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Stabilizing non-trivial solutions of the generalized Kuramoto–Sivashinsky equation using feedback and optimal control

Lighthill–Thwaites Prize

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The problem of controlling and stabilizing solutions to the Kuramoto–Sivashinsky (KS) equation is studied in this paper. We consider a generalized form of the equation in which the effects of an electric field and dispersion are included. Both the feedback and optimal control problems are studied. We prove that we can control arbitrary non-trivial steady states of the KS equation, including travelling wave solutions, using a finite number of point actuators. The number of point actuators needed is related to the number of unstable modes of the equation. Furthermore, the proposed control methodology is shown to be robust with respect to changing the parameters in the equation, e.g. the viscosity coefficient or the intensity of the electric field. We also study the problem of controlling solutions of coupled systems of KS equations. Possible applications to controlling thin film flows are discussed. Our rigorous results are supported by extensive numerical simulations.

Keywords: Kuramoto–Sivashinsky equation; feedback control; optimal control.

1. Introduction

The Kuramoto–Sivashinsky (KS) equation on L-periodic domains

$$\begin{aligned} u_t + u_{xxxx} + u_{xx} + uu_x = 0, \\ u(x,t) = u(x+L,t) \end{aligned} \tag{1.1}$$

is a paradigm evolution equation that has received considerable attention in recent years due to its wide applicability as well as the rich and complex dynamics that it supports. The KS equation arises in many physical problems, including falling film flows (Benney, 1966; Sivashinsky & Michelson, 1980; Shlang & Sivashinsky, 1982; Hooper & Grimshaw, 1985), two-fluid core-annular flows (Papageorgiou *et al.*, 1990; Coward *et al.*, 1995), flame front instabilities and reaction–diffusion–combustion dynamics (Sivashinsky, 1977, 1983), propagation of concentration waves in chemical physics applications (Kuramoto & Tsuzuki, 1975, 1976; Kuramoto, 1978) and trapped ion mode dynamics in plasma physics (Cohen *et al.*, 1976). The KS equation (1.1) is one of the simplest partial differential equations (PDEs) that can produce complex dynamics including chaos; see, for example, the numerical experiments in Hyman & Nicolaenko (1986), Hyman *et al.* (1986), Jolly *et al.* (1990), Kevrekidis *et al.* (1990), Papageorgiou & Smyrlis (1991), Smyrlis & Papageorgiou (1991), Wittenberg (2002) and Wittenberg & Holmes (1999). Routes to chaos have been shown numerically to follow a Feigenbaum period-doubling cascade; see Smyrlis & Papageorgiou (1991) where the two universal Feigenbaum constants are also computed for the KS with three-digit accuracy. A detailed knowledge of the stationary, travelling and time-oscillatory solutions (typically chaotic) of (1.1) is significant in technological applications that

seek to enhance heat or mass transfer, for example. In this sense, certain solutions are better than others and a description of the solution phase space is a crucial step in constructing relevant control strategies that can access unstable states, for instance, that may be desirable in applications.

In many studies, equation (1.1) is scaled to 2π-periodic domains according to the rescaling

$$x^* = \frac{2\pi}{L}x, \quad t^* = \left(\frac{2\pi}{L}\right)^2 t, \quad u^* = \frac{L}{2\pi}u, \quad \delta^* = \frac{2\pi}{L}\delta, \quad \mu^* = \frac{2\pi}{L}\mu, \tag{1.2}$$

to take the form (we drop the stars and use the same symbols for dependent and independent variables)

$$\begin{aligned} u_t + \nu u_{xxxx} + u_{xx} + uu_x = 0, \\ u(x,t) = u(x+2\pi, t), \end{aligned} \tag{1.3}$$

where $\nu = (2\pi/L)^2$ is a positive parameter that decreases as the system size L increases. The mathematical interest in the KS equation (and related models, see below) resides in the fact that it is a simple, 1D equation exhibiting complex dynamics making it amenable to analysis and also a good case study in the area of infinite-dimensional dynamical systems and their control. The equation is of the active-dissipative type and instabilities are present depending on the value of ν. If $\nu > 1$, it is well known (Tadmor, 1986; Temam, 1988; Robinson, 2001; Sell & You, 2002) that the zero solution representing a flat film is unique. However, when $\nu < 1$, the zero solution is linearly unstable and bifurcates into nonlinear states, including steady states, travelling waves and solutions exhibiting spatiotemporal chaos—the dynamical complexity increasing as ν decreases. Some of these solutions are stable, and others are unstable Kevrekidis *et al.* (1990). In Frisch *et al.* (1986), Kevrekidis *et al.* (1990) and Papageorgiou *et al.* (1993), one can find studies of the stability of steady states of the KS equation.

There is an extensive literature on the behaviour of the solutions to the KS equation. Well-posedness of solutions is studied, for instance, in Robinson (2001), Tadmor (1986) and Temam (1988). It was proved in Constantin *et al.* (1989) that the long-time dynamics of the KS equation are finite-dimensional in the sense that they are governed by a dynamical system of finite dimension which is at least as large as the number of linearly unstable modes (this number scales with L or $\nu^{-1/2}$ for (1.1) or (1.3), respectively); these authors also proved that the solutions are attracted by a global attractor, a set of finite dimension. Boundedness of solutions for general initial conditions was proved independently and by using distinct methods by Collet *et al.* (1993b), Goodman (1994) and Il'yashenko (1992). These studies also focussed on finding bounds for the dimension of the global attractor by estimating L^2-norms of the solutions, starting with the odd-parity results of Nicolaenko *et al.* (1985) and those for general initial data by Collet *et al.* (1993b) along with more recent improvements in Bronski & Gambill (2006) and Otto (2009). Analyticity of solutions in a strip in the complex plane around the real axis was also proved in Collet *et al.* (1993a) and Akrivis *et al.* (2013) using different methods.

In the context of falling film flows, there have been several studies to extend the KS equation by including additional physical effects. Of most interest to the present study are the derivations in Tseluiko & Papageorgiou (2006, 2010) for film flow over flat walls in the presence of electric fields applied perpendicular to the undisturbed interface. The resulting equation, which also incorporates the effects of dispersion, is a generalization of (1.3) and takes the form

$$\begin{aligned} &u_t + \nu u_{xxxx} + \mu \mathcal{H}[u_{xxx}] + \delta u_{xxx} + u_{xx} + uu_x = 0, \\ &u(x,t) = u(x+2\pi, t), \quad u(x,0) = u_0(x), \end{aligned} \tag{1.4}$$

where $\mu \geqslant 0$ measures the strength of the applied electric field and the parameter δ measures dispersive effects. The linear operator $\mathcal{H}$ is the Hilbert transform operator and represents flow destabilization due to the electric field. On 2π-periodic domains the definition of $\mathcal{H}$ is

$$\mathcal{H}[u](x) = \frac{1}{2\pi}\mathrm{PV}\int_0^{2\pi} u(\xi)\cot\left(\frac{x-\xi}{2}\right)\mathrm{d}\xi, \tag{1.5}$$

where PV stands for the Cauchy principal value integral. In the model analysed, the electric field needs to be found by solving a harmonic problem above the film and calculating the Dirichlet to Neumann map of the solution to construct the Maxwell stresses that interact with the hydrodynamics; see Tseluiko & Papageorgiou (2006, 2010) for the details. In fact, for the linearized problem the eigenvalues λ corresponding to the eigenfunctions $\exp(\mathrm{i}kx)$ are

$$\lambda = k^2 + \mu k^2|k| - \nu k^4 + \mathrm{i}\delta k^3, \tag{1.6}$$

showing that the presence of the electric field destabilizes the flow and increases the number of linearly unstable modes. Note that instability is possible if $|k| < k_c = (\mu + \sqrt{\mu^2 + 4\nu})/2\nu$, and so there are $2l + 1$ unstable modes, where l is the integer part of k_c (this fact is used in Proposition 1 later). This additional destabilization is important in what follows and makes the control problem more challenging. The modified equation (1.4), in the absence of dispersion ($\delta = 0$), has a similar dynamical behaviour to the KS equation (1.3) but with chaotic dynamics appearing at higher values of ν as μ increases - see Fig. 1. In fact, boundedness of solutions and an estimate of the dimension of the global attractor have been proved in Tseluiko & Papageorgiou (2007) for a class of more general operators whose symbols in Fourier space are such that the electric field term in (1.6) is $|k|^\alpha$ with $3 \leqslant \alpha < 4$. On the other hand, in the absence of an electric field but with dispersion present, it is established that dispersion acts to regularize the dynamics (even chaotic ones) into non-linear travelling wave pulses; see Kawahara (1983), Kawahara & Toh (1985) and Akrivis *et al.* (2012) as well as Pradas *et al.* (2012) and Tseluiko & Kalliadasis (2014) for a weak interaction theory between pulses that are sufficiently separated.

More recently, Antoniades & Christofides (2001), Armaou & Christofides (2000b), Christofides (1998, 2000); Christofides & Armaou (2000), Dubljevic (2010) and Lou & Christofides (2003) showed how to stabilize the zero solution of the KS equation by using state feedback controls. They also proved that using linear feedback controls, it is possible to stabilize the zero solution using only five point actuated controls. In addition, they prove that the stabilization is possible using only a certain number of observations of the solution instead of full knowledge of the solution at all times as long as the number of observations is equal to or exceeds the number of unstable modes. In further work utilizing non-linear feedback controls (Christofides, 2000; Antoniades & Christofides, 2001), Christofides and co-workers formulated optimization techniques and computed possible optimal states by analysing a large number of runs; a proof of the existence of these optimal positions was not given, however.

In this work, we use linear feedback controls and techniques similar to those in Armaou & Christofides (2000b) and Christofides (2000) to stabilize non-uniform unstable steady states of generalized versions of the KS equation at small values of ν that have not been attempted yet. The mathematical complication is due to the increase of the number of linearly unstable modes as ν decreases and μ increases, see (1.6). We achieve stabilization of non-uniform states by stabilizing the zero solution of a modified PDE that is satisfied by a perturbation to the desired steady state. The resulting equation to

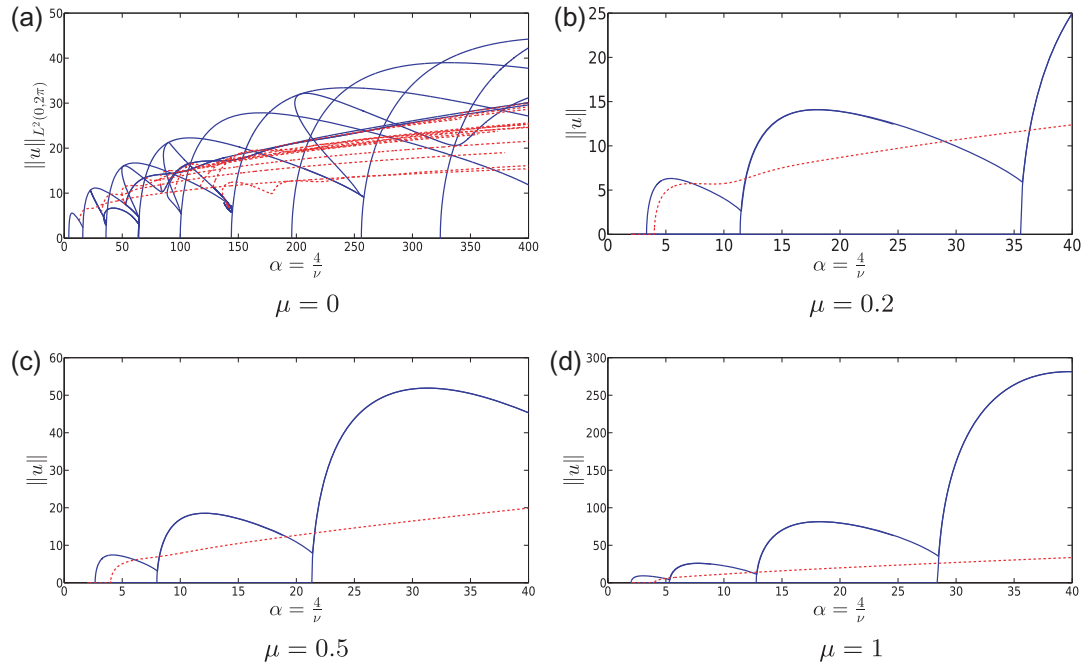


FIG. 1. Bifurcation diagram of the L^2-norm of the steady-state solutions (solid curves—blue online) and travelling wave solutions (dashed curves—red dashed online) to the gKS equation (1.4) (with $\delta = 0$) in the presence of an electric field for $0.1 \leqslant \nu \leqslant 1$, and $\mu = 0$ (a), $\mu = 0.2$ (b), $\mu = 0.5$ (c) and $\mu = 1$ (d). Note that, for $\mu \neq 0$, only a few of the branches are shown in these diagrams.

be controlled is

$$u_t + \nu u_{xxxx} + \mu \mathcal{H}[u_{xxx}] + \delta u_{xxx} + u_{xx} + u u_x = \sum_{i=1}^{m} b_i(x) f_i(t).$$

All our results are still valid in this as well as other cases as long as the linear operator of the PDE has a self-adjoint part, a well-defined separation between stable and unstable modes, and a bounded non-linearity $\mathcal{N}(u)$ in an appropriate functional space. Such conditions are fairly generic in physically derived systems and pose little restriction to our methodology.

In applications, there may be some uncertainties in the estimation of the parameters of the equation, for example, if the intensity of the electric field or the dispersion parameters are not known exactly. It is important, therefore, that the controls applied are robust, that is they still work even when these uncertainties are present. We use results from control theory (Kautsky & Nichols, 1985) to prove analytically that the controls are robust to uncertainties in ν, δ and μ, as long as the error in the prediction of the parameters is small enough, and present numerical simulations that demonstrate this point.

A natural question to address after robust stabilization of the zero solution to the modified PDE is achieved is whether this can be done in an optimal manner. By this, we mean stabilization while minimizing a cost functional that measures how close we are to the desired solution and how much energy we are spending with the controls. This cost functional is of the form

$$\mathcal{C}(u, F) = \frac{1}{2} \int_0^T \left\| u(\cdot, t) - \bar{u} \right\|^2 \mathrm{d}t + \frac{1}{2} \left\| u(\cdot, T) - \bar{u} \right\|^2 + \frac{\gamma}{2} \int_0^T \sum_{i=1}^{m} f_i(t)^2 \, \mathrm{d}t, \tag{1.7}$$

where T is the final time of integration and $\bar{u}$ is the desired steady state we are controlling. As our control variables we will consider the positions of the control actuators, following Lou & Christofides (2003). Note that the presence of a non-linearity in the PDE makes the problem non-convex and therefore we do not expect an optimal control to be unique. However, we only wish to prove the existence of an optimal control and to find computationally such optimal controls.

The methodology developed and implemented here can also be applied to systems of non-linear coupled PDEs. Of particular interest are systems of coupled KS equations that arise in applications to interfacial fluid dynamics problems. Such equations were derived systematically using asymptotic methods in Papaefthymiou *et al.* (2013) to model the non-linear stability of three immiscible viscous fluids of different properties flowing in a stratified arrangement in a plane channel under the action of gravity and/or a driving pressure gradient. The ensuing dynamics is very rich and in fact instabilities can emerge even in the absence of inertia, unlike analogous two-fluid flows. Coupled non-linear systems are mathematically significantly more challenging than scalar PDEs since analytical results on global existence and estimates of solution norms, for example, are poorly understood. Detailed computational results into the complexity of the solutions (especially their zero diffusion limits) of such coupled systems of KS equations can be found in Papaefthymiou & Papageorgiou (2015). In the present study, we consider the problem of feedback and of optimal control when the equations are coupled through the second derivatives alone. This is a special case but arises in the derivation of the equations, see Papaefthymiou *et al.* (2013); more generally, the non-linear terms are also coupled and can cause hyperbolic–elliptic transitions by supporting complex eigenvalues of the non-linear flux functions, see Papaefthymiou & Papageorgiou (2015). We find that solutions to such a system of coupled PDEs can also be controlled through a linear feedback loop and also optimally.

The rest of the paper is organized as follows. In Section 2, we prove rigorously that non-trivial solutions to the generalized KS (gKS) equation can be stabilized and analyse the robustness of the proposed controls to variations in the parameters of the equation and in Section 3 we present numerical experiments that confirm our results. In Section 4, we prove that there exist optimal distributed controls for the KS equation, construct an algorithm to optimize the placement of the control actuators and show our numerical results. Finally, in Section 5 we extend our results to a system of coupled KS equations. We discuss our results in Section 6.

2. Stabilization of non-trivial unstable steady states using linear feedback controls

Our goal is to stabilize non-uniform unstable steady states or steady-state travelling wave solutions of equation (1.4). For the theoretical analysis of the feedback control problem for the KS equation, we need L^∞ bounds on the solution and its derivatives. To establish such estimates, we will use well-known L^2 bounds, together with the Sobolev embedding theorem. Optimal estimates for the solution of the KS equation (1.1) in $(0, L)$ were obtained by Otto (2009), and for the re-scaled 2π-periodic KS equation (1.3) these estimates can be expressed in terms of $\nu = (2\pi/L)^2$ to find

$$\limsup_{t\to\infty} \|u(\cdot,t)\| \leqslant \mathcal{O}\left(\nu^{-1/6}\right), \tag{2.1a}$$

$$\limsup_{t\to\infty} \|u_x(\cdot,t)\| \leqslant \mathcal{O}\left(\nu^{1/2}\ln^{5/3}\left(\nu^{-1/2}\right)\right), \tag{2.1b}$$

$$\limsup_{t\to\infty} \|u_{xx}(\cdot,t)\| \leqslant \mathcal{O}\left(\nu\ln^{5/3}\left(\nu^{-1/2}\right)\right), \tag{2.1c}$$

where $\|\cdot\| = \left(\int_0^{2\pi} (\cdot)^2 \,\mathrm{d}x\right)^{1/2}$ denotes the L^2-norm of the solution. For the generalized equation (1.4) but in the absence of dispersion ($\delta = 0$), Tseluiko & Papageorgiou (2007) use the background flow method to obtain similar estimates in the presence of an electric field. Their estimates are of the form

$$\|u\| \leqslant (\|u_0\| + \|\varphi\|)\ \mathrm{e}^{-Dt} + C(\nu, \mu) + \|\varphi\|, \tag{2.2}$$

where C and D are constants depending on μ and ν, and φ is a constructed function with finite L^2-norm (we do not need to give it here). They also proved that, for $u_0 \in \dot{H}_p^1(0, 2\pi)$, the first and second derivatives of the solution are bounded, and therefore $u \in \dot{H}_p^2(0, 2\pi)$, where $\dot{H}_p^s$ is the Sobolev space of s-times differentiable functions that are periodic and have zero mean.

It is also important to remark that in the case of the gKS equation (1.4), with both δ and μ non-zero, it was proved in Frankel & Roytburd (2008a,b) that the L^2-norm of the solution is also bounded. In fact,

$$\limsup_{t\to\infty} \|u(\cdot, t)\| \leqslant \begin{cases} \mathcal{O}\left(\nu^{-17/4}\right) & \text{if } \nu < \nu_0, \\ C & \text{if } \nu \geqslant \nu_0, \end{cases} \tag{2.3}$$

where ν_0 depends on the symbol of the linear operator. Note that these are not optimal bounds. The authors also prove boundedness in L^2 of spatial derivatives of u up to order 4. The estimates (2.1–2.3) imply (by use of the Sobolev embedding theorem) that there exist constants $C_1,\ C_2$ that depend only on ν and μ such that

$$\|u\|_\infty \leqslant C_1 \|u\|_{H^2}, \quad \|u_x\|_\infty \leqslant C_2 \|u_x\|_{H^1}. \tag{2.4}$$

The controlled gKS equation can now be introduced and will form the basis of our analysis and computations. This consists of a forced version of (1.4) and reads

$$\begin{cases} u_t + \nu u_{xxxx} + \mu \mathcal{H}[u_{xxx}] + \delta u_{xxx} + u_{xx} + u u_x = \displaystyle\sum_{i=1}^{m} b_i(x) f_i(t), \quad x \in (0, 2\pi),\ t > 0, \\ u(x, 0) = u_0(x), \quad x \in (0, 2\pi), \\ \dfrac{\partial^j u}{\partial x^j}(x + 2\pi, t) = \dfrac{\partial^j u}{\partial x^j}(x, t), \quad x \in (0, 2\pi),\ t > 0, \\ f_i(t) \in L^2(0, T). \end{cases} \tag{2.5}$$

We assume that the initial condition satisfies $u_0 \in \dot{H}_p^2(0, 2\pi)$, m denotes the number of controls, $b_i(x)$, $i = 1, \ldots, m$ are the control actuator functions and $f_i(t)$, $i = 1, \ldots, m$ are the controls. We will use point actuator functions, which means that the functions $b_i(x)$ are delta functions centred at positions x_i, i.e. $b_i(x) = \delta(x - x_i)$, or a smooth approximation of such delta functions.

We use an argument similar to Armaou & Christofides (2000a), Christofides (1998) and Christofides & Armaou (2000) to prove that it is possible to stabilize non-trivial steady states of the gKS equation (1.4). Using the Galerkin representation of u,

$$u(x, t) = \frac{u_0(t)}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} u_n^s(t) \frac{\sin(nx)}{\sqrt{\pi}} + \sum_{n=1}^{\infty} u_n^c(t) \frac{\cos(nx)}{\sqrt{\pi}}, \tag{2.6}$$

substituting into (2.5), and taking the inner product with the functions $1/\sqrt{2\pi}$, $\sin(nx)/\sqrt{\pi}$ and $\cos(nx)/\sqrt{\pi}$, $n = 1, \ldots, \infty$, we obtain the following infinite system of ordinary differential equations

(ODEs) (dots denote time derivatives):

$$\begin{cases} \dot{u}_n^s = \left(-\nu n^4 + \mu n^3 + n^2\right) u_n^s + \delta n^3 u_n^c + g_n^s + \displaystyle\sum_{i=1}^{m} b_{\text{in}}^s f_i(t) \quad n = 1, \ldots, \infty, \\ \dot{u}_n^c = \left(-\nu n^4 + \mu n^3 + n^2\right) u_n^c - \delta n^3 u_n^s + g_n^c + \displaystyle\sum_{i=1}^{m} b_{\text{in}}^c f_i(t) \quad n = 0, \ldots, \infty, \end{cases} \tag{2.7}$$

where $b_{\text{in}}^s = \int_0^{2\pi} b_i(x) \sin(nx)\,\mathrm{d}x$ and $b_{\text{in}}^c = \int_0^{2\pi} b_i(x) \cos(nx)\,\mathrm{d}x$. The non-linearities g_n^s and g_n^c are given by Akrivis *et al.* (2011)

$$g_n^s = \frac{n}{4\sqrt{\pi}} \sum_{j+k=n} \left(u_j^c u_k^c - u_j^s u_k^s\right) + \frac{n}{2\sqrt{\pi}} \sum_{j-k=n} \left(u_j^c u_k^c + u_j^s u_k^s\right), \quad n = 1, \ldots, \infty,$$

$$g_n^c = -\frac{n}{2\sqrt{\pi}} \sum_{j+k=n} u_j^c u_k^s + \frac{n}{2\sqrt{\pi}} \sum_{j-k=n} \left(u_j^c u_k^s - u_j^s u_k^c\right), \quad n = 0, \ldots, \infty.$$

In deriving the system (2.7), we used $\mathcal{H}[\sin(x)](x) = -\cos(x)$ and $\mathcal{H}[\cos(x)](x) = \sin(x)$; these formulas can be derived using contour integrations in the complex plane, for example.

We now define

$$z^u = \begin{bmatrix} z_u^u \\ z_s^u \end{bmatrix},$$

where

$$z_u^u = \begin{bmatrix} u_0^c & u_1^s & u_1^c & \cdots & u_l^s & u_l^c \end{bmatrix}^\top, \quad z_s^u = \begin{bmatrix} u_{l+1}^s & u_{l+1}^c & \cdots \end{bmatrix}^\top,$$

where z_u^u contains the coefficients of the (slow) unstable modes and z_s^u those of the (fast) stable modes. In addition,

$$G = \begin{bmatrix} 0 & g_1^s & g_1^c & g_2^s & g_2^c & \cdots \end{bmatrix}^\top, \quad D = \operatorname{diag}\left(0, \delta, -\delta, \ldots, \delta n^3, -\delta n^3, \ldots\right),$$
$$F = \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_m(t) \end{bmatrix}^\top.$$

Furthermore, we introduce the notation

$$A = \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_u \\ B_s \end{bmatrix}, \tag{2.8}$$

where

$$A_u = \operatorname{diag}\left(0, -\nu + \mu + 1, -\nu + \mu + 1, \ldots, -l^4\nu + \mu l^3 + l^2, -l^4\nu + \mu l^3 + l^2\right),$$
$$A_s = \operatorname{diag}\left(-(l+1)^4\nu + \mu(l+1)^3 + (l+1)^2, -(l+1)^4\nu + \mu(l+1)^3 + (l+1)^2, \ldots\right)$$

and

$$B_u = \begin{bmatrix} b^c_{10} & b^c_{20} & \cdots & b^c_{m0} \\ b^s_{11} & b^s_{21} & \cdots & b^s_{m1} \\ b^c_{11} & b^c_{21} & \cdots & b^c_{m1} \\ \vdots & \vdots & \cdots & \vdots \\ b^s_{1l} & b^c_{2l} & \cdots & b^c_{ml} \\ b^s_{1l} & b^s_{2l} & \cdots & b^s_{ml} \end{bmatrix}, \quad B_s = \begin{bmatrix} b^s_{1(l+1)} & b^s_{2(l+1)} & \cdots & b^s_{m(l+1)} \\ b^c_{1(l+1)} & b^c_{2(l+1)} & \cdots & b^c_{m(l+1)} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}. \tag{2.9}$$

We can rewrite the infinite-dimensional system of ODEs (2.7) as

$$\dot{z}^u = Az^u + Dz^u + G + BF. \tag{2.10}$$

We have the following result.

PROPOSITION 1 Let $\bar{u}$ be a linearly unstable steady state or travelling wave solution of (1.4) and let $2l+1$ be the number of unstable eigenvalues of the system

$$u_t = -\nu u_{xxxx} - \mu \mathcal{H}[u_{xxx}] - u_{xx}, \tag{2.11}$$

i.e. $l+1 > (\mu + \sqrt{\mu^2 + 4\nu})/2\nu > l$. If $m = 2l+1$ and there exists a matrix $K \in \mathbb{R}^{m\times m}$ such that all of the eigenvalues of the matrix $A_u + B_u K$ have negative real part, then the state feedback controls

$$[f_1 \quad \cdots \quad f_m]^\top = F = K\left(z^u_u - z^{\bar{u}}_u\right) \tag{2.12}$$

stabilize $\bar{u}$.

Proof. Let $u = \bar{u} + v$ be a solution to (1.4). Substituting into (1.4) and using the fact that $\bar{u}$ is a steady state or travelling wave solution of (2.5), we obtain the following PDE for the perturbation v:

$$v_t + \nu v_{xxxx} + \mu \mathcal{H}[v_{xxx}] + \delta v_{xxx} + v_{xx} + v v_x + \left(\bar{u}v\right)_x = 0, \tag{2.13}$$

and in controlled form we have

$$v_t + \nu v_{xxxx} + \mu \mathcal{H}[v_{xxx}] + \delta v_{xxx} + v_{xx} + v v_x + \left(\bar{u}v\right)_x = \sum_{i=1}^{m} b_i(x) f_i(t). \tag{2.14}$$

First, we will prove that the given controls stabilize the zero solution of

$$v_t = -\nu v_{xxxx} + \mu \mathcal{H}[v_{xxx}] + v_{xx}. \tag{2.15}$$

Note that the dispersion term does not affect instability and it is not necessary to include it in this part of the analysis. After applying a Galerkin truncation and the controls given by (2.12), we obtain

$$\dot{z}^v = \begin{bmatrix} A_u + B_u K & 0 \\ B_s K & A_s \end{bmatrix} z^v = C z^v. \tag{2.16}$$

Since the eigenvalues of $A_u + B_u K$ have negative real part and the matrix multiplying z^v is triangular, it follows that the zero solution to (2.16) is exponentially stable.

Next, following Armaou & Christofides (2000a), Christofides (1998) and Christofides & Armaou (2000), we use a Lyapunov argument to show that these controls stabilize the zero solution to equation (2.13). We first use the fact that exponential stability of the system (2.16) implies that there exists a positive constant a such that the operator $\mathcal{A}(v) = -\nu v_{xxxx} - \mu\mathcal{H}[v_{xxx}] - v_{xx} + \sum_{i=1}^{m} b_i(x)K_{i\cdot}z_u^v$, where $K_{i\cdot}$ denotes the ith row of the matrix K, satisfies

$$(\mathcal{A}v, v) \leqslant -a\|v\|^2. \tag{2.17}$$

Defining $V(v) = \int_0^{2\pi}(v^2/2)\,\mathrm{d}x$, it is easy to verify that $V(0) = 0$ and $V(v) > 0$, $\forall v > 0$. Multiplying (2.13) by v and integrating gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\frac{v^2}{2}\,\mathrm{d}x = \int_0^{2\pi} vv_t\,\mathrm{d}x = (\mathcal{A}v, v) - \delta\int_0^{2\pi} v_{xxx}v\,\mathrm{d}x - \int_0^{2\pi} v^2v_x\,\mathrm{d}x - \int_0^{2\pi} v\left(\bar{u}v\right)_x\mathrm{d}x. \tag{2.18}$$

Integration by parts and use of periodicity show that the first two integrals on the right-hand side of (2.18) are zero. It remains to obtain an estimate for the third integral. Again, using integration by parts and periodicity gives

$$\begin{aligned} -\int_0^{2\pi}\left(\bar{u}v\right)_x v\,\mathrm{d}x &= -\int_0^{2\pi}\bar{u}v_xv\,\mathrm{d}x - \int_0^{2\pi}\bar{u}_xv^2\,\mathrm{d}x = -\frac{1}{2}\int_0^{2\pi}\bar{u}_xv^2\,\mathrm{d}x \\ &\leqslant -\frac{\inf\bar{u}_x}{2}\int_0^{2\pi}v^2\,\mathrm{d}x = -\frac{\inf\bar{u}_x}{2}\|v\|^2. \end{aligned}$$

Adding everything up, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 \leqslant -\left(a + \frac{\inf\bar{u}_x}{2}\right)\|v\|^2. \tag{2.19}$$

If the eigenvalues of the matrix $A_u + B_uK$ are chosen such that $2a + \inf\bar{u}_x \geqslant 0$, we obtain that $(\mathrm{d}/\mathrm{d}t)V(v(t)) \leqslant 0$, which proves that V is a Lyapunov function for the system and therefore the zero solution is stable.

Using controls (2.12), we can therefore stabilize the non-trivial steady state $\bar{u}$ of the original equation. □

Using Proposition 1, we can conclude that in order to stabilize the steady state $\bar{u}$ of equation (1.4), we should solve the PDE

$$u_t + \nu u_{xxxx} + \mu\mathcal{H}[u_{xxx}] + \delta u_{xxx} + u_{xx} + uu_x = \sum_{i=1}^{m} b_i(x)K_{i\cdot}\left(z_u^u - z_u^{\bar{u}}\right). \tag{2.20}$$

Remark 1 Since the solutions to the gKS equation are taken to be periodic with mean zero, it follows that $\inf\bar{u}_x < 0$. Therefore, the constant a in (2.19) must be chosen large enough so that $a + \inf\bar{u}_x/2$ is positive. In the case when $\delta > 0$, we also need to account for the fact that the amplitude of the solutions (and therefore the absolute value of their derivatives) grows with δ (Kawahara & Toh, 1998; Akrivis *et al.*, 2012). Further details can be found in Section 3.

REMARK 2 The above proposition is clearly valid for the case when $\bar{u} = 0$, in which case the controls are $f_i(t) = K_{i\cdot} z_u^u$, as presented in Armaou & Christofides (2000a), Christofides (1998) and Christofides & Armaou (2000). In the case when $\bar{u}$ is a travelling wave, the result follows using a time-dependent $z^{\bar{u}}$. See Section 3 and in particular equation (3.6).

REMARK 3 From estimates (2.1), it follows that the value $\inf |\bar{u}_x|$ is finite and therefore we can conclude (2.19).

REMARK 4 Armaou & Christofides (2000a), Christofides (1998) and Christofides & Armaou (2000) argued that due to the multiplicity of the eigenvalues of the KS equation being $\leqslant 4$, one would only need five controls to stabilize the zero solution of that equation. The same holds in our case: the multiplicity of the eigenvalues of the linear operator in (2.15) is also $\leqslant 4$, but numerical results suggest that we need to use $m = 2l + 1$ controls, or at best $m = 2l - 1$ controls, see Fig. 2 and the discussion below.

REMARK 5 The fact that we are separating the system between stable and unstable modes implies that the matrix B_u is square ($B_u \in \mathbb{R}^{m\times m}$), and using $b_i(x) = \delta(x - x_i)$ means that B_u has full rank. It follows that the Kalman rank condition (Zabczyk, 1992) is automatically verified and the matrix K needed for the stabilization will always exist.

2.1 *Robustness of controls*

A natural and important question is whether the proposed control methodology is robust with respect to changes (or uncertainty) in the parameters ν, μ and δ that appear in the equation. The robustness of our method can be proved rigorously using techniques from control theory, e.g. Kautsky & Nichols (1985, Theorem 6), and we take this up next.

PROPOSITION 2 Let λ_i, $i = 1, \ldots, N$ be the eigenvalues of the matrix C appearing in (2.16), and X be the matrix of eigenvectors of C and let $\kappa(\cdot)$ denote its condition number. Then, we have

$$\|K\|_2 \leqslant \frac{\left(\|A\|_2 + \max_j \left(|\lambda_j|\right) \kappa(X)\right)}{\sigma_m(B)}, \tag{2.21}$$

where $\sigma_m(B)$ is the mth smallest singular value of B, which is defined in equations (2.8) and (2.9), and the solution z^v to equation (2.16) satisfies

$$\|z^v(t)\| \leqslant \kappa(X) \max_j \left(\left|e^{\lambda_j t}\right|\right) \|z^v(0)\|. \tag{2.22}$$

PROPOSITION 3 If the feedback matrix K is such that equation (2.16) is exponentially stable, then the perturbed closed-loop system matrix $A + BK + \Delta$ remains stable for all disturbances Δ which satisfy

$$\|\Delta\|_2 < \min_{s=i\omega} \sigma_N \left(sI - (A + BK)\right) =: \zeta(K), \tag{2.23}$$

where

$$\zeta(K) \geqslant \min_j \mathrm{Re}\left(\frac{-\lambda_j}{\kappa(X)}\right),$$

and $\mathrm{Re}(\cdot)$ denotes the real part.

In particular, if there is an error in the estimation of the parameters ν and μ, then the feedback matrix K will still stabilize the zero solution as long as the error in the parameter estimation is bounded

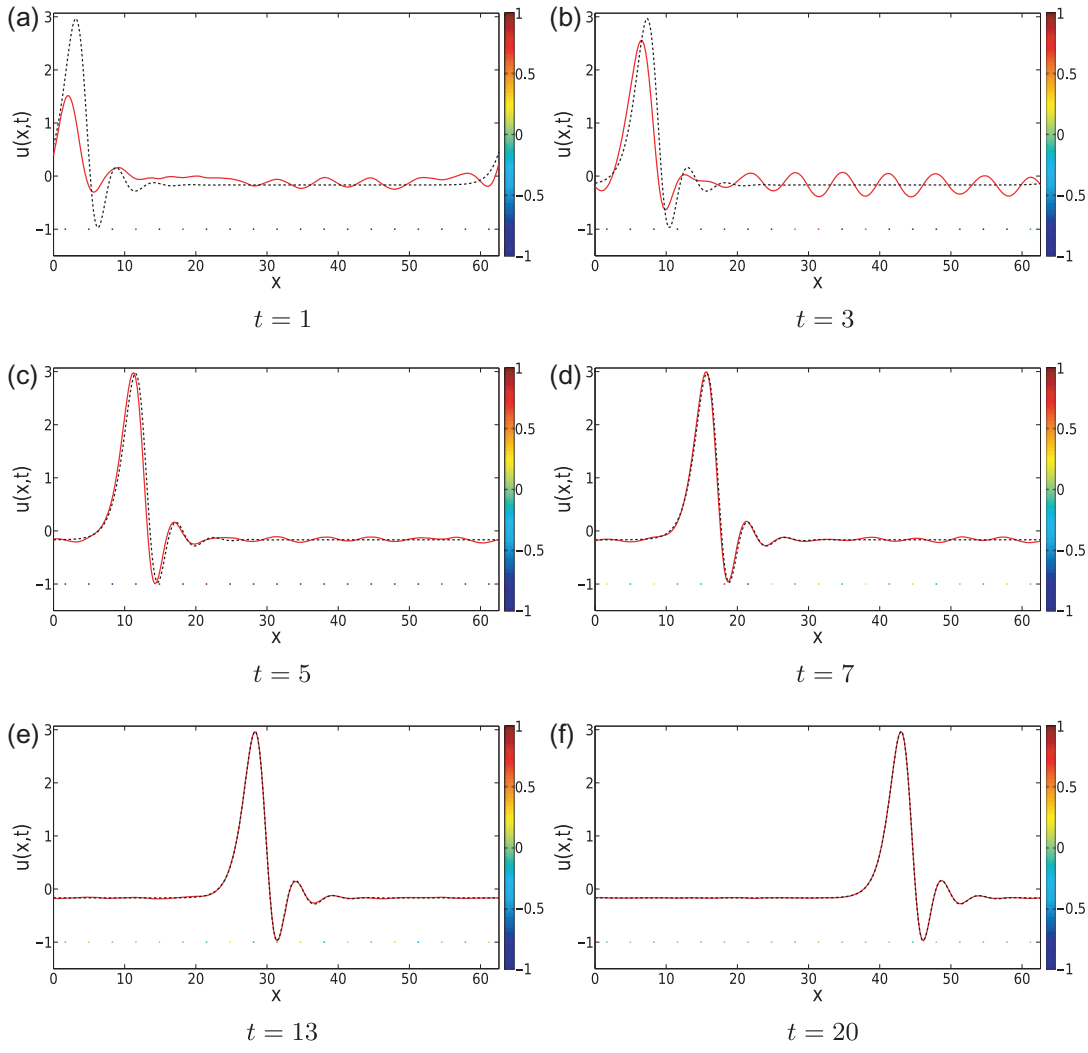


FIG. 2. Snapshots of the time evolution of the stabilized travelling wave solution in Fig. 5(b) using $m = 19$ controls instead of $m = 21$ at different times. The full line is the controlled solution, dashed lines are the desired travelling wave and the dots represent the controls and their intensity. (a) $t = 1$, (b) $t = 3$, (c) $t = 5$, (d) $t = 7$, (e) $t = 13$, (f) $t = 20$.

by $\zeta(K)$. We have studied the robustness of the controls for stabilizing steady states and travelling waves by combining Propositions 2 and 3. We now present a summary of our results.

Variations in δ. As seen from the dispersion relation (1.6), variations in δ do not affect the stability of the solutions and consequently they do not affect the matrix K. This implies that the matrix Δ is zero, and the zero solution to system (2.16) is still stable. In the case when we are interested in stabilizing travelling waves, we need to take into account the fact that the amplitude of the travelling waves increases with δ; see, for example, Kawahara & Toh (1998, Fig. 1). Hence, if we overestimate the value of $\inf \bar{u}_x$ and take this into account when choosing the new eigenvalues, the stabilized solution should remain close to the desired travelling wave as demonstrated by our numerical experiments in Fig. 3.

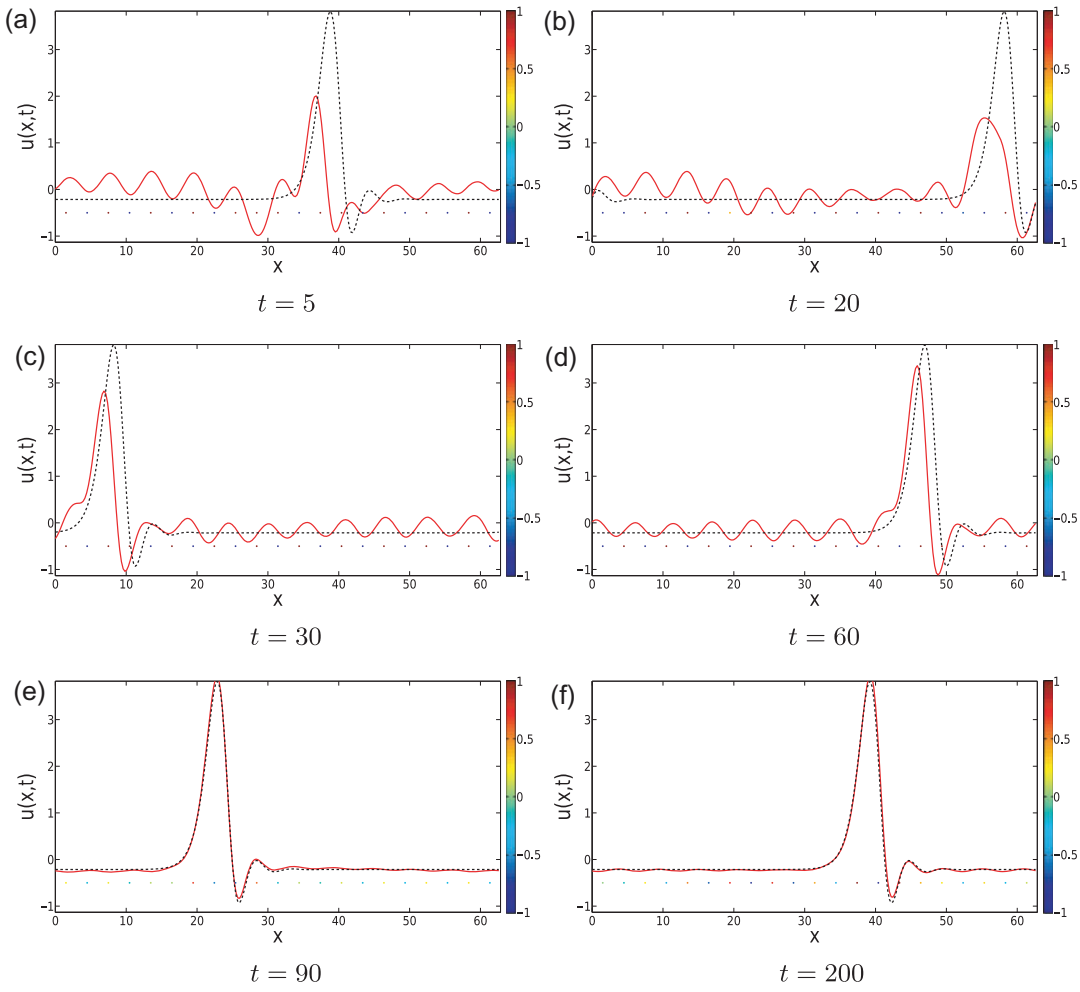


FIG. 3. Snapshots of the time evolution of a stabilized travelling wave solution for $\mu = 0$, $\nu = 0.01$ and assuming uncertainty in the parameter δ. The dashed lines is the desired travelling wave (which is the correct solution for $\delta = 0.03$) and the full lines is the controlled solution assuming $\delta = 0.04$. (a) $t = 5$, (b) $t = 20$, (c) $t = 30$, (d) $t = 60$, (e) $t = 90$, (f) $t = 200$.

Variations in ν and μ. As seen from (1.6) variations in ν and μ can affect the stability of the solutions and the number of unstable modes. An increase in unstable modes in turn affects the number of controls needed since our theoretical results support that we need the same number of controls as unstable modes as stated in Remark 4. However, we have performed numerical experiments (see Fig. 2) that show that using two less controls than predicted theoretically does not affect the stability of the solutions.

Now we consider the case where we have some uncertainty of amplitude ϵ_1 and ϵ_2 in the values of ν and μ, respectively,

$$u_t = -(\nu + \epsilon_1)u_{xxxx} - (\mu + \epsilon_2)\mathcal{H}[u_{xxx}] - u_{xx} - uu_x + \sum_{i=1}^{m} b_i(x)K_i.\left(z^u - z^{\bar{u}}\right). \tag{2.24}$$

The controls have been chosen so that the solution to the equation is stabilized when $\epsilon_1 = \epsilon_2 = 0$. Multiplying (2.24) by u, integrating by parts and using Young's inequality, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|^2 \leqslant -\kappa\|u\|^2 - \epsilon_1\|u_{xx}\|^2 + \frac{\epsilon_2}{2}\left(\|u_x\|^2 + \|u_{xx}\|^2\right),$$

where $\kappa = a + \inf \bar{u}_x/2$ is a constant. On the other hand, the perturbation $-\epsilon_1 u_{xxxx} - \epsilon_2 \mathcal{H}[u_{xxx}]$ can be discretized and written as

$$\Delta = \mathrm{diag}(0, -\epsilon_1 k^4 + \epsilon_2 k^3, -\epsilon_1 k^4 + \epsilon_2 k^3), \tag{2.25}$$

$k = 1, \ldots, N/2$, and it follows that its Fröbenius norm is given by

$$\|\Delta\|_2^2 = 2\sum_{k=1}^{N/2} k^6 \left(-\epsilon_1 k + \epsilon_2\right)^2 = 2\sum_{k=1}^{N/2} k^6 \left(\epsilon_1^2 k^2 - 2\epsilon_1\epsilon_2 k + \epsilon_2^2\right). \tag{2.26}$$

For stability, we need (2.26) to satisfy estimate (2.23), see Proposition 3. Therefore, we have the following proposition.

PROPOSITION 4 Let K be a matrix such that $A_u + B_u K$ has the prescribed (negative real part) eigenvalues, $\lambda_1, \ldots, \lambda_m$, with $m = 2l + 1$, and let

$$BK = \begin{bmatrix} B_u K & 0 \\ B_s K & 0 \end{bmatrix}.$$

Then the perturbed system $A + BK + \Delta$, where Δ is given by (2.25), is stable, provided that

$$\left(2\sum_{k=1}^{N/2} k^6 \left(\epsilon_1^2 k^2 - 2\epsilon_1\epsilon_2 k + \epsilon_2^2\right)\right)^{1/2} \leqslant \min_{s=i\omega} \sigma_N \left(sI - (A + BK)\right).$$

We have performed numerical experiments to test the robustness of the controls, and in particular we focussed on robustness with respect to the parameters δ and ν. Numerical results are presented in Figs 2–4 (results in these figures as well as in Fig. 5 are shown in the original unscaled domain of length L, see (1.2) for the transformations). In Fig. 2, we use the same parameter values as in Fig. 5(b), but we use 19 controls instead of 21, i.e. two controls less than the number of unstable eigenvalues. The dashed curve is the desired travelling wave solution and the solid curve (red online) is the controlled solution with 19 controls. We conclude, therefore, that our control methodology is robust with respect to a slight decrease in the number of controls. Note, however, that the number of controls cannot be significantly smaller than the number of unstable eigenvalues—for example, running the same numerical experiment with 17 controls did not yield satisfactory results in the sense that wavy perturbations observed in panels (b) and (c) were not suppressed.

A robustness test with respect to changes in ν (with $\delta = \mu = 0$) is depicted in Fig. 4. We begin with an unstable travelling wave at $\nu = 0.013$ and wish to control it but by solving the KS equation with a reduced value of $\nu = 0.01$, i.e. we impose an uncertainty in the value of the parameter ν or equivalently in the shape of the desired solution. The results again show robust behaviour with the two solutions being almost indistinguishable. Finally, in Fig. 3 we present robustness experiments for $\mu = 0$, $\nu = 0.01$ and changes in the dispersion parameter δ from 0.03 to 0.04, with equally accurate performance as before.

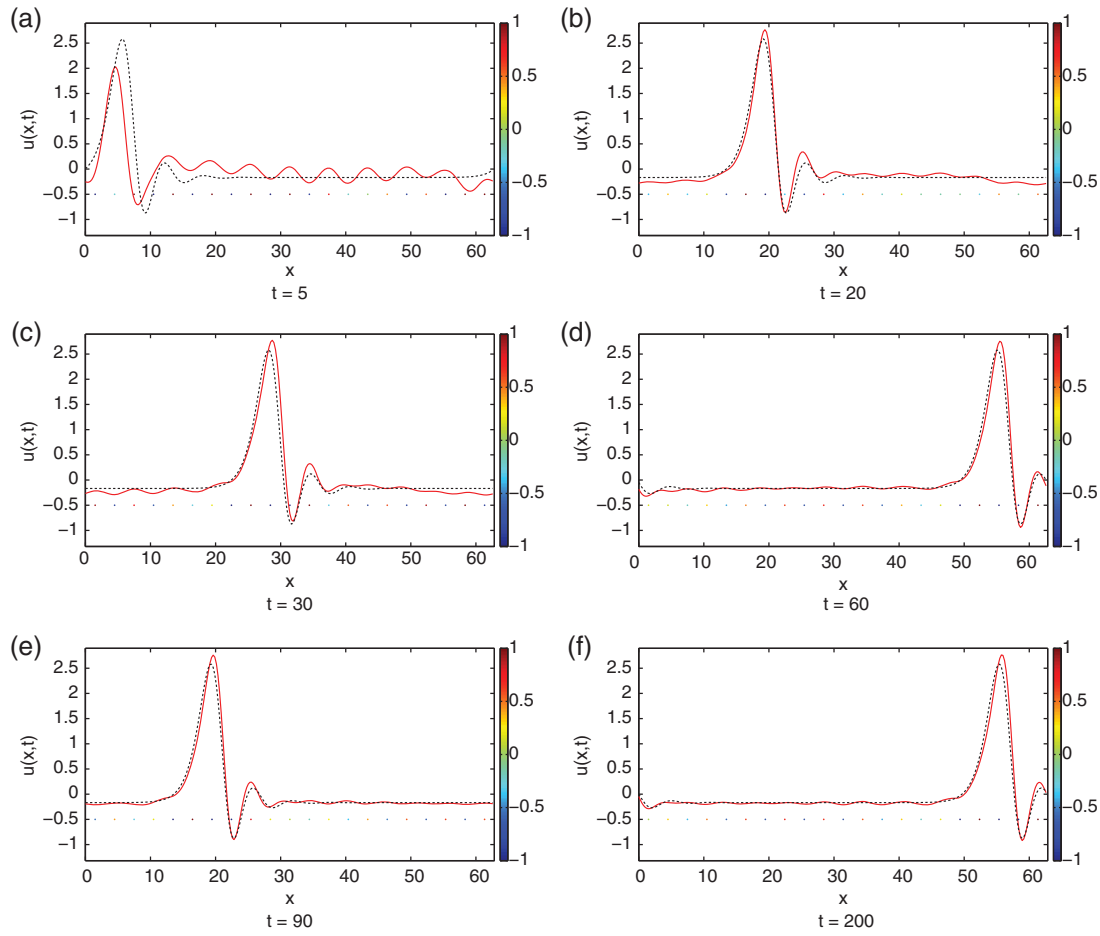


FIG. 4. Snapshots of the time evolution of a stabilized travelling wave solution for $\delta = \mu = 0$ and assuming uncertainty in the parameter ν. The dashed lines is the desired travelling wave (which is the correct solution for $\nu = 0.013 \Leftrightarrow L \approx 55$) and the full lines is the controlled solution assuming $\nu = 0.01 \Leftrightarrow L \approx 62$. (a) $t = 5$, (b) $t = 20$, (c) $t = 30$, (d) $t = 60$, (e) $t = 90$, (f) $t = 200$.

3. Numerical results

Section 2 was devoted to proving rigorously that steady states and steady-state travelling wave solutions of the gKS equation can be stabilized using linear feedback controls. The number of controls is predicted to be at least as large as the number of linearly unstable modes, and robustness with respect to uncertainty in the parameters ν and δ was also proved. In this section, we implement the linear feedback controls numerically and undertake an extensive computational study of the stabilization and control in practical situations.

3.1 *Computation of non-uniform steady states and travelling waves*

One of the main objectives of the present work is the stabilization of unstable solutions of the gKS equation. To obtain steady-state solutions $\bar{u}(x)$ (in the absence of dispersion), we need to solve the

equation

$$\nu\bar{u}_{xxxx} + \mu\mathcal{H}[\bar{u}_{xxx}] + \bar{u}_{xx} + \bar{u}\bar{u}_x = 0, \tag{3.1}$$

in the interval $[0, 2\pi]$, subject to periodic boundary conditions. Travelling waves of speed c are found by looking for solutions of the form $\bar{u}(x,t) = U(x-ct) = U(\xi)$ and solving

$$-cU' + \nu U'''' + \mu\mathcal{H}\left[U'''\right] + \delta U''' + U'' + UU' = 0, \tag{3.2}$$

subject to periodic boundary conditions, where primes denote differentiation with respect to ξ. We note that equation (3.1) is a particular case of (3.2). Expressing the solutions in Fourier series

$$U(\xi) = \sum_{n=1}^{\infty} U_n^s \sin(n\xi) + U_n^c \cos(n\xi), \tag{3.3}$$

and substituting into (3.1) and (3.2), we obtain an infinite system of non-linear algebraic equations for the coefficients U_n^s, U_n^c, $n = 1, \ldots, \infty$, or for the coefficients and the velocity c, in the case of travelling waves. The resulting system of equations for steady states is

$$\left(\nu n^4 - \mu n^3 - n^2\right) U_n^c + g_n^c = 0, \quad n = 1, \ldots, \infty, \tag{3.4a}$$

$$\left(\nu n^4 - \mu n^3 - n^2\right) U_n^s + g_n^s = 0, \quad n = 1, \ldots, \infty. \tag{3.4b}$$

For travelling waves we can assume, without loss of generality due to translation invariance, that $U_1^s = 0$, to obtain

$$-\left(cn + \delta n^3\right) U_n^s + \left(\nu n^4 - \mu n^3 - n^2\right) U_n^c + g_n^c = 0, \quad n = 1, \ldots, \infty, \tag{3.5a}$$

$$\left(cn + \delta n^3\right) U_n^c + \left(\nu n^4 - \mu n^3 - n^2\right) U_n^s + g_n^s = 0, \quad n = 2, \ldots, \infty, \tag{3.5b}$$

$$(c + \delta) U_1^c + g_1^s = 0. \tag{3.5c}$$

The systems were truncated and solved using a non-linear solver (e.g. Matlab's *fsolve*) to find solutions to system (3.4) by first setting $\mu = 0$ and carrying out a numerical continuation on ν, and secondly by fixing the desired value of ν and varying μ. Additional computations were done using the continuation software AUTO-07p (Doedel & Oldman, 2009). For travelling waves, we used continuation on ν, μ and δ. Without loss of generality, we also impose $c > 0$: if $U(x-ct)$ is a solution of (3.2) with $c < 0$, then $-U(-x-(-c)t)$ is also a solution with $c > 0$.

Given the Fourier coefficients and the velocity of a travelling wave, we can write the solution of the KS equation as

$$\bar{u}(x,t) = U(x-ct) = \sum_{n=1}^{\infty} \left(U_n^s \cos(nct) + U_n^c \sin(nct)\right) \sin(nx) + \sum_{n=1}^{\infty} \left(U_n^c \cos(nct) - U_n^s \sin(nct)\right) \cos(nx). \tag{3.6}$$

Our computational results are presented in the bifurcation diagram in Fig. 1 that depicts the variation of the L^2-norm with ν of the steady states and travelling wave solutions of the gKS equation (1.4) in the absence of dispersion ($\delta = 0$). Panels (a)–(d) correspond to $\mu = 0,\ 0.2,\ 0.5,\ 1.0$; steady states are

plotted with solid curves (blue online) and travelling waves with dashed curves (red dashed online). We observe that the presence of the Hilbert transform increases the value of ν for which instability arises (Tseluiko & Papageorgiou, 2006), but it does not change the shape of the bifurcation diagram. This is because the Hilbert transform term acts as a negative diffusion, see equation (1.6), and therefore its presence acts to shift the bifurcation diagram to higher ν, i.e. lower $\alpha = 4/\nu$ as seen in the figure. We emphasize the fact that the bifurcation diagrams in Fig. 1 are not complete and we expect additional unstable branches, in analogy with known results for the KS equation (Kevrekidis *et al.*, 1990). This is not a restriction here, since we are interested in demonstrating the stabilization of unstable steady or travelling wave solutions, rather than the stabilization of all such branches. For the branches computed here, we analysed their stability numerically by adding a small perturbation to the initial condition (about 10% or smaller of the amplitude of the steady-state solution) and studied the time evolution to ensure that we identified unstable steady solutions to be stabilized using linear feedback controls.

3.2 *Time-dependent simulations and feedback control*

We used a Galerkin truncation (Trefethen, 2000) for the spatial discretization of the PDE, with the number of modes varying between 32, 64 and 128 depending on the number of unstable modes. Time integration is carried out using second-order implicit–explicit backward differentiation formulae schemes (Akrivis *et al.*, 2011, 2012).

To construct the matrix K necessary for the stabilization of the steady states, we used *Matlab*'s command *place*. Given the matrices A and B, we sought a matrix K such that the eigenvalues of the matrix $A + BK$ were:

- -1 if it is the eigenvalue corresponding to the constant eigenfunction $1/\sqrt{2\pi}$;
- $\pm\lambda$ if λ is an eigenvalue of A with negative/positive real part;
- $-10\delta\lambda$ instead of $-\lambda$ if $\delta > 0$. We do this because the amplitude of the solutions grows with δ—Kawahara & Toh (1998), so we need to account for this when building the controls.

We begin by presenting numerical results in the absence of electric fields and dispersion ($\mu = 0$, $\delta = 0$) and for two values of $\nu = 0.2$ and $\nu = 0.4$ (note that the number of unstable eigenvalues is $2l + 1$, where $l = [\nu^{-1/2}]$, where $[\cdot]$ denotes the integer part). The number of controls used is 5 and 3, respectively, i.e. equal to $2l + 1$; these are placed equidistantly and the initial condition is

$$u_0(x) = \frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{5} (\sin(nx) + \cos(nx)) .$$

The results are presented in Fig. 6 and clearly show that the system is controlled to the zero solution long before the final computed time of $t = 5$; our results are also in good agreement with those in Christofides & Armaou (2000).

Results analogous to those presented in Fig. 6 were found regarding the stabilization of the zero solution to the KS equation in the presence of an electric field. In what follows, we use the following initial condition unless stated otherwise:

$$u_0(x) = \frac{1}{\sqrt{\pi}} (\sin(x) + \cos(x)) . \tag{3.7}$$

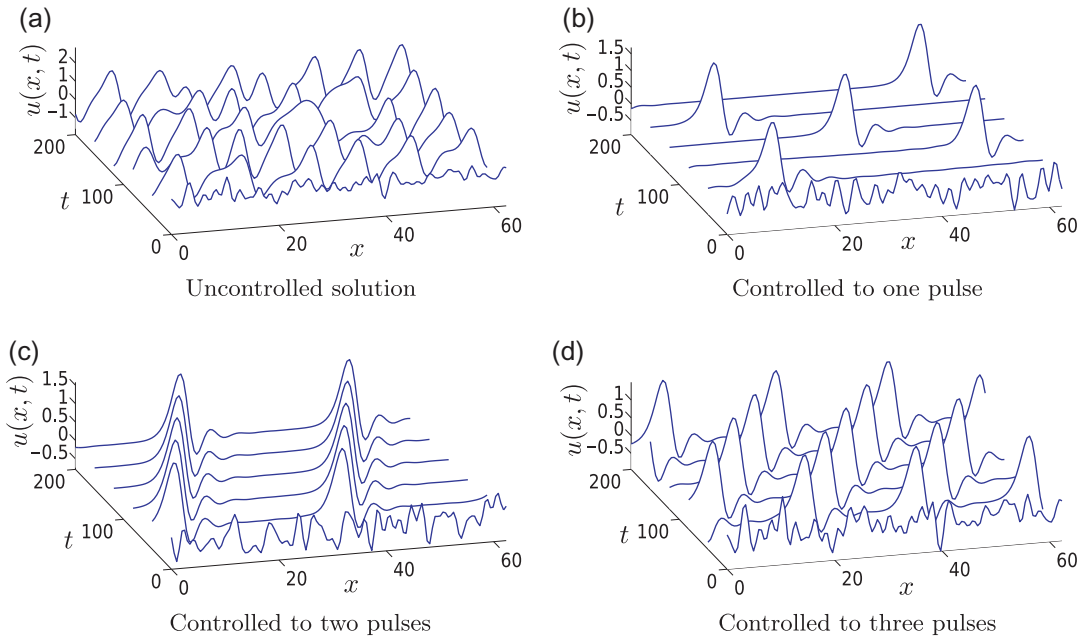


FIG. 5. Solution to the KS equation for $\nu = 0.01$ (a) with no controls, and controlled to (b) one solitary pulse, (c) two solitary pulses and (d) three solitary pulses.

Note that the number of unstable modes is $2l+1$, where $l = [(\mu + \sqrt{\mu^2 + 4\nu})/2\nu]$: see Proposition 1 and this is the number of controls used in the numerical experiments. The numerical results for $\nu = 0.2$ and $\mu = 0.5$ with five equidistant controls are shown in Fig. 7, where we again clearly observe stabilization to the zero solution.

Having shown the stabilization of zero states for relatively small values of ν, we turn next to the stabilization of non-trivial steady states of the gKS equation (1.4), in the absence of dispersion. We illustrate the feasibility of our control methodology for two typical cases that yield unstable steady states as computed in the bifurcation diagram of Fig. 1. In the first case, we use $\nu = 0.1115$, $\mu = 0$, and in the second $\nu = 0.35$, $\mu = 0.3$. In both cases, we used $2l+1$ equidistant controls, i.e. the same as the number of unstable eigenvalues of the system. The results of our numerical experiments are presented in Figs 8 and 9, respectively. When $\nu = 0.1115$, $\mu = 0$, i.e. $\alpha \approx 35.87$, both stable and unstable steady states coexist and the solution of the PDE with a given initial condition, e.g. (3.7), evolves to the most attracting stable state. This is shown in Fig. 8(a) where it is seen that the solution evolves to a stable bimodal steady state, marked with a circle in Fig. 10. We are interested in using feedback control to stabilize one of the coexisting unstable steady states, and the results of achieving this are presented in Fig. 8(b,c); Fig. 8(b) shows the evolution of the initial condition (3.7) using $2l+1 = 5$ equidistant controls and stabilization of the steady state marked with a + in Fig. 10 is achieved relatively quickly after ~ 2 time units. The evolution of the amplitudes of the five applied controls is shown in Fig. 8(c), and we see that the required energy tends to values very close to zero as time evolves. Note that the control amplitudes remain small and close to zero once the unstable controlled state is reached, but they cannot be identically zero due to the unstable nature of the controlled solution. Figure 9 shows the results for $\nu = 0.35$, $\mu = 0.3$. The solution we choose to stabilize at these

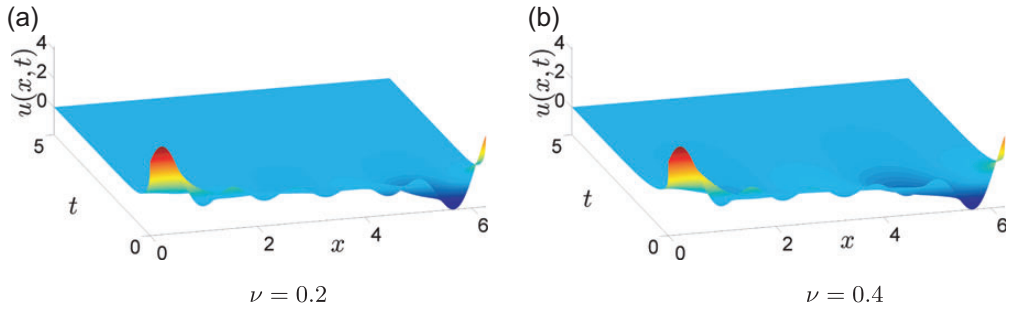


FIG. 6. Spatiotemporal evolution showing stabilization to the zero solution of the KS equation for (a) $\nu = 0.2$ ($\alpha = 20$), and (b) $\nu = 0.4$ ($\alpha = 10$).

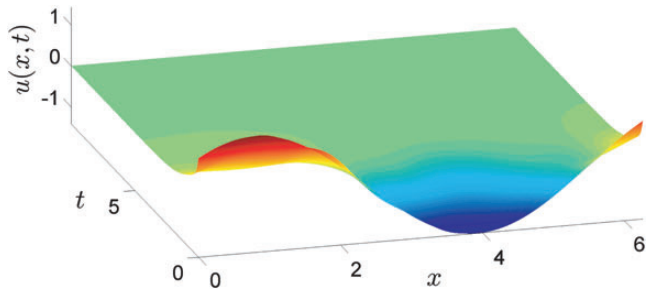


FIG. 7. Spatiotemporal evolution showing stabilization to the zero solution of the KS equation in the presence of an electric field with $\mu = 0.5$ and $\nu = 0.2$ ($\alpha = 20$).

values is an unstable bimodal steady state and Fig. 9 shows how it is stabilized using $2l + 1 = 5$ controls.

Our last task is to stabilize travelling wave solutions of the equation with and without dispersion and electric field. Figure 5 illustrates the stabilization of three different travelling wave solutions to the KS equation (1.4) with no dispersion or electric field ($\delta = \mu = 0$) and for a small value of ν ($\nu = 0.01$) which corresponds to a very large domain ($L = 20\pi \approx 62$) that enables the existence of single pulse travelling waves as well as two- or three-pulse bound states. However, due to the small value of ν, when solving the PDE, the initial condition evolves to a solution that exhibits the spatiotemporal chaotic behaviour that is characteristic of this equation. Figure 5(a) shows this chaotic behaviour while Figure 5(b–d) show the evolution of the controlled solution to 1, 2 and 3 pulses, respectively. We used $m = 21$ equidistant controls in each case.

The control algorithms presented and analysed here also apply to the gKS equation with $\delta > 0$. The findings are similar and for brevity we do not present them here. Detailed results and animations can be found in Gomes *et al.* (2015) and its supplemental material.

4. Optimal control for the gKS equation

In practical applications, it is important to apply controls that minimize the cost associated with their use, and this leads to considerations of an optimal control problem based on some measure of the energy cost of the controls. In what follows, we consider the energy of the controls given by their L^2-norm.

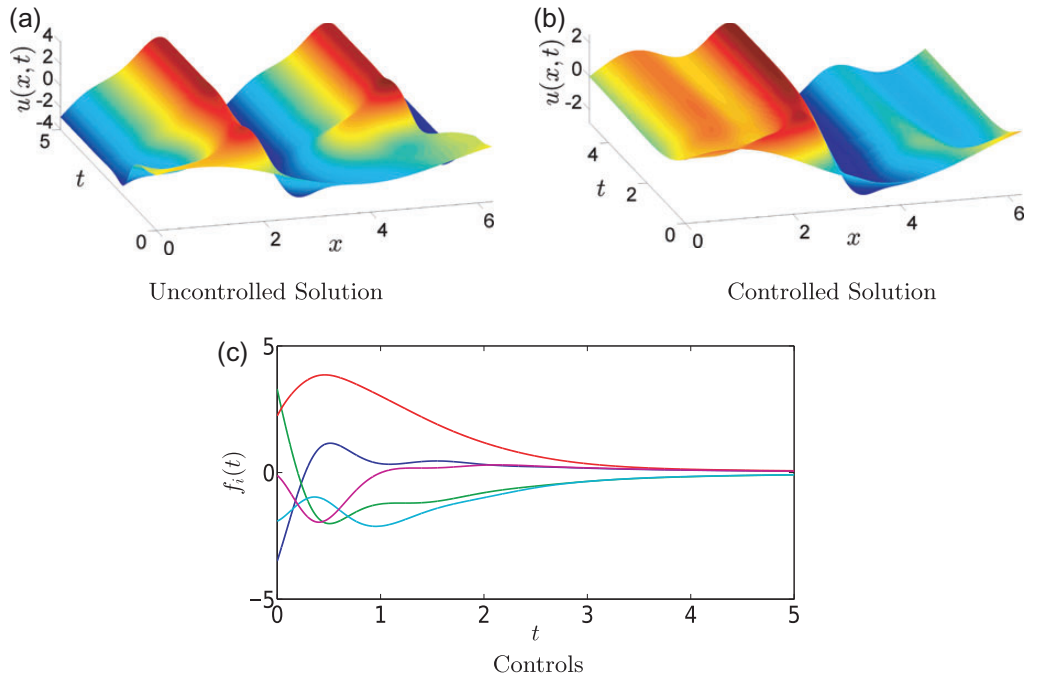


FIG. 8. Control of non-uniform solutions of the KS equation for $\nu = 0.1115$; (a) spatiotemporal evolution without controls (the solution belongs to branch 1 of the bifurcation diagram in Fig. 1(a)); (b) controlled to the steady state in branch 4 of the bifurcation diagram in Fig. 1(a); (c) evolution of the amplitude of the five applied controls.

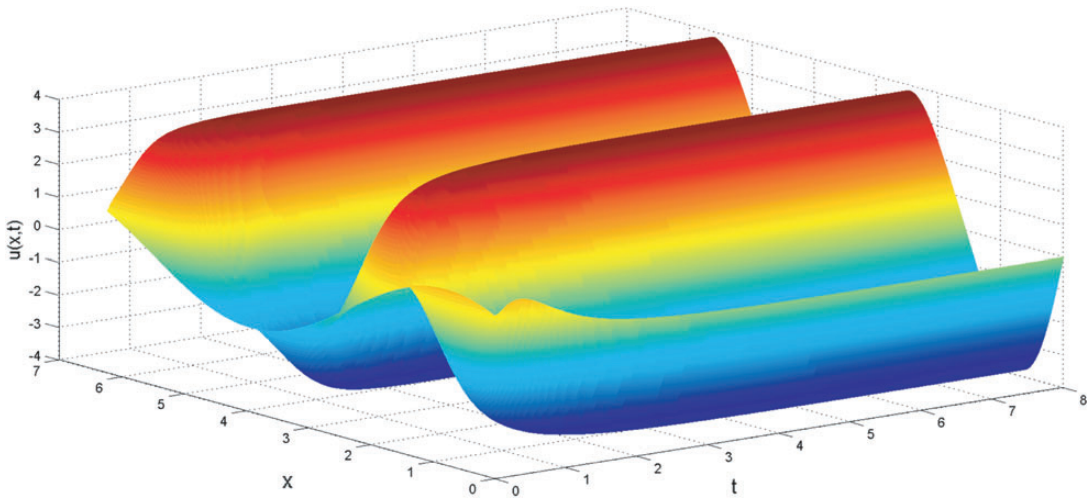


FIG. 9. Spatiotemporal evolution of the stabilized steady state of the KS equation for $\nu = 0.35$ ($\alpha \approx 11.43$), $\mu = 0.3$.

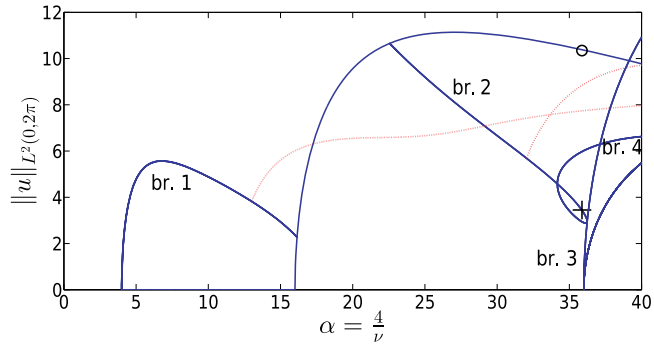


FIG. 10. Zoom in of Figur 1(a) with $\nu \in [0.1, 1]$. Branches are labelled as used in Tables 4–6 in Section 4.2: Branch 1–unimodal steady states; branch 2—bimodal steady states; branch 3—trimodal steady states; branch 4—tetramodal steady states. The cross and open circle symbols indicate the steady states (stable and unstable, respectively) that are shown in the Fig. 8.

Since the controls decay to almost zero relatively fast in time (see Fig. 9, for instance), we expect that minimizing their L^2-norm should decrease their amplitude.

The objective, then, is to achieve control of the zero solution or other unstable steady-state or travelling wave solutions of the gKS, and to do this while spending the least energy possible. To that end, we consider a cost functional that includes the distance between the solution and the desired state as well as the L^2-norm of the controls used. Different distance norms $\|u - \bar{u}\|$ can be used and in our computations we employ the L^2-, H^1- or H^2-norms. The reason we consider different norms is that the solutions are expected to belong to $H^2(0, 2\pi)$, and we wish to analyse the effects of the regularity and the oscillations of the solutions on the cost functional.

In flow control, it is possible to use point actuator functions (Christofides, 2000; Christofides & Armaou, 2000; Antoniades & Christofides, 2001; Dubljevic, 2010), implying that we can take $b_i(x) = \delta(x - x_i)$. Assuming that the cost of placing a control at x_i is the same for all actuator positions $x \in (0, 2\pi)$, it makes sense to seek a solution that minimizes the norms of the control functions $f_i(t)$. Since the delta functions in the feedback controls are not L^2 functions, the standard results of constrained optimization for PDEs (Lions, 1971; Troltzsch, 2010) do not apply. Because of this hurdle, we will first prove existence of optimal controls for the case of general controls, $f(x, t) \in L^2(0, T; \dot{L}^2(0, 2\pi))$, i.e. mean zero spatially periodic controls in $L^2(0, 2\pi)$ that are also L^2 functions of time, and focus on the case of feedback controls where we can apply standard optimization techniques.

We consider cost functionals of the form

$$\mathcal{C}(u, F) = \frac{1}{2}\int_0^T \left\|u(\cdot, t) - \bar{u}\right\|^2 \mathrm{d}t + \frac{1}{2}\left\|u(\cdot, T) - \bar{u}\right\|^2 + \frac{\gamma}{2}\int_0^T \sum_{i=1}^{m} f_i(t)^2 \,\mathrm{d}t, \tag{4.1}$$

where $\bar{u}$ is the desired steady state, $F = [f_1 \ \cdots \ f_m]$ and the norm (e.g. L^2, H^1 or H^2) is left unspecified. The choice of the parameter γ depends on how large we are willing to allow the norm of the controls to become: if we need to maintain a small norm of the controls while allowing the solution to be considerably different from the steady state, then we use $\gamma > 1$. If, on the other hand, we have a very large amount of energy to spend on the controls and want the solution to be as close as possible to the desired steady state, then we choose $\gamma \gg 1$ so that the weight of the controls does not influence significantly the value of the cost functional. The terminal time term $\frac{1}{2}\|u(\cdot, T) - \bar{u}\|^2$ is introduced to

provide us with a condition for the final-value problem obtained when solving the adjoint equation for the optimization problem. Before proceeding to the optimization problem, it is useful to introduce the following definitions.

DEFINITION 1 The space of admissible controls, F_{ad}, is a bounded convex subset of $L^2(0,T;\dot{L}^2(0,2\pi))$.

DEFINITION 2 A control $f^* \in F_{\text{ad}}$ is said to be optimal, and $u^* = u(f^*)$ is the associated optimal state, if $\mathcal{C}(u(f^*),\bar{u},f^*) \leqslant \mathcal{C}(u(f),\bar{u},f)\ \forall f \in F_{\text{ad}}$.

Our numerical experiments presented in Section 4.2 suggest that, given an initial condition and a desired steady state, there exists at least one optimal placement of the control actuators for every value of ν and μ. However, here we prove existence of an optimal control in the case of an open-loop control using the quadratic cost functional

$$\mathcal{C}(u,f) = \frac{1}{2}\int_0^T \left\|u(\cdot,t)-\bar{u}\right\|_{L^2}^2 \mathrm{d}t + \frac{1}{2}\left\|u(\cdot,T)-\bar{u}\right\|_{L^2}^2 + \frac{\gamma}{2}\int_0^T\int_0^{2\pi} f(x,t)^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.2}$$

The point actuated controls in the form of delta functions are not in L^2 and hence an analogous proof in this case requires distribution theory which is beyond the scope of the present study. The optimization problem is

$$\begin{aligned}
&\text{minimize} && \mathcal{C}(u,f) && \text{(4.3a)}\\
&\text{subject to} && u_t + \nu u_{xxxx} + u_{xx} + uu_x = f(x,t), && \text{(4.3b)}\\
& && u(x,0) = u_0(x) \in \dot{H}_p^2(0,2\pi), && \text{(4.3c)}\\
& && \frac{\partial^j u}{\partial x^j}(x+2\pi) = \frac{\partial^j u}{\partial x^j}(x), \quad j=0,1,2,3, && \text{(4.3d)}\\
& && f \in F_{\text{ad}}. && \text{(4.3e)}
\end{aligned}$$

The main result of this section is the following theorem.

THEOREM 1 Assume that $F_{\text{ad}} \subset L^2(0,T;\dot{L}^2(0,2\pi))$. Then (4.3) has at least one optimal control f^* with associated optimal state u^*.

REMARK 6 Since the Hilbert transform and third derivative terms are linear functionals of u, Theorem 1 can be easily generalized to the case when $\mu, \delta > 0$.

REMARK 7 The presence of the Burgers non-linearity in the PDE makes the optimization problem no longer convex. Consequently, we do not expect the solution of the optimal control problem to be unique.

The non-linearity in our problem, defined by $\mathcal{N}(u) = uu_x$, is twice Fréchet differentiable with respect to u but is neither an increasing functional of u, nor is it globally Lipschitz continuous. Furthermore, it depends explicitly on the derivative u_x. Consequently, the well-developed theory of optimal control for systems of reaction–diffusion equations (Lions, 1971; Troltzsch, 2010) does not apply to our problem; see equation (4.3).

Proof of Theorem 1. Let $X = H^1(0,T;\dot{H}^2_p(0,2\pi)) \times F_{\text{ad}}$ and $e(\cdot,\cdot)$ be a functional defined in X by

$$e(u,f) = \begin{bmatrix} u_t + \nu u_{xxxx} + u_{xx} + uu_x - f \\ u(\cdot,0) - u_0(x) \end{bmatrix}. \tag{4.4}$$

Our optimization problem is that of minimizing the cost functional $\mathcal{C}$ subject to $e(u,f) = 0$, periodic boundary conditions and $(u,f) \in X$; see equation (4.3).

Let $(u,f) \in X$ satisfy $e(u,f) = 0$. Since $\mathcal{C}$ is a function of the sum of the norms of u and f, it is clear that $\mathcal{C}$ is non-negative and

$$\mathcal{C}(u,f) \to \infty \quad \text{for } \|(u,f)\|_X \to \infty. \tag{4.5}$$

Therefore, there exists a constant $c \geqslant 0$ such that $c = \inf_{e(u,f)=0} \mathcal{C}(u,f) = \lim_{n\to\infty} \mathcal{C}(u^n,f^n)$, where (u^n,f^n) is a minimizing sequence in X, which exists due to the reflexivity of L^2. From equation (4.5), we can conclude that $\{(u^n,f^n)\}_{n\in\mathbb{N}}$ is bounded, and therefore there exists $(u^*,f^*) \in X$ such that $(u^n,f^n) \rightharpoonup (u^*,f^*)$ for $n \to \infty$. This means that all the linear functionals of u^n and f^n, and in particular their derivatives, also converge weakly to the same functionals of u^* and f^* in the appropriate space. Hence, we only need to prove the convergence of the non-linearity.

Following an argument similar to that in Volkwein (2000) for the Burgers equation, we note that since, for every $t \in [0,T]$, we have $u^*(\cdot,t) \in \dot{H}^2_p(0,2\pi)$; then $u^*(\cdot,t) \in C([0,2\pi])$ and therefore if $\varphi \in X$, $(u^*\varphi)(\cdot,t) \in L^2([0,2\pi])$. Hence, $u^*\varphi \in H^1(0,T;L^2(0,2\pi))$ and

$$\int_0^T \int_0^{2\pi} \left(u^n - u^*\right)_x u^*\varphi \,\mathrm{d}x\,\mathrm{d}t \longrightarrow_{n\to\infty} 0 \quad \forall \varphi \in H^1\left(0,T;\dot{H}^2_p(0,2\pi)\right). \tag{4.6}$$

Finally, from the estimates (2.1) we know that $\|u^n_x\|$ is bounded, and since $H^2(\Omega)$ is compactly embedded in $L^2(\Omega)$, we deduce that

$$\int_0^T \int_0^{2\pi} \left(u^n - u^*\right) u^n_x \varphi \,\mathrm{d}x\,\mathrm{d}t \leqslant \|u^n - u^*\| \|u^n_x\| \|\varphi\|_{L^\infty(0,2\pi)} \longrightarrow_{n\to\infty} 0 \quad \forall \varphi \in \dot{H}^2_p(0,2\pi). \tag{4.7}$$

Hence, by adding and subtracting appropriate terms we have

$$\begin{aligned} &\int_0^T \int_0^{2\pi} \left(u^n u^n_x - u^* u^*_x\right)\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^{2\pi} \left(u^n - u^*\right) u^n_x \varphi + \left(u^n - u^*\right)_x u^*\varphi \,\mathrm{d}x \longrightarrow_{n\to\infty} 0 \quad \forall \varphi \in \dot{H}^2_p(0,2\pi), \end{aligned} \tag{4.8}$$

and therefore the non-linearity $u^n u^n_x$ is weakly convergent to $u^* u^*_x$ in X. Now, noting that u^* and u^*_x are continuous in $[0,2\pi] \times [0,T]$, we observe that u^* satisfies the periodic boundary conditions and the initial condition. If we now consider $\varphi \in X$ satisfying $\varphi(x,T) = 0$, and use the weak convergence of the derivatives of u and equation (4.8), we conclude that (u^*,f^*) is a weak solution of the state equation. The optimality of the pair (u^*,f^*) follows from the weak lower semi-continuity of $\mathcal{C}$ (cf. proof of Theorem 4.15 in Troltzsch (2010)). □

4.1 *Algorithm and numerical experiments*

We note that the dependence of the cost functional on the positions x_i, $i = 1, \ldots, m$ is in the controls, since the matrix K necessary to define them depends on the positions chosen. However, when defining the Lagrangian, we will assume that only the functions $b_i(x)$ depend on x_i, and treat the controls $f_i(t)$ as if they were independent of the positions x_i. Under this assumption, we are able to obtain very satisfactory results, as evidenced by the results presented in the tables below. We begin by introducing the Lagrangian

$$\begin{aligned}\mathcal{L}\left(u, p, [x_1, x_2, \ldots, x_m]^\top\right) = \frac{1}{2}\int_0^T \left\|u(\cdot,t) - \bar{u}\right\|^2 \mathrm{d}t + \frac{1}{2}\left\|u(\cdot,T) - \bar{u}\right\|^2 + \frac{\gamma}{2}\int_0^T \sum_{i=1}^m f_i(t)^2 \,\mathrm{d}t \\ - \int_0^T \int_0^{2\pi} \left(u_t + \nu u_{xxxx} + u_{xx} + uu_x\right) p(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \int_0^{2\pi} \sum_{i=1}^m \delta(x - x_i)_i f_i(t) p(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.9}$$

Integrating by parts in space and time and computing the Fréchet derivative with respect to u (with test functions $h(x,t)$ satisfying $h(x,0) = 0$), we obtain the adjoint equation

$$\begin{cases} -p_t + \nu p_{xxxx} + p_{xx} - up_x = \displaystyle\sum_{i=1}^m \delta(x - x_i) K_{i\cdot} z^p + u - \bar{u}, \\ p(x,T) = u(x,T), \\ \dfrac{\partial^j p}{\partial x^j}(x + 2\pi) = \dfrac{\partial^j p}{\partial x^j}(x), \end{cases} \tag{4.10}$$

where $x \in [0, 2\pi]$ and $t \in [0, T]$. This PDE is backwards in time but is well-posed since it is a final-value problem. To solve it, we obtain the discretized ODE system $-\dot{z}^p = \mathcal{A} z^p + G^{\mathrm{adj}}(z^p, z^u) + z^u - z^{\bar{u}}$, where the elements of $G^{\mathrm{adj}}(z^p, z^u)$ are given by

$$g^s_{n,adj} = \frac{1}{2\sqrt{\pi}} \sum_{j+k=n} k\left(u^s_j p^s_k - u^c_j p^c_k\right) + \frac{1}{2\sqrt{\pi}} \sum_{j-k=n} \left(k\left(u^s_j p^s_k + u^c_j p^c_k\right) - j\left(u^s_k p^s_j + u^c_k p^c_j\right)\right),$$

$$g^c_{n,adj} = \frac{1}{2\sqrt{\pi}} \sum_{j+k=n} k\left(u^c_j p^s_k + u^s_j p^c_k\right) + \frac{1}{2\sqrt{\pi}} \sum_{j-k=n} \left(k\left(u^c_j p^s_k - u^s_j p^c_k\right) + j\left(u^c_k p^s_j - u^s_k p^c_j\right)\right),$$

and we have used the Fourier series representation $p(x,t) = p^c_0/\sqrt{2\pi} + \sum_{n=1}^\infty p^s_n(t)\left(\sin(nx)/\sqrt{\pi}\right) + \sum_{n=1}^\infty p^c_n(t)\left(\cos(nx)/\sqrt{\pi}\right)$.

Differentiating with respect to the positions of the control actuators, we also obtain a descent direction using the variational inequality, or first variation

$$\int_0^T [f_1(t) p_x(\bar{x}_1, t) \cdots f_m(t) p_x(\bar{x}_m, t)]^\top \cdot (x - \bar{x})\,\mathrm{d}t \geqslant 0 \quad \forall x = [x_1 \cdots x_m]^\top, \tag{4.11}$$

where $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_m]$ are the optimal positions. To proceed with the optimization, we will use a gradient descent method, see Troltzsch (2010, Section 5.9), Borzi & Schulz (2012), and consider $F_{\mathrm{ad}} = (0, 2\pi)^m$. The algorithm is as follows.

Algorithm for optimal control for the KS equation

Given ν, μ, γ, T, $u_0(x)$, A, x^0, B^0, $\bar{u}$, compute the matrix K^0.

while $\mathcal{C}$(current iteration) $<$ $\mathcal{C}$(previous iteration) **do**

1. Solve the state equation to obtain u^{k-1} and compute $\mathcal{C}(u^{k-1}, \bar{u}, F(x^{k-1}))$;
2. Solve the adjoint equation to obtain p^{k-1};
3. Define $P_x = \left[p_x^{k-1}(x_1^{k-1}, t) \cdots p_x^{k-1}(x_m^{k-1}, t)\right]$,
   $P_{k-1} = \displaystyle\int_0^T K^{k-1}(z^{u^{k-1}} - z^{\bar{u}})P_x \, dt$ and $h_k = -P_{k-1}$;
4. Find $s = \min_{s>0} \left\{C(u(x^{k-1} + sh_k), \bar{u}, F(x^{k-1} + sh_k))\right\}$;
5. Project $x^{k-1} + sh_k$ into $(0, 2\pi)^m$, obtaining x^k;
6. Compute the matrix B^k;
7. Compute the matrix K^k with *Matlab*'s command *place*.

end

FIG. 11. Algorithm for optimal control of the KS equation.

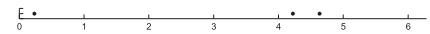
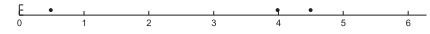
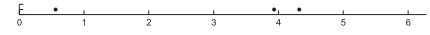
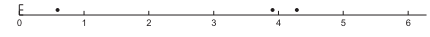
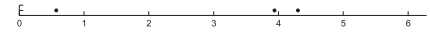
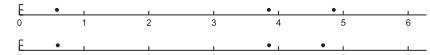
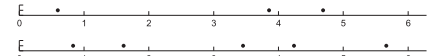
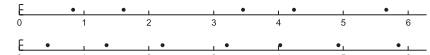
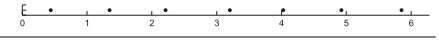
Note that as mentioned earlier we consider the following three different cost functionals:

$$\mathcal{C}_1(u, \bar{u}, f) = \frac{1}{2}\int_0^T \left\|u(\cdot, t) - \bar{u}\right\|^2 \mathrm{d}t + \frac{1}{2}\left\|u(\cdot, T) - \bar{u}\right\|^2 + \frac{\gamma}{2}\sum_{i=1}^m \left\|f_i(t)\right\|_{L^2(0,T)}, \tag{4.12}$$

$$\begin{aligned}\mathcal{C}_2(u, \bar{u}, f) = {} & \frac{1}{2}\int_0^T \left(\left\|u(\cdot, t) - \bar{u}\right\|^2 + \left\|u_x(\cdot, t) - \bar{u}_x\right\|^2\right) \mathrm{d}t \\ & + \frac{1}{2}\left(\left\|u(\cdot, T) - \bar{u}\right\|^2 + \left\|u_x(\cdot, T) - \bar{u}_x\right\|^2\right) + \frac{\gamma}{2}\sum_{i=1}^m \left\|f_i(t)\right\|_{L^2(0,T)},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\mathcal{C}_3(u, \bar{u}, f) = {} & \frac{1}{2}\int_0^T \left(\left\|u(\cdot, t) - \bar{u}\right\|^2 + \left\|u_x(\cdot, t) - \bar{u}_x\right\|^2 + \left\|u_{xx}(\cdot, t) - \bar{u}_{xx}\right\|^2\right) \mathrm{d}t \\ & + \frac{1}{2}\left(\left\|u(\cdot, T) - \bar{u}\right\|^2 + \left\|u_x(\cdot, T) - \bar{u}_x\right\|^2 + \left\|u_{xx}(\cdot, T) - \bar{u}_{xx}\right\|^2\right) + \frac{\gamma}{2}\sum_{i=1}^m \left\|f_i(t)\right\|_{L^2(0,T)}.\end{aligned} \tag{4.14}$$

TABLE 1 *Optimal positions and value of the cost functional considered in the L^2-norm for different values of ν when stabilizing the zero solution to the KS equation*

ν	Cost $\mathcal{C}_1$	Cost of controls	Iterations	Optimal positions
0.9	8.9647	0.5592	3	
0.8	6.5012	1.1274	6	
0.7	5.5760	1.7204	4	
0.6	5.2803	2.3018	3	
0.5	5.2230	2.8204	5	
0.4	5.9339	3.8404	5	
0.3	6.2152	3.9813	2	
0.2	6.3127	4.5261	2	
0.1	7.1652	5.5759	2	

4.2 *Numerical experiments*

Computations were carried out using the algorithm presented in Fig. 11 for various values of ν and μ. The number of controls used was equal to the number of unstable eigenvalues, and m equidistant points were used as an initial guess for the position of the controls. In all the computations the initial condition is $u_0(x) = \sin(x)/\sqrt{\pi} + \cos(x)/\sqrt{\pi}$, and the final time is $T = 10$. Numerical results are presented in Tables 1–3 for the stabilization of the zero solution of the KS equation, and in Tables 4–6 for the stabilization on non-trivial unstable steady states as computed in the bifurcation diagram of Fig. 1(a). Each entry in Tables 1–3 contains the value of ν, the value of the cost functional ($\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ for Tables 1–3, respectively), the cost of the controls $\sum_{i=1}^{m} \|f_i(t)\|_{L^2(0,T)}$, the number of iterations required to obtain an optimal state and in the last column the spatial distribution of the controls over the domain $[0, 2\pi]$—a heavy dot is placed where a control acts. Tables 4–6 are presented in an analogous manner, with the difference that the first column provides information on the unstable solution that is being controlled, and in particular the branch on Fig. 1(a) where the solution was taken from is stated along with the value of ν. As the results indicate, several distinct unstable solutions at a given value of ν are controlled (e.g. for $\nu = 0.1$, three solutions are stabilized coming from branches 1, 3 and 4, respectively).

As expected we observe that the value of the cost functionals $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ given by (4.12–4.14) increases as ν decreases. Furthermore, the value of the cost functional also increases as we increase the desired regularity of the solution from L^2- to H^1- to H^2-norms. Comparing the results and in particular the positions of the optimal controls for the three different cost functionals in Tables 1–3, we can conclude that, for the stabilization of the zero steady states, the optimal control problem is more robust (in the sense that the optimal positions of the controls do not change much as ν is reduced) when the L^2 cost functional $\mathcal{C}_1$ is used.

Turning now to the results of Tables 4–6 that deal with the stabilization of unstable non-uniform steady states, we observe once again that there is an increase in the cost functionals as ν decreases. We also observe that in this case (and in contrast to the stabilization of the zero solution) the higher-order norms give optimal controls that are more robust, with respect to changing ν, in comparison to utilizing the L^2 cost functional.

Similar numerical experiments were performed for the optimal control problem for the KS equation in the presence of an electric field, $\mu > 0$. The results are quite similar to the ones already presented in

TABLE 2 *Optimal positions and value of the cost functional considered in the H^1-norm for different values of ν when stabilizing the zero solution to the KS equation*

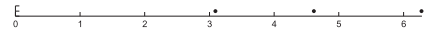
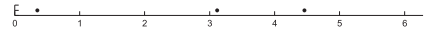
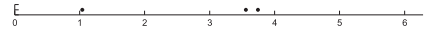
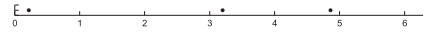
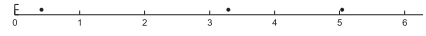
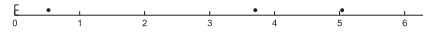
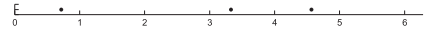
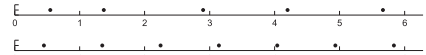
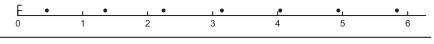
ν	Cost $\mathcal{C}_2$	Cost of controls	Iterations	Optimal positions
0.9	17.6354	0.9698	3	
0.8	12.2553	1.5486	6	
0.7	10.8270	2.0016	2	
0.6	10.2340	4.0181	6	
0.5	10.1517	4.5407	4	
0.4	9.3345	4.2298	2	
0.3	10.9671	4.8110	2	
0.2	10.7154	5.5308	3	
0.1	9.7088	5.6463	1	

TABLE 3 *Optimal positions and value of the cost functional considered in the H^2-norm for different values of ν when stabilizing the zero solution to the KS equation*

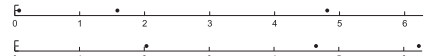
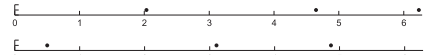
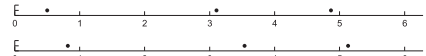
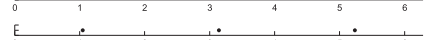
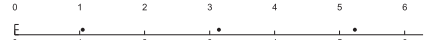
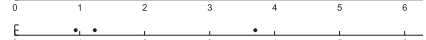
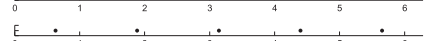
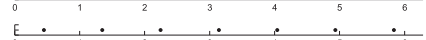
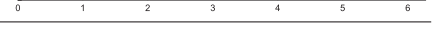
ν	Cost $\mathcal{C}_3$	Cost of controls	Iterations	Optimal positions
0.9	27.2098	1.1313	3	
0.8	19.3431	2.3490	5	
0.7	15.8522	2.5815	4	
0.6	14.0865	3.3384	5	
0.5	17.0462	6.4166	1	
0.4	20.7720	8.3217	1	
0.3	14.4393	5.3865	3	
0.2	21.5856	6.1456	1	
0.1	21.1636	5.6463	1	

TABLE 4 *Optimal positions and value of the cost functional considered in the L^2-norm for different values of ν when stabilizing some of the non-trivial steady states from the bifurcation diagram* 1(a)

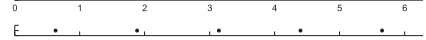
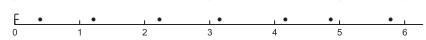
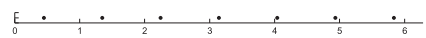
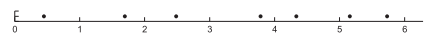
ν	Cost $\mathcal{C}_1$	Cost of controls	Iterations	Optimal positions
0.3	14.3164	9.7419	5	
0.2, br.1	30.0588	21.8691	1	
0.2, br.3	24.0520	16.2832	2	
0.1, br.1	28.5591	32.9859	2	
0.1, br.3	37.8902	32.4264	1	
0.1, br.4	62.3916	51.6820	4	

TABLE 5 *Optimal positions and value of the cost functional considered in the H^1-norm for different values of ν when stabilizing some of the non-trivial steady states from the bifurcation diagram* 1(a)

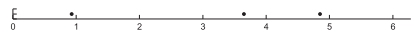
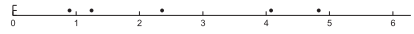
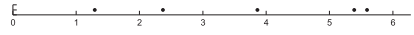
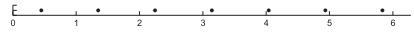
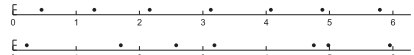
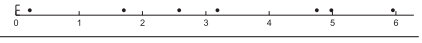
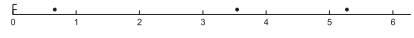
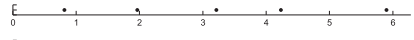
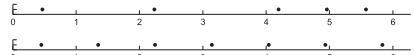
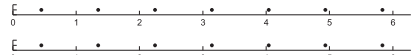
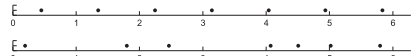
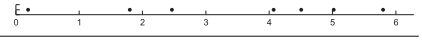
ν	Cost $\mathcal{C}_2$	Cost of controls	Iterations	Optimal positions
0.3	24.6922	9.6582	3	
0.2, br.1	53.7672	17.9929	2	
0.2, br.3	52.2630	15.4089	3	
0.1, br1	87.1636	35.2169	1	
0.1, br.3	83.9787	33.7581	2	
0.1, br.4	171.6040	62.1381	2	

TABLE 6 *Optimal positions and value of the cost functional considered in the H^2-norm for different values of ν when stabilizing some of the non-trivial steady states from the bifurcation diagram* 1(a)

ν	Cost $\mathcal{C}_3$	Cost of controls	Iterations	Optimal positions
0.3	54.5441	13.8436	2	
0.2, br.1	262.7363	21.1679	3	
0.2, br.3	266.0515	32.4603	3	
0.1, br.1	702.4697	35.2169	1	
0.1, br.3	745.6007	32.4264	1	
0.1, br.4	1384.6689	63.1272	2	

this section and we omit providing additional graphs and tables. However, we summarize the conclusions drawn from the non-zero electric field numerical experiments as follows: with few exceptions, an increase in the intensity of the electric field parameter μ increases the cost of the controls. In addition, it is found that the optimal controls for stabilizing zero steady states are more robust, with respect to changes in μ, when using the L^2 cost functional. Similarly, when stabilizing non-trivial steady states, more robust optimal positions for the controls arise when the H^1 and H^2 cost functionals are used. Both of these findings are analogous to those for the non-electrified control problem.

A more detailed comparison of the energy required to control different solutions using equidistant actuators or optimally computed positions as described above is provided in Figs. 12 and 13. Figure 12 shows the stabilization of a non-uniform steady state for the KS equation, $\nu = 0.3$ and $\mu = 0$, while Fig. 13 shows analogous results but for the electrified problem with parameters $\nu = 0.5$ and $\mu = 0.4$ (in both cases dispersion is absent, $\delta = 0$). Figure 13(a) shows the spatiotemporal evolution to the desired state in the presence of controls, while Fig. 13(b,c) depict the evolution of the control amplitudes (there are three controls in each case) for equidistant or optimally positioned actuators, respectively. The results show that the amplitudes of optimally placed controls decay to zero faster than those of the equidistantly placed ones.

4.2.1 *Optimal control of travelling waves* We also performed similar numerical experiments to find the optimal position of the control actuators when stabilizing travelling waves. We found that in most cases, we cannot do better than equidistant controls.

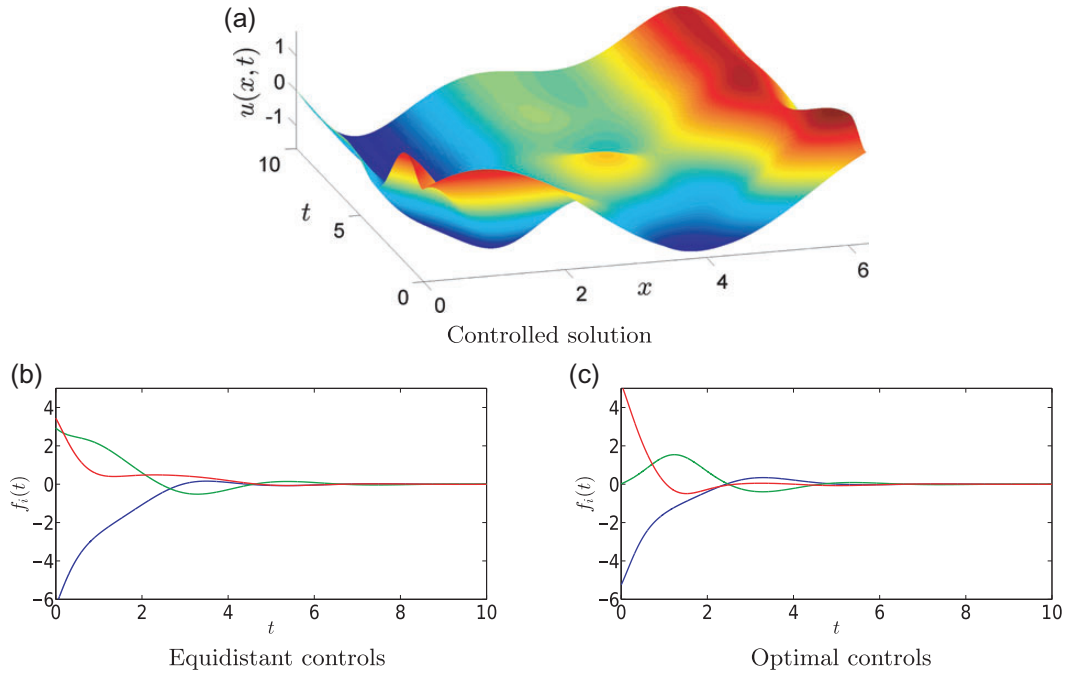


FIG. 12. Controlled steady state of the KS equation for $\nu = 0.3$ (a) and controls applied: (b) equidistant and (c) optimal.

We believe that this is due to the following reasons. First, the length of the domain needed for the existence of (unstable) travelling waves is long, and therefore the number of unstable modes (and hence of the number of controls) is large (e.g. in the example of Fig. 5 we are using $m = 21$ controls); thus, shifting the position of the controls in a relatively large domain should not have a big effect on their amplitudes. Secondly, solitary pulses on long domains necessarily have large flat regions which are susceptible to linear instabilities leading to the non-linear wavy perturbations seen in panels (b) and (c) in Fig. 2. It is interesting to note that there are 10 wavy structures corresponding to the number of linearly unstable modes; thus, we expect optimality when the controls are approximately equally spaced, thus guaranteeing one control under each wavy structure. Shifting the controls can introduce instability and non-linear growth to a different state.

5. Feedback and optimal control for coupled KS equations

In Sections 2–4, we studied analytically and computationally the feedback control and optimal control problems for the gKS equation. In applications, systems of equations emerge with two or more non-linear coupled PDEs, and this section is concerned with the control of such systems. As an example, we refer to the system of two coupled KS equations that arises in the weakly non-linear asymptotic analysis of a three-layer flow of immiscible viscous fluids stratified in a channel and driven by gravity and/or a stream wise pressure gradient, see Papaefthymiou *et al.* (2013). The fully coupled system is a challenging PDE problem and questions such as the existence and uniqueness of solutions, steady states and bifurcation theory are still poorly understood. Such problems are currently under investigations and our findings will be reported elsewhere. In this section, we consider

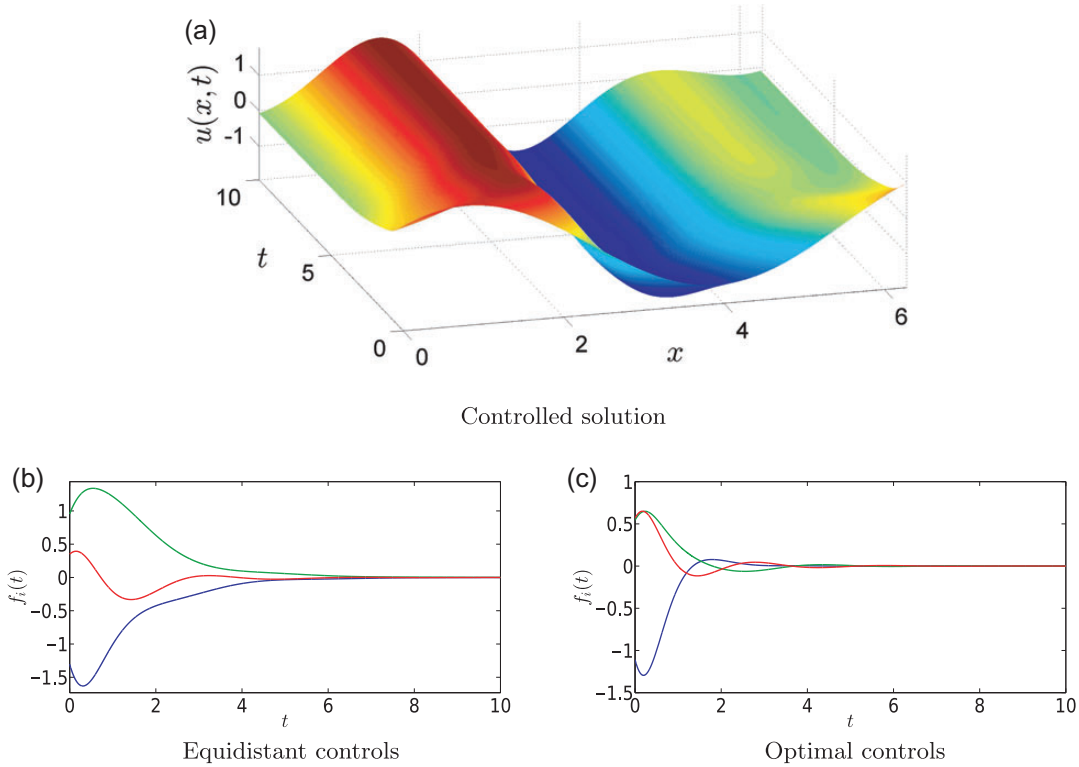


FIG. 13. Controlled steady state of the KS equation for $\nu = 0.5$, $\mu = 0.4$ (a) and controls applied: (b) equidistant and (c) optimal.

the problem of feedback and of optimal control for a system of KS equations that are coupled only through the second derivatives. Such a coupling is special but can arise in the application of three-layer flow. More generally, the non-linearities are also coupled and in fact the non-linear flux functions can generically have real or complex eigenvalues implying hyperbolic elliptic transitions, thus complicating the analysis significantly; see Papaefthymiou & Papageorgiou (2015) for a detailed study of such effects.

In what follows, we consider the following coupled system of KS equations:

$$\begin{cases} u_{1,t} = -\nu u_{1,xxxx} - u_{1,xx} - u_1 u_{1,x} - \alpha_1 u_{2,xx}, \\ u_{2,t} = -\nu u_{2,xxxx} - u_{2,xx} - u_2 u_{2,x} - \alpha_2 u_{1,xx}. \end{cases} \tag{5.1}$$

We consider the equations in the interval $(0, 2\pi)$ with periodic boundary conditions and initial conditions $u_1(x, 0) = u_{10}(x)$ and $u_2(x, 0) = u_{20}(x)$ and $u_{10},\ u_{20} \in \dot{H}^2_p(0, 2\pi)$.

We can prove using the background flow method (Collet *et al.*, 1993b; Nicolaenko *et al.*, 1985; Tseluiko & Papageorgiou, 2007) that the solutions to the system are bounded.

PROPOSITION 5 Assume that $u_{10},\ u_{20} \in \dot{H}^2_p(0, 2\pi)$. Then there exists a constant $C = C(\nu, \alpha_1, \alpha_2)$ such that

$$\|u_1\|_{L^2} + \|u_2\|_{L^2} \leqslant C. \tag{5.2}$$

Similar bounds can be obtained for the H^1- and H^2-norm, and we can therefore conclude that the solutions to system (5.1) are in $L^\infty(0, 2\pi)$.

The proof is not included here due to space constraints and will appear elsewhere. A similar result can be proved for the case when the coupling comes only through the fourth-order derivatives (i.e. there is a non-diagonal negative definite fourth-order viscosity matrix), and we believe that similar results can also be proved for the fully coupled system with non-diagonal second- as well as fourth-order viscosity matrices.

5.1 *Feedback control for the coupled KS equations*

Since equations (5.1) are coupled linearly, analogous results to the ones presented earlier for the single KS are obtained. First, we can prove that it is possible to stabilize any steady-state solution (either the zero solution or any non-trivial steady state) for this system. We proceed in the same way as for the single KS equation and write the controlled system

$$\begin{cases} u_{1,t} = -\nu u_{1,xxxx} - u_{1,xx} - u_1 u_{1,x} - \alpha_1 u_{2,xx} + \sum_{j_1=1}^{m} \delta(x - x_{j_1}) f_{j_1}(t), \\ u_{2,t} = -\nu u_{2,xxxx} - u_{2,xx} - u_2 u_{2,x} - \alpha_2 u_{1,xx} + \sum_{j_2=1}^{m} \delta(x - x_{j_2}) f_{j_2}(t). \end{cases} \tag{5.3}$$

Defining

$$U(x,t) = \begin{bmatrix} u_1(x,t) \\ u_2(x,t) \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} u_{1n}^s(t) \\ u_{2n}^s(t) \end{bmatrix} \sin(nx) + \sum_{n=0}^{\infty} \begin{bmatrix} u_{1n}^c(t) \\ u_{2n}^c(t) \end{bmatrix} \cos(nx), \tag{5.4}$$

and taking the inner product with the functions $1/\sqrt{2\pi}$, $\sin(nx)/\sqrt{\pi}$ and $\cos(nx)/\sqrt{\pi}$ yields the following infinite system of ODEs:

$$\begin{cases} \dot{u}_{\text{in}}^s = \left(-\nu n^4 + n^2\right) u_{\text{in}}^s + \alpha_i n^2 u_{jn}^s + g_{\text{in}}^s + \sum_{j_i=1}^{m} b_{j_i n}^s f_{j_i}(t), \quad n = 1, \ldots, \infty, \\ \dot{u}_{\text{in}}^c = \left(-\nu n^4 + n^2\right) u_{\text{in}}^c + \alpha_i n^2 u_{jn}^c + g_n^c + \sum_{j_i=1}^{m} b_{j_i n}^c f_{j_i}(t), \quad n = 0, \ldots, \infty, \end{cases} \tag{5.5}$$

where $i, j = 1, 2$, $i \neq j$, and the functions b and g are defined as in the single KS case. We truncate the system at N modes and define

$$\begin{aligned} z^U &= \begin{bmatrix} u_{10}^c & u_{11}^s & u_{11}^c & \cdots & u_{1N}^s & u_{1N}^c & u_{20}^c & u_{21}^s & u_{21}^c & \cdots & u_{2N}^s & u_{2N}^c \end{bmatrix}^\top, \\ G &= \begin{bmatrix} 0 & g_{11}^s & g_{11}^c & \cdots & g_{1N}^s & g_{1N}^c & 0 & g_{21}^s & g_{21}^c & \cdots & g_{2N}^s & g_{2N}^c \end{bmatrix}^\top, \\ F &= \begin{bmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1m}(t) & f_{21}(t) & f_{22}(t) & \cdots & f_{2m}(t) \end{bmatrix}^\top. \end{aligned}$$

Next we write

$$A = \begin{bmatrix} A_0 & A_1 \\ A_2 & A_0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

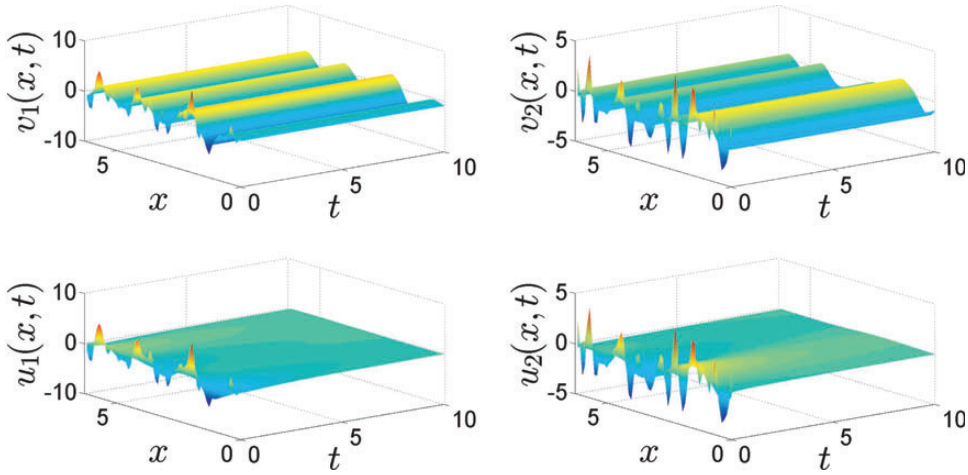


FIG. 14. Uncontrolled solution (v_i, $i=1,2$) and controlled zero solution (u_i, $i=1,2$) of the system of coupled KS equations for $\nu=0.5$, $\alpha_1=0.8$ and $\alpha_2=0.5$.

where $A_0=\text{diag}(0,-\nu+1,-\nu+1,\ldots,-\nu n^4+n^2,-\nu n^4+n^2,\ldots)$, $A_i=\text{diag}(0,\alpha_i,\alpha_i,\ldots,\alpha_i n^2,\alpha_i n^2,\ldots)$ and

$$B_i=\begin{bmatrix} b^c_{1_i0} & b^c_{2_i0} & \cdots & b^c_{m_i0} \\ b^s_{1_i1} & b^s_{2_i1} & \cdots & b^s_{m_i1} \\ b^c_{1_i1} & b^c_{2_i1} & \cdots & b^c_{m_i1} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix},$$

for $i=1,2$. Hence the infinite system of ODEs can be written as

$$\dot{z}^U = Az^U + G + BF. \tag{5.6}$$

We can prove a result similar to Proposition 1.

PROPOSITION 6 Let $\bar{U}=\begin{bmatrix}\bar{u}_1\\ \bar{u}_2\end{bmatrix}$ be a non-trivial (unstable) steady-state solution of (5.1), and let $l=l_1+l_2$ be the number of unstable eigenvalues of the linearized system, i.e. $l_1^2<(1+\sqrt{\alpha_1\alpha_2})/\nu<(l_1+1)^2$ and $l_2^2<(1-\sqrt{\alpha_1\alpha_2})/\nu<(l_2+1)^2$. If $m=2(l+1)$ and there exists a matrix K such that all of the eigenvalues of the matrix $A+BK$ have negative real part, then the state feedback controls

$$\begin{bmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1m}(t) & f_{21}(t) & f_{22}(t) & \cdots & f_{2m}(t)\end{bmatrix}^\top = F = K\left(z^U - z^{\bar{U}}\right), \tag{5.7}$$

stabilize this non-trivial steady-state solution of system (5.1).

The proof of this result follows the same argument as the proof of Proposition 1.

We present in Figs 14 and 15 the numerical results of the stabilization of the zero solution and a steady-state solution, respectively, for system (5.1) with $\nu=0.5$, $\alpha_1=0.8$ and $\alpha_2=0.5$. We used $m=4$ equidistant controls to control each solution, corresponding physically to applying four controls in each wall. Upper panels correspond to the uncontrolled solution, and lower panels correspond to the stabilized solution. We clearly observe in both figures the stabilization of the desired steady state.

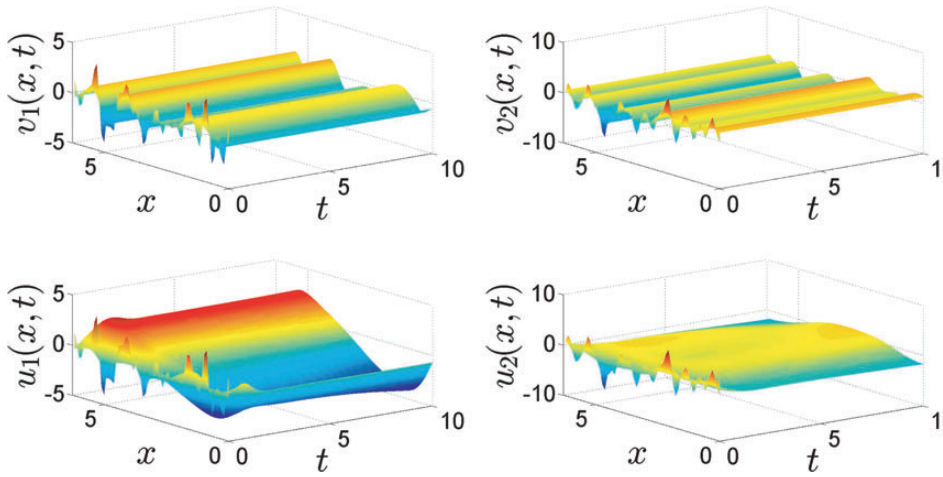


FIG. 15. Uncontrolled solution (v_i, $i=1,2$) and stabilized steady-state solution (u_i, $i=1,2$) of the system of coupled KS equations for $\nu=0.5$, $\alpha_1=0.8$ and $\alpha_2=0.5$.

5.2 *Optimal control of the system of coupled KS equations*

We consider next the problem of controlling an arbitrary steady state $\bar{U}=\left[\begin{smallmatrix}\bar{u}_1\\ \bar{u}_2\end{smallmatrix}\right]$ in an optimal way. We introduce the cost functional

$$\begin{aligned}\mathcal{C}(U,F)=&\frac{1}{2}\int_0^T\left(\left\|u_1(\cdot,t)-\bar{u}_1\right\|_{L^2}^2+\left\|u_2(\cdot,t)-\bar{u}_2\right\|_{L^2}^2\right)\mathrm{d}t\\&+\frac{1}{2}\left(\left\|u_1(\cdot,T)-\bar{u}_1\right\|_{L^2}^2+\left\|u_2(\cdot,T)-\bar{u}_2\right\|_{L^2}^2\right)\\&+\frac{\gamma}{2}\int_0^T\left(\left\|f_1(x,t)\right\|_{L^2}^2+\left\|f_2(x,t)\right\|_{L^2}^2\right)\mathrm{d}t.\end{aligned}\tag{5.8}$$

The optimization problem that we have to solve takes the form

$$\text{minimize}\quad \mathcal{C}(U,F)\tag{5.9a}$$

$$\text{subject to}\quad u_{1,t}+\nu u_{1,xxxx}+u_{1,xx}+u_1u_{1,x}+\alpha_1u_{2,xx}=f_1(x,t),\tag{5.9b}$$

$$u_{2,t}+\nu u_{2,xxxx}+u_{2,xx}+u_2u_{2,x}+\alpha_2u_{1,xx}=f_2(x,t),\tag{5.9c}$$

$$u_i(x,0)=u_{0,i}(x),\quad i=1,2,\tag{5.9d}$$

$$\frac{\partial^j u_i}{\partial x^j}(x+2\pi)=\frac{\partial^j u_i}{\partial x^j}(x),\quad j=0,1,2,3,\ i=1,2,\tag{5.9e}$$

$$f_i\in F_{\text{ad}},\quad i=1,2.\tag{5.9f}$$

Here, $u_{0,i}\in\dot{H}_p^2(0,2\pi)$ and F_{ad} is a bounded, closed and convex subset of $L^2((0,2\pi)\times(0,T))$.

We can prove the following theorem.

THEOREM 2 If $F_{\text{ad}} \subset L^2((0,T);\dot{L}^2(0,2\pi))$, the optimal control problem (5.9a)–(5.9f) has at least one optimal control $F^* = \begin{bmatrix} f_1^* \\ f_2^* \end{bmatrix}$ with associated optimal state $U^* = \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix}$.

Sketch of the proof. The proof follows the same steps as that of Theorem 1 for the single KS equation. For the coupled system, we need to consider a different state space $X = (H^1(0,T;\dot{H}_p^2(0,2\pi)))^2 \times (F_{\text{ad}})^2$, and redefine $e(\cdot,\cdot;\cdot,\cdot)$:

$$e(u_1,u_2;f_1,f_2) = \begin{bmatrix} u_{1,t} + \nu u_{1,xxxx} + u_{1,xx} + u_1 u_{1,x} + \alpha_1 u_{2,xx} - f_1(x,t) \\ u_{2,t} + \nu u_{2,xxxx} + u_{2,xx} + u_2 u_{2,x} + \alpha_2 u_{1,xx} - f_2(x,t) \\ u_1(\cdot,0) - u_{0,1}(x) \\ u_2(\cdot,0) - u_{0,2}(x) \end{bmatrix}. \tag{5.10}$$

The rest of the proof follows Theorem 1, but accounting for the fact that, for every $t \in [0,T]$, we have $U^*(\cdot,t) \in (\dot{H}_p^2(0,2\pi))^2$, and then $U^*(\cdot,t) \in (C([0,2\pi]))^2$ and therefore if $\varphi_i \in X$, $(u_i^*\varphi_i)(\cdot,t) \in L^2([0,2\pi])$ for $i = 1,2$.

Finally, we also need the estimates in Proposition 5 to establish that $\|u_{i,x}^n\|_{L^2}$ is bounded, and since H^2 is compactly embedded in L^2, we deduce that

$$\begin{aligned} &\int_0^T \int_0^{2\pi} \left(u_i^n - u_i^*\right) u_{i,x}^n \varphi_i \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \left\|u_i^n - u_i^*\right\|_{L^2} \left\|u_{i,x}^n\right\|_{L^2} \left\|\varphi_i\right\|_{L^\infty} \longrightarrow_{n\to\infty} 0 \quad \forall \varphi_i \in \dot{H}_p^2(\Omega). \end{aligned} \tag{5.11}$$

□

6. Conclusions

In this paper, we studied the problem of controlling and stabilizing solutions to the gKS equation. We studied both feedback and optimal control problems. For the optimal control problem, we proved existence of an optimal control and we investigated numerically the problem of optimal actuator placement. By extending earlier work by Armaou & Christofides (2000a,b), Christofides (1998) and Christofides & Armaou (2000), we showed rigorously that we can control arbitrary non-trivial steady states of the KS equation, including travelling wave solutions, using only a finite number of point actuators. The number of point actuators needed is related to the number of unstable modes. We also investigated the robustness of the controllers with respect to changing the parameters in the equation. In particular, we showed that our proposed control methodology can be used in the presence of uncertainty. Our results can be extended to coupled systems of KS equations.

We have not discussed about the practical implementation of the control methodologies studied in this paper. For example, we have assumed that complete information about the solution of the KS equation is available. This and other issues related to the implementation of the control algorithm are discussed in Thompson *et al.* (0000).

We considered the case where the entire solution to the KS equation is available to us. It is straightforward, however, to apply our results to the case when only a finite number of observations is available, using the techniques presented in Armaou & Christofides (2000a). For brevity of exposition we have not done this for the KS equation. In a forthcoming paper Thompson *et al.* (0000), we study the feedback control problem when only a finite number of observations is available to us for a more complicated

PDE, the quasilinear Benney equation arising in falling film problems (see Kalliadasis *et al.*, 2012 and references therein).

There are several directions in which the results presented in this paper can be extended. First, the KS equation is a simplified model for thin film flows obtained using weakly non-linear analysis and valid close to criticality (Craster & Matar, 2009; Kalliadasis *et al.*, 2012). We can apply the control methodologies studied in this paper to other simplified models that are closer to the full 2D Navier–Stokes dynamics, such as the Benney equation and the weighted residual model. These equations are more complicated since they are quasilinear and they can include additional degrees of freedom. It is possible to extend our results to such models (Thompson *et al.*, 0000). Furthermore, the control of unstable travelling waves for the KS equation with dispersion can be analysed in detail, leading to an efficient and robust algorithm. This problem is studied further in Gomes *et al.* (2015). Finally, our techniques can be extended so that they apply to the noisy KS equation. Given that the noise itself can sometimes stabilize linearly unstable solutions (Pradas *et al.*, 2011, 2012), the interaction between noise and controls can lead to very interesting dynamic phenomena. We think that this is a particularly interesting direction for further research, since the noisy KS equation is closely related to the noisy Kardar–Parisi–Zhang equation, which is a universal model for weakly asymmetric processes (Hairer, 2013).

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