*_\varepsilon {\rm Id}(y) & \textit{for } y \in B_2 \setminus B_1, \\ \eta(y) & \beta(y) & \textit{for } y \in B_1, \end{cases} \tag{4}$$

where the functions $\mathcal{F}^*_\varepsilon 1(y)$ and $\mathcal{F}^*_\varepsilon {\rm Id}(y)$ in (4) are the so-called push-forward functions given as

$$\mathcal{F}^*_\varepsilon 1(y) = \frac{1}{\det D\mathcal{F}_\varepsilon(x)}; \qquad \mathcal{F}^*_\varepsilon {\rm Id}(y) = \frac{D\mathcal{F}_\varepsilon(x) D\mathcal{F}^\top_\varepsilon(x)}{\det D\mathcal{F}_\varepsilon(x)} \qquad \textit{where } x = \mathcal{F}^{-1}_\varepsilon(y).$$

The estimate (3) makes precise the meaning of $u_{\rm cl}(t,x)$ approximating $u_{\rm hom}(t,x)$ mentioned earlier. Our choice of the domain Ω containing B_2 is arbitrary and Theorem 1 asserts that for any such arbitrary choice, the distance between the solutions $u_{\rm hom}$ and $u_{\rm cl}$ (measured in the ${\rm L}^2(\partial\Omega)$-norm) can be made as small as we wish provided we engineer appropriate cloaking coefficients (for instance via a homogenization approach) – see (4) – in the annulus $B_2 \setminus B_1$. This is the notion of "near-cloak". Unlike the "perfect cloaking" strategies which demand equality between $u_{\rm hom}$ and $u_{\rm cl}$ everywhere outside B_2, "near-cloak" strategies only ask for them to be close in certain norm topologies. "Near-cloaking" strategies is what matters in practice.

Our idea of proof for Theorem 1 goes via the following steps: *Step I*: Using a "change-of-variable" principle, we deduce the equivalence between the evolution (1) and the evolution of temperature field in a uniformly conducting medium with a small defect of extreme conductivity provided that we are interested in the behaviour of the temperature fields outside B_2. *Step II*: We then study the long time behaviour of solutions to certain initial boundary value problems of parabolic type with Neumann data. We further characterise the equilibrium states and prove an exponential decay estimate on the associated semigroup in the Sobolev space ${\rm H}^1(\Omega)$. *Step III*: We close the arguments by borrowing an idea from [7] which deals with the effect of small inhomogeneities with extreme conductivities on boundary measurements in the context of electric impedance tomography.

Rather than giving the technical details of the proof in their entirety (interested readers are to consult [4]), we here focus our attention on an interesting observation on the long time behaviour of a particular parabolic problem which we encounter in the proof of Theorem 1.

For an unknown $v^\varepsilon(t,x)$, consider the initial-boundary value problem

$$\frac{\partial v^\varepsilon}{\partial t} = {\rm div}\Big(\mathcal{D}^\varepsilon(x)\nabla v^\varepsilon\Big) \text{ in } (0,\infty)\times\Omega; \quad \nabla v^\varepsilon \cdot \mathbf{n}(x) = 0 \text{ on } (0,\infty)\times\partial\Omega; \quad v^\varepsilon(0,x) = v^{\rm in}(x) \text{ in } \Omega, \tag{5}$$

with the conductivity $\mathcal{D}^\varepsilon(x)$ being uniform with a small defect. More precisely, the conductivity coefficient $\mathcal{D}^\varepsilon(x) = {\rm Id}$ in $\Omega \setminus B_\varepsilon$ and $\mathcal{D}^\varepsilon(x) = \varepsilon^2 \beta\left(\frac{x}{\varepsilon}\right)$ in B_ε, where β is any bounded positive definite matrix. We prove the following long time asymptotic result.

Proposition 2 *Let $v^\varepsilon(t,x)$ be the solution to the initial-boundary value problem* (5). *Suppose the initial datum $v^{\rm in} \in {\rm H}^1(\Omega)$. Then, there exists a constant $\gamma(\varepsilon) > 0$ such that for all $t > 0$, we have*

$$\left\|v^\varepsilon(t,\cdot) - \langle v^{\rm in}\rangle\right\|^2_{{\rm H}^1(\Omega)} \leq e^{-2t\gamma(\varepsilon)} \left(\left\|v^{\rm in}\right\|^2_{{\rm L}^2(\Omega)} + \varepsilon^{-2}\left\|\nabla v^{\rm in}\right\|^2_{{\rm L}^2(\Omega)}\right) \tag{6}$$

where $\langle v^{\rm in}\rangle$ denotes the averaged initial field.

The proof of the above proposition employs the spectral method. More precisely, for the eigen-pairs $(\mu_k^\varepsilon, \varphi_k^\varepsilon)_{k=1}^\infty$, we consider the Neumann eigenvalue problem

$$-\mathrm{div}\,(\mathcal{D}^\varepsilon \nabla \varphi_k^\varepsilon) = \mu_k^\varepsilon \varphi_k^\varepsilon \quad \text{in } \Omega; \qquad \nabla \varphi_k^\varepsilon \cdot \mathbf{n}(x) = 0 \quad \text{on } \partial\Omega \tag{7}$$

and use the family $\{\varphi_k^\varepsilon\}_{k=1}^\infty$ of eigenfunctions as an orthonormal basis in $\mathrm{L}^2(\Omega)$. It turns out that the constant $\gamma(\varepsilon)$ in (6) is the first non-zero eigenvalue to the above spectral problem. A non-trivial part of the proof of Proposition 2 is to derive the estimate

$$\|\nabla v^\varepsilon\|^2_{\mathrm{L}^2(\Omega)} \leq e^{-2t\gamma(\varepsilon)}\varepsilon^{-2}\left\|\nabla v^{\mathrm{in}}\right\|^2_{\mathrm{L}^2(\Omega)} \tag{8}$$

using the spectral representation via the aforementioned orthonormal basis of eigenfunctions.
We shall illustrate the sharpness of the estimate (8) by a numerical experiment using the finite element software COMSOL MULTIPHYSICS in a two dimensional domain $(-3,3)^2$. Taking the positive definite matrix β to be identity and considering the initial datum $v^{\mathrm{in}}(x_1,x_2) = x_2 + 3$, we compute the solution $v^\varepsilon(t,x)$ for the initial boundary value problem (5) while fixing the regularization parameter $\varepsilon = 10^{-1}$. For the same choice of ε, we also compute the first non-zero eigenvalue (0.522689) and the associated eigenfunction for the spectral problem (7).

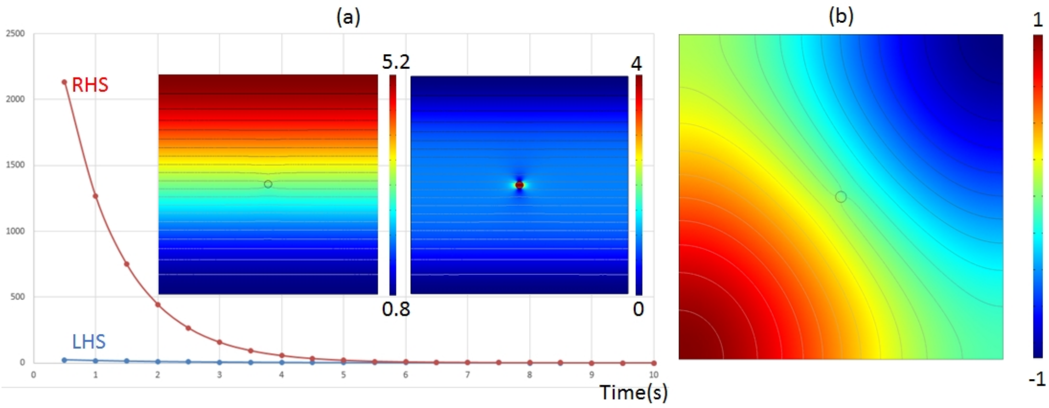

Fig. 1: Panel (a) – Graphs corresponding to the inequality (8): left-hand side of the inequality (blue) and right-hand side of the inequality (red); Insets in Panel (a) – Contour plots at the time instant $t = 0.5s$: field v^ε (left) and field $|\partial_{x_1} v^\varepsilon|^2 + |\partial_{x_2} v^\varepsilon|^2$ (right); Panel (b) – Eigenfunction: plot of the eigenfield corresponding to first non-zero eigenvalue 0.522689.

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