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Maximum likelihood drift estimation for multiscale diffusions

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Abstract

We study the problem of parameter estimation using maximum likelihood for fast/slow systems of stochastic differential equations. Our aim is to shed light on the problem of model/data mismatch at small scales. We consider two classes of fast/slow problems for which a closed coarse-grained equation for the slow variables can be rigorously derived, which we refer to as averaging and homogenization problems. We ask whether, given data from the slow variable in the fast/slow system, we can correctly estimate parameters in the drift of the coarse-grained equation for the slow variable, using maximum likelihood. We show that, whereas the maximum likelihood estimator is asymptotically unbiased for the averaging problem, for the homogenization problem maximum likelihood fails unless we subsample the data at an appropriate rate. An explicit formula for the asymptotic error in the log-likelihood function is presented. Our theory is applied to two simple examples from molecular dynamics.

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1. Introduction

Fitting stochastic differential equations (SDEs) to time-series data is often a useful way of extracting simple model fits which capture important aspects of the dynamics [19]. However, whilst

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the data may well be compatible with an SDE model in many respects, it is often incompatible with the desired model at small scales. Since many commonly applied statistical techniques see the data at small scales this can lead to inconsistencies between the data and the desired model fit. This phenomenon appears quite often in econometrics [1,2,25], where the term market microstructure noise is used to describe the high frequency/small scale part of the data as well as in molecular dynamics [36]. In essence, the problem that we are facing is that there is an inconsistency between the coarse-grained model that we are using and the microscopic dynamics from which the data is generated, at small scales. Similar problems appear quite often in statistical inference, in the context of parameter estimation for misspecified or incorrect models [23, Sec. 2.6].

The aim of this paper is to create a theoretical framework in which it is possible to study this issue, in order to gain better insight into how it is manifest in practice, and how to overcome it. In particular our goal is to investigate the following problem: how can we fit data obtained from the high-dimensional, multiscale full dynamics to a low-dimensional, coarse-grained model which governs the evolution of the resolved ("slow") degrees of freedom? We will study this question for a class of stochastic systems for which we can derive rigorously a coarse-grained description for the dynamics of the resolved variables. More specifically, we will work in the framework of coupled systems of multiscale SDEs for a pair of unknown functions $(x(t), y(t))$. We assume that $y(t)$ is fast, relative to $x(t)$, and that the equations average or homogenize to give a closed equation for $X(t)$ to which $x(t)$ converges in the limit of infinite scale separation. The function $X(t)$ then approximates $x(t)$, typically in the sense of weak convergence of probability measures [13,37]. We then ask the following question: given data for $x(t)$, from the coupled system, can we correctly identify parameters in the averaged or homogenized model for $X(t)$?

Fast/slow systems of SDEs of this form have been studied extensively over the last four decades; see [6,15,17,21,20,29,37] and the references therein. Recently, various methods have been proposed for numerically solving these SDEs [11,18,40]. In these works, the coefficients of the limiting SDE are calculated "on the fly" from simulations of the fast/slow system. There is a direct link between these numerical methods and our approach in that our goal is also to infer information about the coefficients in the coarse-grained equation using data from the multiscale system. However, our interest is mainly in situations where the "microscopic" multiscale system is not known explicitly. From this point of view, we merely use the multiscale stochastic system as our "data generating process"; our goal is to fit this data to the coarse-grained equation for $X(t)$, the limit of the slow variable $x(t)$.

A first step towards the understanding of this problem was taken in [36]. There, the data generating process $x(t)$ was taken to be the path of a particle moving in a multiscale potential under the influence of thermal noise. The goal was to identify parameters in the drift as well as the diffusion coefficient in the homogenized model for $X(t)$, the weak limit of $x(t)$. It was shown that the maximum likelihood estimator is asymptotically biased and that subsampling is necessary in order to estimate the parameters of the homogenized limit correctly, based on a time series (i.e. single observation) of $x(t)$.

In this paper we extend the analysis to more general classes of fast/slow systems of SDEs for which either an averaging or homogenization principle holds [37]. We consider cases where the drift in the averaged or homogenized equation contains parameters which we want to estimate using observations of the slow variable in the fast/slow system. We show that in the case of averaging the maximum likelihood function is asymptotically unbiased and that we can estimate correctly the parameters of the drift in the averaged model from a single path of the slow variable $x(t)$. On the other hand, we show rigorously that the maximum likelihood estimator is asymptotically biased for homogenization problems. In particular, an additional term appears in the likelihood function in the limit of infinite scale separation. We then show that this term vanishes, and hence that the maximum likelihood estimator becomes asymptotically unbiased, provided that we subsample at an appropriate rate.

To be more specific, in this paper we will consider fast/slow systems of SDEs of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x, y) + \alpha_0(x, y)\frac{\mathrm{d}U}{\mathrm{d}t} + \alpha_1(x, y)\frac{\mathrm{d}V}{\mathrm{d}t}, \tag{1.1a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\epsilon}g_0(x, y) + \frac{1}{\sqrt{\epsilon}}\beta(x, y)\frac{\mathrm{d}V}{\mathrm{d}t}; \tag{1.1b}$$

with $t \in [0, T]$ or the SDEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\epsilon}f_0(x, y) + f_1(x, y) + \alpha_0(x, y)\frac{\mathrm{d}U}{\mathrm{d}t} + \alpha_1(x, y)\frac{\mathrm{d}V}{\mathrm{d}t}, \tag{1.2a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\epsilon^2}g_0(x, y) + \frac{1}{\epsilon}g_1(x, y) + \frac{1}{\epsilon}\beta(x, y)\frac{\mathrm{d}V}{\mathrm{d}t} \tag{1.2b}$$

with $t \in [0, T]$. We will refer to Eqs. (1.1) as the **averaging problem** and to Eqs. (1.2) as the **homogenization problem**. In both cases our assumptions on the coefficients in the SDEs are such that a coarse-grained (averaged or homogenized) equation exists, which is of the form

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X; \theta) + K(X)\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{1.3}$$

The slow variable $x(t)$ converges weakly, in the limit as $\epsilon \to 0$, to $X(t)$, the solution of (1.3). We assume that the vector field $F(X; \theta)$ depends on a set of parameters θ that we want to estimate based on data from either the averaging or the homogenization problem. We suppose that the actual drift compatible with the data is given by $F(X) = F(X; \theta_0)$. We ask whether it is possible to identify the value of the parameter $\theta = \theta_0$, in the large time asymptotic, by finding the *maximum likelihood estimator* (MLE) when using a statistical model of the form (1.3), but given data from (1.1) or (1.2). In this paper we will consider, for simplicity, the case where the state space of the fast/slow system is compact: $(x, y) \in \mathcal{X} \times \mathcal{Y} = \mathbb{T}^\ell \times \mathbb{T}^{d-\ell}$ where $\mathbb{T}^\ell$ denotes the unit torus in ℓ dimensions. Our main results can be stated, informally, as follows.

Theorem 1.1. *Assume that we are given continuous time data. The MLE for the averaging problem (i.e. fitting data from* (1.1a) *to* (1.3)*) is asymptotically unbiased. On the other hand, the MLE for the homogenization problem (i.e. fitting data from* (1.2a) *to* (1.3)*) is asymptotically biased and an explicit formula for the asymptotic error in the likelihood, E_∞, can be obtained.*

Precise statements of the above results can be found in Theorems 3.9, 3.11 and 3.12.

The failure of the MLE when applied to the homogenization problem is due to the presence of high frequency data. Naturally, in order to be able to identify the parameter $\theta = \theta_0$ in (1.3), in the large time asymptotic and using data from (1.2a), subsampling at an appropriate rate is necessary.

Theorem 1.2. *The MLE for the homogenization problem becomes asymptotically unbiased if we subsample at an appropriate rate.*

Roughly speaking, the sampling rate should be between the two characteristic time scales of the fast/slow SDEs (1.2), 1 and ϵ^2. The precise statement of this result can be found in Theorems 4.1 and 4.5. In practice real data will not come explicitly from a scale-separated model like (1.1a) or (1.2a). However real data is often multiscale in character. Thus the results in this paper shed light on the pitfalls that may arise when fitting simplified statistical models to multiscale data. Furthermore the results indicate the central, and subtle, role played by subsampling data in order to overcome mismatch between model and data at small scales.

The rest of the paper is organized as follows. In Section 2 we study the fast/slow stochastic systems introduced above, and prove appropriate averaging and homogenization theorems. In Section 3 we introduce the maximum likelihood function for (1.3) and study its limiting behavior, given data from the averaging and homogenization problems (1.1a) and (1.2a). In Section 4 we show that, when subsampling at an appropriate rate, the maximum likelihood estimator for the homogenization problem becomes asymptotically unbiased. In Section 5 we present examples of fast/slow stochastic systems that fit into the general framework of this paper. Section 6 is reserved for conclusions. Various technical results are proved in the Appendix.

2. Set-up

In this section we describe the basic framework for averaging and homogenization in SDEs. We consider fast/slow systems of SDEs for the variables $\mathcal{X} \times \mathcal{Y} = \mathbb{T}^l \times \mathbb{T}^{d-l}$, where $\mathbb{T}^d$ is the unit torus in d dimensions, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The choice of a compact state space for the fast/slow system considerably simplifies the analysis.[1] In Section 6 we comment on how the proofs of our results have to be modified in order to cover the case where the state space is not compact.

Let $\varphi_t^{y,\xi}$ denote the Markov process which solves the SDE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_t^{y,\xi}\right) = g_0\left(\xi, \varphi_t^{y,\xi}\right) + \beta\left(\xi, \varphi_t^{y,\xi}\right)\frac{\mathrm{d}V}{\mathrm{d}t}, \quad \varphi_0^{y,\xi} = y. \tag{2.1}$$

Here $\xi \in \mathcal{X}$ is a fixed parameter and, for each $t \geq 0$, $\varphi_\xi^t(y) \in \mathcal{Y}$, $g_0 : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{d-l}$, $\beta : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{(d-l)\times m}$ and V is a standard Brownian motion in m dimensions.[2] The generator of the process is

$$\mathcal{L}_0(\xi) = g_0(\xi, y) \cdot \nabla_y + \frac{1}{2}B(\xi, y) : \nabla_y \nabla_y \tag{2.2}$$

with $B(\xi, y) := \beta(\xi, y)\beta(\xi, y)'$ where $'$ denotes the transpose and $:$ denotes the matrix inner product:

$$B(\xi, y) : \nabla_y \nabla_y := \sum_{i,j} B_{ij}(\xi, y)\frac{\partial^2}{\partial y_i \partial y_j}.$$

Notice that $\mathcal{L}_0(\xi)$ is a differential operator in y alone, with ξ a parameter.

Our interest is in data generated by the projection onto the x coordinate of systems of SDEs for (x, y) in $\mathcal{X} \times \mathcal{Y}$. In particular, for U a standard Brownian motion in $\mathbb{R}^n$ we will consider

[1] In order to define diffusion processes on the torus, one can consider the drift as 1-periodic in all directions, take the diffusion on $\mathbb{R}^d$ and then look at its values mod $\mathbb{Z}^d$, i.e. on $\mathbb{T}^d$. See, for example, [4,8,26,6]. Alternatively, one can define a diffusion process on the torus as the Markov process with generator $\mathcal{L}$, equipped with periodic boundary conditions. See [31] and the references therein. See also [12].

[2] Throughout this paper we write stochastic differential equations as identities in fully differentiated form, even though Brownian motion is not differentiable. In all cases the identity should be interpreted as holding in integrated form, with the Itô interpretation of the stochastic integral.

either of the following coupled systems of SDEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x, y) + \alpha_0(x, y)\frac{\mathrm{d}U}{\mathrm{d}t} + \alpha_1(x, y)\frac{\mathrm{d}V}{\mathrm{d}t}, \tag{2.3a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\epsilon}g_0(x, y) + \frac{1}{\sqrt{\epsilon}}\beta(x, y)\frac{\mathrm{d}V}{\mathrm{d}t}; \tag{2.3b}$$

or the SDEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\epsilon}f_0(x, y) + f_1(x, y) + \alpha_0(x, y)\frac{\mathrm{d}U}{\mathrm{d}t} + \alpha_1(x, y)\frac{\mathrm{d}V}{\mathrm{d}t}, \tag{2.4a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\epsilon^2}g_0(x, y) + \frac{1}{\epsilon}g_1(x, y) + \frac{1}{\epsilon}\beta(x, y)\frac{\mathrm{d}V}{\mathrm{d}t}. \tag{2.4b}$$

Here $f_i : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^l$, $\alpha_0 : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{l\times n}$, $\alpha_1 : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{l\times m}$, $g_1 : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{d-l}$ and g_0, β and V are as above.

Assumptions 2.1. Consider Eqs. (2.3) and (2.4) on $\mathcal{X} \times \mathcal{Y} = \mathbb{T}^\ell \times \mathbb{T}^{d-\ell}$.

- All coefficients in equations (2.3) and (2.4) are smooth in both x and y.
- The matrix $B(\xi, y) = \beta(\xi, y)\beta(\xi, y)'$ is positive definite, uniformly in ξ and y: There exists a constant $C > 0$ such that

  $$\langle a, B(\xi, y)a\rangle \geq C|a|^2, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y},\ a \in \mathbb{R}^{d-\ell},$$

  where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product.

Under Assumptions 2.1 we can prove the following result. The proof is based on standard results from the theory of elliptic PDEs in bounded domains. We refer to [37, Ch. 7] and [14, Ch. 6].

Theorem 2.2. *Let Assumptions 2.1 hold. Then*

- *The equation*

  $$-\mathcal{L}_0^*(\xi)\rho(y;\xi) = 0, \qquad \int_{\mathcal{Y}} \rho(y;\xi)\mathrm{d}y = 1,$$

  where $\mathcal{L}_0^*(\xi)$ denotes the L^2-adjoint of $\mathcal{L}_0(\xi)$ (i.e. the Fokker–Planck operator) has a unique non-negative solution $\rho(y;\xi) \in L^1(\mathcal{Y})$ for every $\xi \in \mathcal{X}$; furthermore $\rho(y;\xi)$ is C^∞ in y and ξ.
- *For each $\xi \in \mathcal{X}$ define the weighted Hilbert space $L^2_\rho(\mathcal{Y};\xi)$ with inner product*

  $$\langle a, b\rangle_\rho := \int_{\mathcal{Y}} \rho(y;\xi)a(y)b(y)\mathrm{d}y.$$

  Assume that the function $h(y;\xi)$ is smooth in both of its arguments. Then for all $\xi \in \mathcal{X}$ the Poisson equation

  $$-\mathcal{L}_0(\xi)\Theta(y;\xi) = h(y;\xi), \qquad \int_{\mathcal{Y}} \rho(y;\xi)\Theta(y;\xi)\mathrm{d}y = 0 \tag{2.5}$$

  has a unique solution $\Theta(y;\xi) \in W^{2,p}_\rho(\mathcal{Y};\xi)$, provided that

  $$\int_{\mathcal{Y}} \rho(y;\xi)h(y;\xi)\mathrm{d}y = 0.$$

- *Θ, the solution of (2.5) is smooth in ξ. In particular, there exists a constant $C > 0$ so that*

$$\|\nabla_\xi \Theta(y;\xi)\|_{L^p} \le C, \qquad \|D_\xi^2 \Theta\|_{L^p} \le C \tag{2.6}$$

for every $p \ge 1$, where $D_\xi^2 \Theta$ denotes the Hessian matrix of $\Theta(y;\xi)$ with respect to ξ and $\|\cdot\|_{L^p}$ denotes the $L^p_\rho(\mathcal{Y};\xi)$ norm.

The first part of Theorem 2.2 essentially states that the process (2.1) is ergodic, for each $\xi \in \mathcal{X}$. Let $\mathcal{L}_0 = \mathcal{L}_0(x)$ and define

$$\mathcal{L}_1 = f_0 \cdot \nabla_x + g_1 \cdot \nabla_y + C : \nabla_y \nabla_x,$$
$$\mathcal{L}_2 = f_1 \cdot \nabla_x + \frac{1}{2} A : \nabla_x \nabla_x,$$

where

$$A(x,y) = \alpha_0(x,y)\alpha_0(x,y)' + \alpha_1(x,y)\alpha_1(x,y)',$$
$$C(x,y) = \alpha_1(x,y)\beta(x,y)'.$$

The generators for the Markov processes defined by Eqs. (2.3) and (2.4) respectively are

$$\mathcal{L}_{av} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2, \tag{2.7}$$

$$\mathcal{L}_{hom} = \frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1 + \mathcal{L}_2, \tag{2.8}$$

with the understanding that $f_0 \equiv 0$ and $g_1 \equiv 0$ in the case of $\mathcal{L}_{av}$. We let Ω denote the probability space for the pair of Brownian motions U, V.

In (2.3) (resp. (2.4)) the dynamics for y with x viewed as frozen has solution $\varphi_{t/\epsilon}^{y(0),x}$ (resp. $\varphi_{t/\epsilon^2}^{y(0),x}$). Of course x is not frozen, but since it evolves much more slowly than y, intuition based on freezing x and considering the process (2.1) is useful in understanding how averaging and homogenization arise for Eqs. (2.3) and (2.4) respectively. Specifically, for (2.3) on time scales long compared with ϵ and short compared to 1, x will be approximately frozen and y will traverse its invariant measure with density $\rho(y;x)$. We may thus average over this measure and eliminate y. Similar ideas hold for Eq. (2.4), but are complicated by the presence of the term $\epsilon^{-1} f_0$. These ideas underlie the averaging and homogenization results contained in the next two subsections. The averaging result is essentially the classical Bogolubov averaging principle (see [20]) whilst the homogenization result is a variant of the results contained in, for example, the works of Papanicolaou and coworkers [29,28,27,30]. We include proofs of the results because we then employ the basic methodology, based around repeated use of application of the Itô formula to Poisson equations, when studying parameter estimation problems in subsequent sections. However we emphasize that the results in this section are classical and well known. In particular, proofs of ergodic theorems for diffusion processes on the torus (or, more generally, on compact state spaces) can be found in many papers and textbooks. See for instance [6, Ch. 3], [37, Ch. 6], [4,7,5,31,16,8,26,13]. In these references both PDE-based proofs (based on proving that the generator of the process has a spectral gap in the appropriate weighted L^2-space) or probabilistic techniques (based on Markov chain techniques and coupling arguments) can be found.

2.1. Averaging

Define $F : \mathcal{X} \to \mathbb{R}^l$ and $K : \mathcal{X} \to \mathbb{R}^{l\times l}$ by

$$F(x) := \int_{\mathcal{Y}} f_1(x, y)\rho(y; x)\mathrm{d}y$$

and

$$K(x)K(x)' := \int_{\mathcal{Y}} \big(\alpha_0(x, y)\alpha_0(x, y)' + \alpha_1(x, y)\alpha_1(x, y)'\big)\rho(y; x)\mathrm{d}y.$$

Note that $K(x)K(x)'$ is positive semidefinite and hence $K(x)$ is well defined via, for example, the Cholesky decomposition.

Theorem 2.3. *Let Assumptions 2.1 hold and let $x(0) = X(0)$. Then $x \Rightarrow X$ in $C([0, T], \mathcal{X})$ and X solves the SDE*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X) + K(X)\frac{\mathrm{d}W}{\mathrm{d}t}, \tag{2.9}$$

where W is a standard l-dimensional Brownian motion.

We use the notation Ω_0 to denote the probability space for the Brownian motion W.

Proof. Consider the Poisson equation

$$-\mathcal{L}_0\Xi(y; x) = f_1(x, y) - F(x), \quad \int_{\mathcal{Y}} \rho(y; x)\Xi(y; x)\mathrm{d}y = 0$$

with unique solution $\Xi(y; x) \in W^{2,p}_\rho(\mathcal{Y}; x)$. Applying Itô's formula to Ξ we obtain

$$\frac{\mathrm{d}\Xi}{\mathrm{d}t} = \frac{1}{\epsilon}\mathcal{L}_0\Xi + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1\Xi + \mathcal{L}_2\Xi + \frac{1}{\sqrt{\epsilon}}\nabla_y\Xi\beta\frac{\mathrm{d}V}{\mathrm{d}t} + \nabla_x\Xi\alpha_0\frac{\mathrm{d}U}{\mathrm{d}t} + \nabla_y\Xi\alpha_1\frac{\mathrm{d}V}{\mathrm{d}t}.$$

From this we obtain

$$\int_0^t \Big(f_1(x(s), y(s)) - F(x(s))\Big)\mathrm{d}s = e_0(t),$$

where

$$e_0(t) = \sqrt{\epsilon}\int_0^t \big(\mathcal{L}_1\Xi\mathrm{d}s + \nabla_y\Xi\beta\mathrm{d}V\big) + \epsilon\int_0^t \big(\mathcal{L}_2\Xi\mathrm{d}s + \nabla_x\Xi\alpha_0\mathrm{d}U + \nabla_y\Xi\alpha_1\mathrm{d}V\big)$$
$$+\,\epsilon\left(\Xi(y(0); x(0)) - \Xi(y(t); x(t))\right).$$

Thus, by Theorem 2.2 (in particular, the uniform L^p-bounds on $\Xi(y; x)$ and its first two derivatives with respect to both x and y) and the Burkholder–Davis–Gundy inequality,

$$\left(\mathbb{E}\sup_{t\in[0,T]}|e_0|^p\right)^{1/p} \le C(\sqrt{\epsilon} + \epsilon).$$

In particular,

$$\lim_{\epsilon\to 0}\|e_0\|_{L^p(L^\infty)} = 0,$$

where we have introduced the notation

$$\| \cdot \|_{L^p(L^\infty)} := \left(\mathbb{E} \sup_{t \in [0,T]} |\cdot|^p \right)^{1/p}.$$

Hence

$$x(t) = x(0) + \int_0^t F(x(s))\mathrm{d}s + M(t) + e_0(t)$$

with

$$M(t) := \int_0^t \alpha_0(x(s), y(s))\mathrm{d}U(s) + \int_0^t \alpha_1(x(s), y(s))\mathrm{d}V(s).$$

The quadratic variation process for $M(t)$ is

$$\langle M \rangle_t = \int_0^t A(x(s), y(s))\mathrm{d}s,$$

where

$$A(x, y) = \alpha_0(x, y)\alpha_0(x, y)' + \alpha_1(x, y)\alpha_1(x, y)'.$$

By use of the Poisson equation technique applied above to show that $f_0(x, y)$ can be approximated by $F(x)$ (its average against the fast y process), we can show similarly that

$$\int_0^t A(x(s), y(s))\mathrm{d}s = \int_0^t K(x(s))K(x(s))'\mathrm{d}s + e_1(t)$$

where, as above,

$$\lim_{\epsilon \to 0} \|e_1\|_{L^p(L^\infty)} = 0.$$

Let

$$B(t) = x(0) + \int_0^t F(x(s))\mathrm{d}s + e_0(t),$$

$$q(t) = \int_0^t K(x(s))K(x(s))'\mathrm{d}s + e_1(t),$$

then

$$x(t) = B(t) + M(t),$$

where $M(t)$ and $M(t)M(t)' - q(t)$ are $\mathcal{F}_t$ martingales, with $\mathcal{F}_t$ the filtration generated by $\sigma((U(s), V(s)), s \leq t)$. Let $C_c^\infty(\mathcal{X})$ denote the space of compactly supported C^∞ functions. The martingale problem for

$$\mathcal{A} = \{(f, K : F \cdot \nabla f + \nabla_x \nabla_x f) : f \in C_c^\infty(\mathcal{X})\}$$

is well posed and $x(s)$, $y(s)$ and $X(s)$ are continuous. By L^2 convergence of the e_i to 0 in $C([0, T], \mathcal{X})$ we deduce convergence to 0 in probability, in the same space. Hence by a slight generalization of Theorem 4.1 in Chapter 7 of [13] we deduce the desired result. $\square$

2.2. Homogenization

In order for Eq. (2.4) to produce a sensible limit as $\epsilon \to 0$ it is necessary to impose a condition on f_0. Specifically we assume the following which, roughly, says that $f_0(x, y)$ averages to zero against the invariant measure of the fast y process, with x fixed. It is sometimes termed the *centering condition.*

Assumption 2.4. The function f_0 satisfies the centering condition

$$\int_{\mathcal{Y}} \rho(y; x) f_0(x, y) \mathrm{d}y = 0.$$

Let $\Phi(y; x) \in L^2_\rho(\mathcal{Y}; x)$ be the solution of the equation

$$-\mathcal{L}_0 \Phi(y; x) = f_0(x, y), \quad \int_{\mathcal{Y}} \rho(y; x) \Phi(y; x) \mathrm{d}y = 0, \tag{2.10}$$

which is unique by Assumption 2.4 and Theorem 2.2. Define

$$\begin{aligned} F_0(x) &:= \int_{\mathcal{Y}} (\mathcal{L}_1 \Phi)(x, y) \rho(y; x) \mathrm{d}y \\ &= \int_{\mathcal{Y}} \Big(\big(\nabla_x \Phi f_0\big)(x, y) + \big(\nabla_y \Phi g_1\big)(x, y) + \big(\alpha_1 \beta' : \nabla_y \nabla_x \Phi\big)(x, y) \Big) \rho(y; x) \mathrm{d}y, \end{aligned}$$

$$F_1(x) := \int_{\mathcal{Y}} f_1(x, y) \rho(y; x) \mathrm{d}y \quad \text{and}$$

$$F(x) = F_0(x) + F_1(x).$$

Also define

$$A_1(x) A_1(x)' := \int_{\mathcal{Y}} \Big(\big(\nabla_y \Phi \beta + \alpha_1\big) \big(\nabla_y \Phi \beta + \alpha_1\big)' \Big)(x, y) \rho(y; x) \mathrm{d}y,$$

$$A_0(x) A_0(x)' := \int_{\mathcal{Y}} \alpha_0(x, y) \alpha_0(x, y)' \rho(y; x) \mathrm{d}y \quad \text{and}$$

$$K(x) K(x)' = A_0(x) A_0(x)' + A_1(x) A_1(x)'.$$

Note that $K(x)K(x)'$ is positive semidefinite by construction so that $K(x)$ is well defined by, for example, the Cholesky decomposition.

Theorem 2.5. *Let Assumptions 2.1, 2.4 hold. Then $x \Rightarrow X$ in $C([0, T], \mathcal{X})$ and X solves the SDE*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X) + A(X) \frac{\mathrm{d}W}{\mathrm{d}t} \tag{2.11}$$

where W is a standard l-dimensional Brownian motion.

Proof. We consider three Poisson equations: that for Φ given above and

$$-\mathcal{L}_0 \chi(y; \xi) = f_1(x, y) - F_1(x), \quad \int_{\mathcal{Y}} \rho(y; x) \chi(y; x) \mathrm{d}y = 0, \tag{2.12a}$$

$$-\mathcal{L}_0 \Psi(y; \xi) = (\mathcal{L}_1 \Phi)(x, y) - F_0(x), \quad \int_{\mathcal{Y}} \rho(y; x) \Psi(y; x) \mathrm{d}y = 0. \tag{2.12b}$$

All of these equations have a unique solution since the right-hand sides average to zero against the density $\rho(y; x)$ by assumption (Φ) or by construction (χ, Ψ).

By the Itô formula we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{1}{\epsilon^2}\mathcal{L}_0\Phi + \frac{1}{\epsilon}\mathcal{L}_1\Phi + \mathcal{L}_2\Phi + \frac{1}{\epsilon}\nabla_y\Phi\beta\frac{\mathrm{d}V}{\mathrm{d}t} + \nabla_x\Phi\alpha_0\frac{\mathrm{d}U}{\mathrm{d}t} + \nabla_x\Phi\alpha_1\frac{\mathrm{d}V}{\mathrm{d}t}.$$

From this we obtain, using arguments similar to those in the proof of Theorem 2.3,

$$\frac{1}{\epsilon}\int_0^t f_0(x,y)\mathrm{d}s = \int_0^t (\mathcal{L}_1\Phi)(x(s),y(s))\mathrm{d}s + \int_0^t (\nabla_y\Phi\beta)(x(s),y(s))\mathrm{d}V(s) + e_0(t)$$

where

$$\lim_{\epsilon\to 0}\|e_0\|_{L^p(L^\infty)} = 0$$

and where, recall, Ω is the probability space for (U, V). Applying Itô's formula to χ, the solution of (2.12a), we may show that

$$\int_0^t \Big(f_1(x(s),y(s)) - F_1(x(s))\Big)\mathrm{d}s = e_1(t)$$

where

$$\lim_{\epsilon\to 0}\|e_1\|_{L^p(L^\infty)} = 0.$$

Thus

$$x(t) = x(0) + \int_0^t \big(\mathcal{L}_1\Phi\big)(x(s),y(s))\mathrm{d}s + \int_0^t F_1(x(s))\mathrm{d}s + \int_0^t \big(\nabla_y\Phi\beta\big)(x(s),y(s))\mathrm{d}V(s)$$
$$+ \int_0^t \alpha_0(x(s),y(s))\mathrm{d}U(s) + \int_0^t \alpha_1(x(s),y(s))\mathrm{d}V(s) + e_2(t)$$

and

$$\lim_{\epsilon\to 0}\|e_2\|_{L^p(L^\infty)} = 0.$$

By applying Itô's formula to Ψ, the solution of (2.12b), we obtain

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{1}{\epsilon^2}\mathcal{L}_0\Psi + \frac{1}{\epsilon}\mathcal{L}_1\Psi + \mathcal{L}_2\Psi + \frac{1}{\epsilon}\nabla_y\Psi\beta\frac{\mathrm{d}V}{\mathrm{d}t} + \nabla_x\Psi\alpha_0\frac{\mathrm{d}U}{\mathrm{d}t} + \nabla_x\Psi\alpha_1\frac{\mathrm{d}V}{\mathrm{d}t}.$$

From this we obtain

$$\int_0^t \Big(\mathcal{L}_1\Phi - F_0\Big)(x,y)\mathrm{d}s = e_3(t)$$

where

$$\lim_{\epsilon\to 0}\|e_3\|_{L^p(L^\infty)} = 0.$$

Thus

$$x(t) = x(0) + \int_0^t F(x(s))\mathrm{d}s + M(t) + e_4(t) \quad \text{and}$$

$$M(t) := \int_0^t \alpha_0(x(s),y(s))\,\mathrm{d}U(s) + \big(\nabla_y\Phi\beta + \alpha_1\big)(x(s),y(s))\,\mathrm{d}V(s).$$

Here

$$\lim_{\epsilon \to 0} \|e_4\|_{L^p(L^\infty)} = 0.$$

Define

$$A_2(x, y) = \big(\nabla_y \Phi\beta + \alpha_1\big)\big(\nabla_y \Phi\beta + \alpha_1\big)'(x, y) + \alpha_0(x, y)\alpha_0(x, y)'.$$

The quadratic variation of $M(t)$ is

$$\langle M\rangle_t = \int_0^t A_2(x(s), y(s))\mathrm{d}s.$$

By use of the Poisson equation technique we can show that

$$\int_0^t A_2(x(s), y(s))\mathrm{d}s = \int_0^t K(x(s))K(x(s))'\mathrm{d}s + e_5(t)$$

where, as above,

$$\lim_{\epsilon \to 0} \|e_5\|_{L^p(L^\infty)} = 0.$$

The remainder of the proof proceeds as in Theorem 2.3. □

3. Parameter estimation

In this section we study parameter estimation problems for data with a multiscale character. Recall that Ω_0 is the probability space for W. Imagine that we try to fit data $\{x(t)\}_{t\in[0,T]}$ from (2.3) or (2.4) to a homogenized or averaged equation of the from (2.9) or (2.11), but with unknown parameter $\theta \in \Theta$, where Θ is an open subset of $\mathbb{R}^k$, in the drift:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X;\theta) + K(X)\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{3.1}$$

Suppose that the actual drift compatible with the data is given by $F(X) = F(X;\theta_0)$. We ask whether it is possible to correctly identify $\theta = \theta_0$ by finding the *maximum likelihood estimator* (MLE) when using a statistical model of the form (3.1), but given data from (2.3) or (2.4). Recall that the averaging and homogenization techniques from the previous section show that $x(t)$ from (2.3) and (2.4) converges weakly to the solution of an equation of the form (3.1). We make the following assumptions concerning the model equations (3.1) which will be used to fit the data.

Assumptions 3.1. We assume that K is uniformly positive definite on $\mathcal{X}$. We also assume that (3.1) is ergodic with invariant measure $\nu(\mathrm{d}x) = \pi(x)\mathrm{d}x$ at $\theta = \theta_0$ and that

$$A_\infty := \int_{\mathcal{X}} \Big(K(x)^{-1}F(x) \otimes K(x)^{-1}F(x)\Big)\,\pi(x)\mathrm{d}x \tag{3.2}$$

is invertible.

Given data $\{z(t)\}_{t\in[0,T]}$, the log-likelihood function for θ satisfying (3.1) is given by

$$\mathbb{L}(\theta; z) = \int_0^{\mathrm{T}} \langle F(z;\theta), \mathrm{d}z\rangle_{a(z)} - \frac{1}{2}\int_0^{\mathrm{T}} |F(z;\theta)|^2_{a(z)}\mathrm{d}t, \tag{3.3}$$

where

$$\langle p, q\rangle_{a(z)} = \langle K(z)^{-1}p, K(z)^{-1}q\rangle.$$

To be precise

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_0} = \exp\left(\mathbb{L}(\theta; X)\right)$$

where $\mathbb{P}$ is the path space measure for (3.1) and $\mathbb{P}_0$ is the path space measure for (3.1) with $F \equiv 0$ [38].[3] The MLE is

$$\hat{\theta} = \arg\max_{\theta} \mathbb{L}(\theta; z). \tag{3.4}$$

As preliminary to understanding the effect of using multiscale data, we start by exhibiting an underlying property of the log likelihood when confronted with data from the model (3.1) itself. The following theorem shows that, in this case: (i) in the limit $T \to \infty$ the log likelihood is asymptotically independent of the particular sample path of (3.1) chosen — it depends only on the invariant measure π; (ii) as a consequence we see that, asymptotically, time-ordering of the data is irrelevant to parameter estimation; (iii) under some additional assumptions, the large T expression also shows that choosing data from the model (3.1) leads to the correct estimation of drift parameters, in the limit $T \to \infty$.

Theorem 3.2. *Let Assumptions 3.1 hold and let $\{X(t)\}_{t\in[0,T]}$ be a sample path of (3.1) with $\theta = \theta_0$. Then, in $L^2(\Omega_0)$ and almost surely with respect to $X(0)$,*

$$\lim_{T\to\infty} \frac{2}{T}\mathbb{L}(\theta; X) = \int_{\mathcal{X}} |F(X;\theta_0)|^2_{a(X)}\pi(X)\mathrm{d}X - \int_{\mathcal{X}} |F(X;\theta) - F(X;\theta_0)|^2_{a(X)}\pi(X)\mathrm{d}X.$$

This expression is maximized by choosing $\hat{\theta} = \theta_0$, in the limit $T \to \infty$.

Proof. By Lemmas A.2 and A.3 in the Appendix we deduce that, with all limits in $L^2(\Omega)$,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\mathbb{L}(\theta; X) &= \lim_{T\to\infty}\Big(\frac{1}{T}\int_0^{\mathrm{T}} \langle F(X;\theta), F(X;\theta_0)\rangle_{a(X)}\mathrm{d}t \\ &\quad + \frac{1}{T}\int_0^{\mathrm{T}} \langle F(X;\theta), K(X)\mathrm{d}W\rangle_{a(X)}\mathrm{d}t - \frac{1}{2T}\int_0^{\mathrm{T}} |F(X;\theta)|^2_{a(X)}\mathrm{d}t\Big) \\ &= \int_{\mathcal{X}} \langle F(X;\theta), F(X;\theta_0)\rangle_{a(X)}\pi(X)\mathrm{d}X - \frac{1}{2}\int_{\mathcal{X}} |F(X;\theta)|^2_{a(X)}\pi(X)\mathrm{d}X.\end{aligned}$$

Completing the square provides the proof. □

In the particular case where the parameter θ appears linearly in the drift it can be viewed as an $\mathbb{R}^{l\times l}$ matrix Θ and

$$F(X;\theta) = \Theta F(X). \tag{3.5}$$

The correct value for Θ is thus the $\mathbb{R}^{l\times l}$ identity matrix I. The maximum likelihood estimator is

$$\hat{\Theta}(z; T) = A(z; T)^{-1}B(z; T) \tag{3.6}$$

[3] For a proof of Girsanov's theorem for diffusion processes on manifolds, including the unit torus, see [12, Ch. iX].

where

$$A(z;T) = \frac{1}{T}\int_0^{\mathrm{T}} K(z)^{-1}F(z) \otimes K(z)^{-1}F(z)\mathrm{d}t,$$
$$B(z;T) = \frac{1}{T}\int_0^{\mathrm{T}} K(z)^{-1}\mathrm{d}z \otimes K(z)^{-1}F(z);$$

if $A(z;T)$ is not invertible then we set $\hat{\Theta}(z;T) = 0$. A result closely related to Theorem 3.2 is the following [4]:

Theorem 3.3. *Let Assumptions 3.1 hold and let $\{X(t)\}_{t\in[0,T]}$ be a sample path of (3.1) with $\theta = \theta_0$ so that $F(X;\theta) = F(X)$. Then*

$$\lim_{T\to\infty} \hat{\Theta}(X;T) = I$$

in probability.

Proof. We observe that

$$B(X;T) = A(X;T) + J_1$$

where

$$J_1 = \frac{1}{T}\int_0^{\mathrm{T}} \mathrm{d}W \otimes K(X)^{-1}F(X)$$

and where $\mathbb{E}|J_1|^2 = \mathcal{O}(1/T)$ by Lemma A.2. By ergodicity, and Lemma A.3, we have that

$$A(X;T) = A_\infty + J_2$$

where $\mathbb{E}|J_2|^2 = \mathcal{O}(1/T)$ and A_∞ is given by (3.2). By Assumptions 3.1 and for T sufficiently large, $A(z;T)$ is invertible and we have

$$\hat{\Theta}(X;T) = I + (A_\infty + J_2)^{-1}J_1$$

and the result follows. $\square$

Remark 3.4. The invertibility of A_∞ is necessary in order to be able to successfully estimate the drift of the linear system.

In order to prove an analogue of Theorem 3.3 when the drift depends nonlinearly on the parameter θ we need to make additional assumptions.

Assumptions 3.5.
- We assume that

$$\inf_{|u|>\delta}\int_{\mathcal{X}} |F(X;\theta_0+u) - F(X;\theta_0)|^2_{a(X)}\pi(X)\mathrm{d}X > \kappa(\delta) > 0, \quad \forall\delta > 0. \tag{3.7}$$

  When (3.7) holds we will say that the system is identifiable.
- There exist an $\alpha > 0$ and $\hat{F} : \mathcal{X} \to \mathbb{R}$, square integrable with respect to the invariant measure, i.e. $\int_{\mathcal{X}} \hat{F}(X)^2\pi(X)\mathrm{d}X < \infty$, such that

$$|F(X;\theta) - F(X;\theta')|_{a(X)} \le |\theta-\theta'|^\alpha \hat{F}(X). \tag{3.8}$$

[4] The proof is standard and we outline it only for comparison with the situation in the next subsection where data from a multiscale model is employed.

Under the above assumption we can prove convergence of the MLE to the correct value θ_0.

Theorem 3.6. *Suppose that Assumptions 3.1 and 3.5 hold. If, in addition, the parameter space Θ is compact, then*

$$\lim_{T\to\infty} \hat{\theta}(X;T) = \theta_0$$

in probability.

Proof. This is a straightforward application of the results in [39]. □

We now ask whether the likelihood behaves similarly when confronted with data $\{x(t)\}$ from the underlying multiscale systems (2.3) or (2.4). To address this issue we make use of the properties of the invariant measure for these underlying multiscale systems. We will need to assume that the fast/slow system is uniformly elliptic.

Assumption 3.7. Define the matrix field $\Sigma = \gamma\gamma^{\mathrm{T}}$ where

$$\gamma = \begin{pmatrix} \alpha_0 & \alpha_1 \\ 0 & \frac{1}{\epsilon}\beta \end{pmatrix}.$$

Assume that there exists a constant $C_\gamma > 0$, independent of $\epsilon \to 0$ such that

$$\langle \xi, \Sigma(x,y)\xi \rangle \geq C_\gamma |\xi|^2 \quad \forall (x,y) \in \mathcal{X} \times \mathcal{Y}, \xi \in \mathbb{R}^d.$$

The proof of the theorem below follows from properties of periodic functions. e.g [9], and from elliptic PDEs theory in bounded domains, e.g. [37, Ch. 7], [14, Ch. 6]. Related results were proved in [36].

Theorem 3.8. *Let Assumptions 2.1 and 3.7 hold. Then.*

- *The fast/slow SDE (2.3) (resp. (2.4)) is ergodic with invariant measure $\mu^\epsilon(\mathrm{d}x\mathrm{d}y)$ which is absolutely continuous with respect to the Lebesgue measure on $\mathcal{X} \times \mathcal{Y}$ with smooth density $\rho^\epsilon(x,y)$.*
- *The limiting SDE (2.9) or (2.11) is ergodic with invariant measure $\nu(\mathrm{d}x)$ which is absolutely continuous with respect to the Lebesgue measure on $\mathcal{X}$ with smooth density $\pi(x)$.*
- *The measure $\mu^\epsilon(\mathrm{d}x\mathrm{d}y) = \rho^\epsilon(x,y)\mathrm{d}x\mathrm{d}y$ converges weakly to the measure $\mu(\mathrm{d}x\mathrm{d}y) = \pi(x)\rho(y;x)\mathrm{d}x\mathrm{d}y$ where $\rho(y;x)$ is the invariant density of the fast process (2.1) given in Assumptions 2.1 and $\pi(x)$ is the invariant density for (2.9) (resp. (2.11)).*
- *The invariant measure $\mu^\epsilon(\mathrm{d}x\mathrm{d}y) = \rho^\epsilon(x,y)\mathrm{d}x\mathrm{d}y$ satisfies a Poincaré inequality with a constant independent of ϵ: there exists a constant C_p independent of ϵ such that for every mean zero $H^1(\mathcal{X} \times \mathcal{Y}; \mu^\epsilon(\mathrm{d}x\mathrm{d}y))$ function f we have that*

  $$\|f\| \leq C_p \|\nabla f\| \tag{3.9}$$

  where ∇ represents the gradient with respect to $(x', y')'$ and $\|\cdot\|$ denotes the $L^2(\mathcal{X} \times \mathcal{Y}; \mu^\epsilon(\mathrm{d}x\mathrm{d}y))$ norm.

3.1. Averaging

In the limit $\epsilon \to 0$, $X(t)$ from (3.1) approximates $x(t)$ from (2.3), in the sense of weak convergence. It is thus natural to ask what happens when the MLE for the averaged equation (3.1) is confronted with data from the original multiscale equation (2.3). The following result shows that, in this case, the estimator will behave well, for large time and small ϵ. Large time is always required for convergence of drift parameter estimation, even when model and data match (see Theorem 3.2).

Theorem 3.9. *Let Assumptions 2.1, 3.1 and 3.7 hold. Let $\{x(t)\}_{t\in[0,T]}$ be a sample path of (2.3) and $\{X(t)\}_{t\in[0,T]}$ a sample path of (3.1) at $\theta = \theta_0$. Then the following limits, to be interpreted in $L^2(\Omega)$ and $L^2(\Omega_0)$ respectively, and almost surely with respect to $x(0), y(0), X(0)$, are identical:*

$$\lim_{\epsilon\to 0}\lim_{T\to\infty}\frac{1}{T}\mathbb{L}(\theta;x) = \lim_{T\to\infty}\frac{1}{T}\mathbb{L}(\theta;X).$$

Proof. We start by observing that, by Lemma A.3 and Theorem 3.8,

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{T\to\infty}\frac{1}{T}\int_0^{\mathrm{T}} |F(x;\theta)|^2_{a(x)}\mathrm{d}t &= \lim_{\epsilon\to 0}\int_{\mathcal{X}\times\mathcal{Y}} |F(x;\theta)|^2_{a(x)}\rho^{\epsilon}(x,y)\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathcal{X}\times\mathcal{Y}} |F(x;\theta)|^2_{a(x)}\pi(x)\rho(y;x)\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathcal{X}} |F(x;\theta)|^2_{a(x)}\pi(x)\mathrm{d}x,\end{aligned}$$

where the limits are in $L^2(\Omega)$. Now, from Eq. (2.3) it follows that

$$\begin{aligned}&\frac{1}{T}\int_0^{\mathrm{T}}\langle F(x;\theta),\mathrm{d}x\rangle_{a(x)} = \frac{1}{T}\int_0^{\mathrm{T}}\langle F(x;\theta), f_1(x,y)\rangle_{a(x)}\mathrm{d}t\\ &\quad+\frac{1}{T}\int_0^{\mathrm{T}}\langle F(x;\theta),\alpha_0(x,y)\mathrm{d}U\rangle_{a(x)} + \frac{1}{T}\int_0^{\mathrm{T}}\langle F(x;\theta),\alpha_1(x,y)\mathrm{d}V\rangle_{a(x)}.\end{aligned}$$

The last two integrals tend to zero in $L^2(\Omega)$ as $T\to\infty$ by Lemma A.2. In order to analyze the first integral on the right-hand side we consider solution of the Poisson equation

$$-\mathcal{L}_0\Lambda = \langle F(x;\theta), f_1(x,y) - F(x;\theta_0)\rangle_{a(x)},\qquad \int_{\mathcal{Y}}\rho(y;\xi)\Lambda(y)\mathrm{d}y = 0.$$

This has a unique solution $\Lambda(y;x)\in L^2_\rho(\mathcal{Y};x)$ by construction of F.

Applying Itô's formula to Λ gives

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \frac{1}{\epsilon}\mathcal{L}_0\Lambda + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1\Lambda + \mathcal{L}_2\Lambda + \frac{1}{\sqrt{\epsilon}}\nabla_y\Lambda\beta\frac{\mathrm{d}V}{\mathrm{d}t} + \nabla_x\Lambda\alpha_0\frac{\mathrm{d}U}{\mathrm{d}t} + \nabla_x\Lambda\alpha_1\frac{\mathrm{d}V}{\mathrm{d}t}$$

which shows that

$$\begin{aligned}&\frac{1}{T}\int_0^{\mathrm{T}}\langle F(x;\theta), f_1(x,y)\rangle_{a(x)}\mathrm{d}t = \frac{1}{T}\int_0^{\mathrm{T}}\langle F(x;\theta), F(x;\theta_0)\rangle_{a(x)}\mathrm{d}t\\ &\quad+\frac{\epsilon}{T}\int_0^{\mathrm{T}}\big(\mathcal{L}_2\Lambda\big)(x(t),y(t))\mathrm{d}t - \frac{\epsilon}{T}\Big(\Lambda(x(T),y(T)) - \Lambda(x(0),y(0))\Big)\end{aligned}$$

$$+\frac{1}{T}\int_0^{\mathrm{T}}\sqrt{\epsilon}\Big((\nabla_y\Lambda\beta)(x(t),y(t))\mathrm{d}V(t)+(\mathcal{L}_1\Lambda)\,(x(t),y(t))\mathrm{d}t\Big)$$
$$+\frac{1}{T}\int_0^{\mathrm{T}}\epsilon\Big((\nabla_x\Lambda\alpha_0)(x(t),y(t))\mathrm{d}U(t)+(\nabla_y\Lambda\alpha_1)(x(t),y(t))\mathrm{d}V(t)\Big).$$

The stochastic integrals tend to zero in $L^2(\Omega)$ as $T\to\infty$. By Theorem 2.2 Λ is bounded. Furthermore, in $L^2(\Omega)$,

$$\frac{1}{T}\int_0^{\mathrm{T}}\big(\mathcal{L}_i\Lambda\big)(x(t),y(t))\mathrm{d}t\to\int_{\mathcal{X}\times\mathcal{Y}}\big(\mathcal{L}_i\Lambda\big)(x,y)\rho(y;x)\mathrm{d}y,\quad i=1,2.$$

Hence we deduce that

$$\lim_{\epsilon\to0}\lim_{T\to\infty}\frac{1}{T}\int_0^{\mathrm{T}}\langle F(x;\theta),f_1(x,y)\rangle_{a(x)}\mathrm{d}t=\lim_{\epsilon\to0}\lim_{T\to\infty}\frac{1}{T}\int_0^{\mathrm{T}}\langle F(x;\theta),F(x;\theta_0)\rangle_{a(x)}\mathrm{d}t$$
$$=\lim_{\epsilon\to0}\int_{\mathcal{X}\times\mathcal{Y}}\langle F(x;\theta),F(x;\theta_0)\rangle_{a(x)}\rho^{\epsilon}(x,y)\mathrm{d}x\mathrm{d}y$$
$$=\int_{\mathcal{X}}\langle F(x;\theta),F(x;\theta_0)\rangle\pi(x)\mathrm{d}x.$$

The result follows. □

In the particular case of linear parameter dependence, when the MLE is given by (3.6) we have the following result, showing that the MLE recovers the correct answer from high frequency data compatible with the statistical model in an appropriate asymptotic limit.

Theorem 3.10. *Let Assumptions 2.1, 3.1 and 3.7 hold. Assume that $F(X;\theta)$ is given by (3.5). Let $\{x(t)\}_{t\in[0,T]}$ be a sample path of (2.3). Then $\hat{\Theta}$ given by (3.6) satisfies*

$$\lim_{T\to\infty}\hat{\Theta}(x;T)=\Theta_\epsilon\quad\textit{in probability}$$

for some Θ_ϵ to be identified in the proof and

$$\lim_{\epsilon\to0}\Theta_\epsilon=I.$$

Proof. Using Eq. (2.3) we find that

$$B(x;T)=A(x;T)+C_\epsilon+(J_3-C_\epsilon)+J_4,\quad\text{where}$$
$$C_\epsilon=\int_{\mathcal{X}\times\mathcal{Y}}K(x)^{-1}\left(f_1(x,y)-F(x)\right)\otimes K(x)^{-1}F(x)\rho^{\epsilon}(x,y)\mathrm{d}x\mathrm{d}y,$$
$$J_3=\frac{1}{T}\int_0^{\mathrm{T}}K(x)^{-1}\left(f_1(x,y)-F(x)\right)\otimes K(x)^{-1}F(x)\mathrm{d}t,$$
$$J_4=\frac{1}{T}\int_0^{\mathrm{T}}K(x)^{-1}\left(\alpha_0(x,y)\mathrm{d}U+\alpha_1(x,y)\mathrm{d}V\right)\otimes K(x)^{-1}F(x).$$

Here, for fixed $\epsilon>0$, $\mathbb{E}|J_4|^2=\mathcal{O}(1/T)$ by Lemma A.2 and $\mathbb{E}|J_3-C_\epsilon|^2=\mathcal{O}(1/T)$ by ergodicity and Lemma A.3. Also

$$A(x;T)=A_{\infty,\epsilon}+J_5$$

where

$$A_{\infty,\epsilon} := \int_{\mathcal{X}\times\mathcal{Y}} \left(K(x)^{-1}F(x) \otimes K(x)^{-1}F(x)\right) \rho^{\epsilon}(x,y)\mathrm{d}x\mathrm{d}y,$$

with $\mathbb{E}|J_5|^2 = \mathcal{O}(1/T)$, again by ergodicity and Lemma A.3. Thus,

$$\hat{\Theta}(X;T) = I + (A_{\infty,\epsilon} + J_5)^{-1}\left(C_\epsilon + (J_3 - C_\epsilon) + J_4\right),$$

which converges to

$$\lim_{T\to\infty} \hat{\Theta}(X;T) = I + (A_{\infty,\epsilon})^{-1}(C_\epsilon)$$

in probability. If $A_{\infty,\epsilon}$ is not invertible, we set the limit equal to zero.

By Assumptions 3.1 $A_{\infty,\epsilon}$ is invertible for ϵ sufficiently small, and by the weak convergence of the measures with density ρ^{ϵ},

$$\lim_{\epsilon\to 0} A_{\infty,\epsilon} = A_\infty$$

and

$$\lim_{\epsilon\to 0} C_\epsilon = 0.$$

The result follows. □

We would like to show that this also holds for the general case of nonlinear parameter dependence. Let

$$\hat{\theta}(x;T) := \arg\max_{\theta} \mathbb{L}(\theta;x)$$

be the set of maximizers of $\mathbb{L}(\theta;x)$. We show that the following holds:

Theorem 3.11. *Let Assumptions 2.1, 3.1, 3.5 and 3.7 hold and assume that $\theta \in \Theta$, a compact set. Let $\{x(t)\}_{t\in[0,T]}$ be a sample path of (2.3) at $\theta = \theta_0$. Then,*

$$\lim_{T\to\infty} dist\left(\hat{\theta}(x;T), \theta_\epsilon\right) = 0 \quad \textit{in probability},$$

where $dist$ is the asymmetric Hausdorff semi-distance and θ_ϵ a subset of the parameter space that will be identified in the proof. Also

$$\lim_{\epsilon\to 0} d_H(\theta_\epsilon, \theta_0) = 0$$

where d_H is the Hausdorff distance.

Proof. We set $g_{\epsilon,T}(x,\theta) := \frac{1}{T}\mathbb{L}(\theta;x)$ and

$$g_\epsilon(\theta) := \lim_{T\to\infty} \frac{1}{T}\mathbb{L}(\theta;x).$$

This limit exists (in $L_2(\Omega)$), as a result of the ergodicity of the multiscale system. In fact, looking at the proof of Theorem 3.9, one can see that it will be equal to

$$\int_{\mathcal{X}\times\mathcal{Y}} \left(\langle F(x;\theta), f_1(x,y)\rangle_{a(x)} - \frac{1}{2}|F(x;\theta)|^2_{a(x)}\right) \rho^{\epsilon}(x,y)\mathrm{d}x\mathrm{d}y$$
$$= \int_{\mathcal{X}\times\mathcal{Y}} \left(\langle F(x;\theta), F(x;\theta_0)\rangle_{a(x)} - \frac{1}{2}|F(x;\theta)|^2_{a(x)} + \sqrt{\epsilon}\big(\mathcal{L}_1 \Lambda\big)(x,y)\right.$$

$$+ \epsilon\big(\mathcal{L}_2\Lambda\big)(x,y)\big)\,\rho^{\epsilon}(x,y)\mathrm{d}x\mathrm{d}y$$

for Λ as defined in that proof. We also set $g_0(\theta) = \lim_{\epsilon\to 0} g_\epsilon(\theta)$. We have seen that θ_0 maximizes g_0 and by definition $\hat{\theta}(x;T)$ maximizes $g_{\epsilon,T}(x;\theta)$.

Let us define the equivalence relation $\sim$ by

$$\theta_1 \sim \theta_2 \iff g_\epsilon(\theta_1) = g_\epsilon(\theta_2).$$

Let $\tilde{\Theta} = (\Theta/\sim)$ be the corresponding quotient space and define

$$\hat{\theta}_{\epsilon,T}(x) = \arg\max_{\theta\in\tilde{\Theta}} g_\epsilon(x;\theta) \quad \text{and} \quad \hat{\theta}_\epsilon = \arg\max_{\theta\in\tilde{\Theta}} g_\epsilon(\theta).$$

We show that $\hat{\theta}_{\epsilon,T}(x) \to \hat{\theta}_\epsilon$ as $T \to \infty$ in probability, by applying Lemma A.4, replacing ϵ by $\frac{1}{T}$, g_ϵ by $g_{\epsilon,T}$ and g_0 by g_ϵ. We need to verify conditions (A.2)–(A.4). Condition (A.2) follows by the construction of g_ϵ as the limit of $g_{\epsilon,T}$. We can show that (A.3) holds by following the arguments in [39] and using the assumption that α_0, α_1 and f_1 are uniformly bounded. Finally, the identifiability condition (A.4) will be satisfied because we are working on the quotient space where the maximizer is unique.

Let θ_ϵ be the equivalence class of $\hat{\theta}_\epsilon$. We have shown that any maximizer of $g_\epsilon(x,\theta)$ will converge in probability to a point in θ_ϵ, which implies the convergence in probability of the Hausdorff semi-distance between $\hat{\theta}(x,T)$ and θ_ϵ to zero.

We will now show that $\theta \to \theta_0$ as $\epsilon \to 0$ for any $\theta \in \theta_\epsilon$. Since Θ is a compact set, this will be true provided that $g_\epsilon(\theta) \to g_0(\theta)$ uniformly with respect to θ. $F(X;\theta)$ is continuous with respect to θ by Assumption (3.8) and consequently Λ is also continuous by construction. Since Θ is a compact set, both F and Λ are bounded with respect to θ. Thus, terms three and four converge uniformly to zero. We can also show that terms one and two converge uniformly to $g_0(\theta)$, using the boundedness of F and the weak convergence of $\int \rho^\epsilon(x,y)\mathrm{d}y\mathrm{d}x$ to $\pi(x)\mathrm{d}x$. The convergence of every point in the set θ_ϵ to θ_0 implies the convergence of the Hausdorff distance between θ_ϵ and θ_0 (or, more precisely, the singleton containing θ_0) to zero. $\square$

3.2. Homogenization

We now ask what happens when the MLE for the homogenized equation (3.1) is confronted with data from the multiscale equation (2.4), which homogenizes to give (3.1). The situation differs substantially from the case where data is taken from the multiscale equations (2.3) which averages to give (3.1): the two likelihoods are not identical in the large T limit.

In order to state the main result of this subsection we need to introduce the Poisson equation

$$-\mathcal{L}_0\Gamma = \langle F(x;\theta), f_0(x,y)\rangle_{a(x)}, \qquad \int_{\mathcal{Y}} \rho(y;\xi)\Gamma(y;x)\mathrm{d}y = 0 \tag{3.10}$$

which has a unique solution $\Gamma(y;x) \in L^2_\rho(\mathcal{Y};x)$. Note that

$$\Gamma = \langle F(x;\theta), \Phi(x,y)\rangle_{a(x)},$$

where Φ solves (2.10). Define

$$E_\infty(\theta) = \int_{\mathcal{X}\times\mathcal{Y}} \Big(\mathcal{L}_1\Gamma(x,y) - \big\langle F(x;\theta), \big(\mathcal{L}_1\Phi(x,y)\big)\big\rangle_{a(x)}\Big)\pi(x)\rho(y;x)\mathrm{d}x\mathrm{d}y. \tag{3.11}$$

Note that Γ depends on θ but that Φ does not.

The following theorem shows that the correct limit of the log likelihood is not obtained unless E_∞ is a constant in θ, something which will not be true in general. However in the case where $f_0, g_1 \equiv 0$ we do obtain $E_\infty = 0$ and in this case we recover the averaging situation covered in the Theorems 2.3 and 3.9 (with ϵ replaced by ϵ^2).

Theorem 3.12. *Let Assumptions 2.1, 2.4, 3.1, 3.7, and (3.8) hold. Let $\{x(t)\}_{t\in[0,T]}$ be a sample path of (2.4) and $\{X(t)\}_{t\in[0,T]}$ a sample path of (3.1) at $\theta = \theta_0$. Then the following limits, to be interpreted in $L^2(\Omega)$ and $L^2(\Omega_0)$ respectively, and almost surely with respect to $x(0), y(0), X(0)$, are identical:*

$$\lim_{\epsilon\to 0}\lim_{T\to\infty}\frac{1}{T}\mathbb{L}(\theta; x) = \lim_{T\to\infty}\frac{1}{T}\mathbb{L}(\theta; X) + E_\infty(\theta).$$

Proof. As in the averaging case of Theorem 3.9 we have

$$\lim_{\epsilon\to 0}\lim_{T\to\infty}\frac{1}{T}\int_0^T |F(x;\theta)|^2_{a(x)}\mathrm{d}t = \int_{\mathcal{X}} |F(x;\theta)|^2_{a(x)}\pi(x)\mathrm{d}x.$$

Now

$$\frac{1}{T}\int_0^T \langle F(x;\theta), \mathrm{d}x\rangle_{a(x)} = I_1 + I_2 + I_3$$

where

$$I_1 = \frac{1}{\epsilon T}\int_0^T \langle F(x;\theta), f_0(x,y)\rangle_{a(x)}\mathrm{d}t,$$
$$I_2 = \frac{1}{T}\int_0^T \langle F(x;\theta), f_1(x,y)\rangle_{a(x)}\mathrm{d}t,$$
$$I_3 = \frac{1}{T}\int_0^T \langle F(x;\theta), \alpha_0(x,y)\mathrm{d}U + \alpha_1(x,y)\mathrm{d}V\rangle_{a(x)}.$$

Now I_3 is $\mathcal{O}(1/\sqrt{T})$ in $L^2(\Omega)$ by Lemma A.2. Techniques similar to those used in the proof of Theorem 3.9 show that, in $L^2(\Omega)$,

$$\lim_{\epsilon\to 0}\lim_{T\to\infty} I_2 \to \int_{\mathcal{X}} \langle F(x;\theta), F_1(x;\theta_0)\rangle_{a(x)}\pi(\mathrm{d}x).$$

Now consider I_1. Applying Itô's formula to the solution Γ of the Poisson equation (3.10), we obtain

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{1}{\epsilon^2}\mathcal{L}_0\Gamma + \frac{1}{\epsilon}\mathcal{L}_1\Gamma + \mathcal{L}_2\Gamma + \frac{1}{\epsilon}\nabla_y\Gamma\beta\frac{\mathrm{d}V}{\mathrm{d}t} + \nabla_x\Gamma\alpha_0\frac{\mathrm{d}U}{\mathrm{d}t} + \nabla_x\Gamma\alpha_1\frac{\mathrm{d}V}{\mathrm{d}t}.$$

From this we deduce that

$$\frac{1}{\epsilon T}\int_0^T \langle F(x;\theta), f_0(x,y)\rangle\mathrm{d}t = \frac{1}{T}\int_0^T \Big(\mathcal{L}_1\Gamma\Big)\mathrm{d}t + I_4$$

where

$$\lim_{\epsilon\to 0}\lim_{T\to\infty} I_4 = 0.$$

Thus

$$I_1 = \frac{1}{\epsilon T}\int_0^{\mathrm{T}} \langle F(x;\theta), f_0(x,y)\rangle \mathrm{d}t = I_4 + I_5 + I_6$$

where, in $L^2(\Omega)$,

$$I_5 = \frac{1}{T}\int_0^{\mathrm{T}} \left\langle F(x;\theta), \left(\mathcal{L}_1\Phi(x,y)\right)\right\rangle_{a(x)} \mathrm{d}t,$$
$$I_6 = \frac{1}{T}\int_0^{\mathrm{T}} \left(\mathcal{L}_1\Gamma(x,y) - \left\langle F(x;\theta), \left(\mathcal{L}_1\Phi(x,y)\right)\right\rangle_{a(x)}\right)\mathrm{d}t.$$

By the methods used in the proof of Theorem 3.9 we deduce that

$$\lim_{\epsilon\to 0}\lim_{T\to\infty} I_5 \to \int_{\mathcal{X}} \langle F(x;\theta), F_0(x;\theta_0)\rangle_{a(x)}\pi(x)\mathrm{d}x.$$

Putting together all the estimates we deduce that, in L^2,

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{T\to\infty}\frac{1}{T}\mathbb{L}(x;\theta) &= \lim_{T\to\infty}\frac{1}{T}\mathbb{L}(X;\theta) + \lim_{\epsilon\to 0}\lim_{T\to\infty} I_6\\ &= \lim_{T\to\infty}\frac{1}{T}\mathbb{L}(X;\theta) + E_\infty(\theta). \quad \square\end{aligned}$$

4. Subsampling

In the previous section we studied the behavior of estimators when confronted with multiscale data. The data is such that, in an appropriate asymptotic limit $\epsilon \to 0$, it behaves weakly as if it comes from a single scale equation in the form of the statistical model. By considering the behavior of continuous time estimators in the limit of large time, followed by taking $\epsilon \to 0$, we studied the behavior of estimators which do not subsample the data. We showed that in the averaging set-up this did not cause a problem — the likelihood behaves as if confronted with data from the statistical model itself; but in the homogenization set-up the likelihood function was asymptotically biased for large time. In this section we show that subsampling the data can overcome this issue, provided the subsampling rate is chosen appropriately.

In the following we use $\mathbb{E}^{\pi}$ to denote expectation on $\mathcal{X}$ with respect to measure with density π and $\mathbb{E}^{\rho^\epsilon}$ to denote expectation on $\mathcal{X}\times\mathcal{Y}$ with respect to measure with density ρ^ϵ. Recall that, by Assumption 3.7 the latter measure has weak limit with density $\pi(x)\rho(y;x)$. Let $\Omega' = \Omega\times\mathcal{X}\times\mathcal{Y}$ and consider the probability measure induced on paths x, y solving (2.4) by choosing initial conditions distributed according to the measure $\pi(x)\rho(y;x)\mathrm{d}x\mathrm{d}y$. With expectation $\mathbb{E}$ under this measure we will also use the notation

$$\|\cdot\|_p := \left(\mathbb{E}|\cdot|^p\right)^{1/p}.$$

We define the discrete log-likelihood function found from applying the likelihood principle to the Euler–Marayama approximation of the statistical model (3.1). Let $z = \{z_n\}_{n=0}^{N-1}$ denote a time series in $\mathcal{X}$. We obtain the likelihood

$$\mathbb{L}^{\delta,N}(\theta;z) = \sum_{n=0}^{N-1}\langle F(z_n;\theta), z_{n+1}-z_n\rangle_{a(z_n)} - \frac{1}{2}\sum_{n=0}^{N-1}|F(z_n;\theta)|^2_{a(z_n)}\delta.$$

Let $x_n = x(n\delta)$, noting that $x(t)$ depends on ϵ, and set $x = \{x_n\}_{n=0}^{N-1}$. The basic theorem in this section proves the convergence of the discrete log-likelihood function, given multiscale data, to the true log likelihood of the homogenized equation. The theory works provided that we subsample (i.e. choose δ) at an appropriate ϵ-dependent rate. We state and prove the theorem, relying on a pair of intuitively reasonable propositions which we then prove at the end of the section.

Theorem 4.1. *Let Assumptions 2.1, 2.4, 3.1, 3.7 and (3.8) hold. Let $\{x(t)\}_{t\in[0,T]}$ be a sample path of (2.4) and $X(t)$ a sample path of (3.1) at $\theta = \theta_0$. Let $\delta = \epsilon^\alpha$ with $\alpha \in (0, 1)$ and let $N = [\epsilon^{-\gamma}]$ with $\gamma > \alpha$. Then the following limits, to be interpreted in $L^2(\Omega')$ and $L^2(\Omega_0)$ respectively, and almost surely with respect to $X(0)$, are identical:*

$$\lim_{\epsilon\to 0} \frac{1}{N\delta}\mathbb{L}^{N,\delta}(\theta; x) = \lim_{T\to\infty} \frac{1}{T}\mathbb{L}(\theta; X). \tag{4.1}$$

The proof of this theorem is based on the following two technical results, whose proofs are presented in the Appendix.

Proposition 4.2. *Let $(x(t),\ y(t))$ be the solution of (2.4) and assume that Assumptions 2.1 and 2.4 hold. Then, for $\epsilon,\ \delta$ sufficiently small, the increment of the process $x(t)$ can be written in the form*

$$x_{n+1} - x_n = F(x_n; \theta_0)\, \delta + M_n + R(\epsilon, \delta), \tag{4.2}$$

where M_n denotes the martingale term

$$M_n = \int_{n\delta}^{(n+1)\delta} \left(\nabla_y \Phi\beta + \alpha_0\right)(x(s), y(s))\mathrm{d}V + \int_{n\delta}^{(n+1)\delta} \alpha_1(x(s), y(s))\mathrm{d}U$$

with $\|M_n\|_p \le C\sqrt{\delta}$ and

$$\|R(\epsilon, \delta)\|_p \le C(\delta^{3/2} + \epsilon\delta^{\frac{1}{2}} + \epsilon).$$

Proposition 4.3. *Let $g \in C^1(\mathcal{X})$ and let Assumption 3.7 hold. Assume that ϵ and N are related as in Theorem 4.1. Then*

$$\lim_{\epsilon\to 0} \frac{1}{N} \sum_{n=0}^{N-1} g(x_n) = \mathbb{E}^\pi g, \tag{4.3}$$

where the convergence is in $L^2(\Omega')$.

Proof of Theorem 4.1. We define

$$I_1(x, \theta) = \sum_{n=0}^{N-1} \langle F(x_n; \theta), x_{n+1} - x_n\rangle_{a(x_n)}$$

and

$$I_2(x) = \frac{1}{2} \sum_{n=0}^{N-1} |F(x_n; \theta)|^2_{a(x_n)}\delta.$$

By Proposition 4.3 we have that

$$\frac{1}{N\delta} I_2(x) \to \frac{1}{2}\int_{\mathcal{X}} |F(x;\theta)|^2_{a(x)} \pi(\mathrm{d}x).$$

We use Proposition 4.2 to deduce that

$$\begin{aligned}
\frac{1}{N\delta} I_1(x;\theta) &= \frac{1}{N\delta}\sum_{n=0}^{N-1} \langle F(x_n;\theta), F(x_n;\theta_0)\delta + M_n + R(\epsilon,\delta)\rangle_{a(x_n)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \langle F(x_n;\theta), F(x_n;\theta_0)\rangle_{a(x_n)} + \frac{1}{N\delta}\sum_{n=0}^{N-1} \langle F(x_n), M_n\rangle_{a(x_n)} \\
&\quad + \frac{1}{N\delta}\sum_{n=0}^{N-1} \langle F(x_n), R(\epsilon,\delta)\rangle_{a(x_n)} \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

Again using Proposition 4.3 we have that

$$J_1 \to \int_{\mathcal{X}} \langle F(x;\theta), F(x;\theta_0)\rangle_{a(x)}\, \pi(\mathrm{d}x).$$

Furthermore, using the fact that M_n is independent of x_n and has quadratic variation of order δ it follows that

$$\begin{aligned}
\|J_2\|_2^2 &\le \frac{1}{N^2\delta^2}\sum_{n=0}^{N-1} \mathbb{E}\left|\langle F(x_n;\theta), M_n\rangle_{a(x_n)}\right|^2 \\
&\le \frac{C}{N\delta}.
\end{aligned}$$

Here Q is defined to obtain the correct quadratic variation of the M_n. Consequently, and since $\gamma > \alpha$,

$$\|J_2\|_2 \le o(1)$$

as $\epsilon \to 0$. Similarly, using martingale moment inequalities [22, Eq. (3.25) p. 163] we obtain

$$\|J_2\|_p \le o(1).$$

Finally, again using Proposition 4.2, we have, for $q^{-1} + p^{-1} = 1$,

$$\begin{aligned}
\|J_3\|_p &\le \frac{1}{N\delta}\sum_{n=0}^{N-1} \|F(x_n)\|_q \|R(\epsilon,\delta)\|_p \le C\frac{1}{N\delta} N\left(\delta^{3/2} + \epsilon + \epsilon\delta^{1/2}\right) \\
&\le o(1),
\end{aligned}$$

as $\epsilon \to 0$, since we have assumed that $\alpha \in (0,1)$.

We thus have

$$\lim_{\epsilon\to 0} \frac{1}{N\delta}\mathbb{L}^{N,\delta}(\theta;x) = \int_{\mathcal{X}} \langle F(x;\theta), F(x;\theta_0)\rangle_{a(x)}\, \pi(x)\mathrm{d}x - \frac{1}{2}\int_{\mathcal{X}} |F(x;\theta)|^2_{a(x)} \pi(x)\mathrm{d}x.$$

By completing the square we obtain (4.1). □

As before, we would like to use this theorem in order to prove the consistency of our estimator. The theory developed in [39] no longer applies because it is based on the assumption that the function we are maximizing (i.e. the log-likelihood function) is a continuous semimartingale, which is not true for the discrete semimartingale $\mathcal{L}^{N,\delta}(\theta; x)$. The most difficult part in proving consistency is to prove that the martingale converges uniformly to zero (Assumption (A.3) in Lemma A.4). To avoid this difficulty, we make some extra assumptions that allow us to get rid of the martingale part:

Assumptions 4.4. 1. There exists a function $V : \mathcal{X} \times \Theta \to \mathbb{R}$ such that for each $\theta \in \Theta$, $V(\cdot, \theta) \in C^3(\mathcal{X})$ and

$$\nabla V(z; \theta) = \left(K(z)K(z)'\right)^{-1} F(z; \theta), \quad \forall z \in \mathcal{X}, \theta \in \Theta. \tag{4.4}$$

2. Define $G : \mathcal{X} \times \Theta \to \mathbb{R}$ as follows:

$$G(z; \theta) := D^2 V(z; \theta) : (K(z)K(z)'),$$

where $D^2 V$ denotes the Hessian matrix of V. Then there exist an $\beta > 0$ and $\hat{G} : \mathcal{X} \to \mathbb{R}$ that is square integrable with respect to the invariant measure, such that

$$|G(z; \theta) - G(z; \theta')| \le |\theta - \theta'|^\beta \hat{G}(z).$$

Suppose that the above assumption is true and $\{X(t)\}_{t\in[0,T]}$ is a sample path of (3.1). Then, if we apply Itô's formula to function V, we get that for every $\theta \in \Theta$:

$$\mathrm{d}V(X(t); \theta) = \langle \nabla V(X(t); \theta), \mathrm{d}X(t)\rangle + \frac{1}{2} G(X(t); \theta)\mathrm{d}t.$$

But from (4.4) we have that

$$\begin{aligned}\langle \nabla V(X(t); \theta), \mathrm{d}X(t)\rangle &= \left\langle \left(K(X(t))K(X(t))'\right)^{-1} F(X(t); \theta), \mathrm{d}X(t)\right\rangle \\ &= \langle F(X(t); \theta), \mathrm{d}X(t)\rangle_{a(X(t))}\end{aligned}$$

and thus

$$\langle F(X(t); \theta), \mathrm{d}X(t)\rangle_{a(X(t))} = \mathrm{d}V(X(t)) - \frac{1}{2} G(X(t); \theta)\mathrm{d}t.$$

Using this identity, we can write the log-likelihood function (3.3) in the form

$$\begin{aligned}\mathbb{L}(\theta; X(t)) = {} & (V(X(T); \theta) - V(X(0); \theta)) \\ & - \frac{1}{2}\int_0^{\mathrm{T}} \left(|F(X(t); \theta)|^2_{a(X(t))} + G(X(t); \theta)\right) \mathrm{d}t.\end{aligned}$$

Using this version of the log-likelihood function, we define

$$\tilde{\mathbb{L}}^{N,\delta}(\theta; z) = -\frac{1}{2} \sum_{n=0}^{N-1} \left(|F(z_n; \theta)|^2_{a(z_n)} + G(z_n; \theta)\right) \delta. \tag{4.5}$$

Now we can prove asymptotic consistency of the MLE, provided that we subsample at the appropriate sampling rate.

Theorem 4.5. *Let Assumptions 2.1, 2.4, 3.1, 3.5, 3.7 and 4.4 hold and assume that $\theta \in \Theta$, a compact set. Let $\{x(t)\}_{t\in[0,T]}$ be a sample path of (2.4) at $\theta = \theta_0$. Define*

$$\hat{\theta}(x;\epsilon) := \arg\max_{\theta} \tilde{\mathbb{L}}^{N,\delta}(\theta; x)$$

with N and δ as in Theorem 4.1 above and $\tilde{\mathbb{L}}^{N,\delta}(\theta; x)$ defined in (4.5). Then,

$$\lim_{\epsilon\to 0} \hat{\theta}(x;\epsilon) = \theta_0, \quad \textit{in probability}.$$

Proof. We apply Lemma A.4 with $g_\epsilon(x,\theta) = \frac{1}{N\delta}\tilde{\mathcal{L}}^{N,\delta}(\theta; x)$ and $g_0(\theta)$ its limit. Note that

$$\lim_{\epsilon\to 0} \frac{1}{N\delta}\tilde{\mathbb{L}}^{N,\delta}(\theta; x) = \lim_{T\to\infty} \frac{1}{T}\mathbb{L}(\theta; X)$$

by Proposition 4.3 and the fact that

$$\lim_{T\to\infty} \frac{1}{T}\left(V(X(T);\theta) - V(X(0);\theta)\right) = 0,$$

which follows from the ergodicity of X. As in Theorem 4.1, the limits are interpreted in $L^2(\Omega')$ and $L^2(\Omega_0)$ respectively, and almost surely with respect to $X(0)$. As we have already seen, the maximizer of $g_0(\theta)$ is θ_0. So, Assumption (A.2) is satisfied. Also, Assumptions 3.5 is equivalent to (A.4). To prove consistency, we need to prove (A.3), which can be viewed as uniform ergodicity. The proof is again similar to that in [39]. First, we note that by Assumptions 3.5 and 4.4, both $g_\epsilon(\cdot,\theta)$ and $g_0(\theta)$ are continuous with respect to θ, so it is sufficient to prove (A.3) on a countable dense subset $\Theta^\star$ of Θ. Then, uniform ergodicity follows from [10, Thm. 6.1.5], provided that

$$N_{[\,]}\left(\epsilon, \mathcal{F}, \|\cdot\|_{L^1(\pi)}\right) < \infty,$$

i.e. the number of balls of radius ϵ with respect to $\|\cdot\|_{L^1(\pi)}$ needed to cover

$$\mathcal{F} := \{|F(z;\theta)|^2_{a(z)} + G(z;\theta);\ \theta \in \Theta^\star\}$$

is finite. As demonstrated in [39], this follows from the Hölder continuity of $|F(z;\theta)|^2_{a(z)}$ and $G(z;\theta)$. □

5. Examples

Numerical experiments, illustrating the phenomena studied in this paper, can be found in the paper [36]. The experiments therein are concerned with a particular case of the general homogenization framework considered in this paper and illustrate the failure of the MLE when the data is sampled too frequently, and the role of subsampling to ameliorate this problem. In this section we construct two examples which identify the term E_∞ responsible for the failure of the MLE.

5.1. Langevin equation in the high friction limit

We consider the Langevin equation in the high friction limit[5]:

$$\epsilon^2 \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -\nabla_q V(q;\theta) - \frac{\mathrm{d}q}{\mathrm{d}t} + \sqrt{2\beta^{-1}}\frac{\mathrm{d}W}{\mathrm{d}t}, \tag{5.1}$$

where $V(q;\theta)$ is a smooth confining potential depending on a parameter $\theta \in \Theta \subset \mathbb{R}^\ell$,[6] β stands for the inverse temperature and $W(t)$ is a standard Brownian motion on $\mathbb{R}^d$. We write this equation as a first order system

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{\epsilon}p, \qquad \frac{dp}{\mathrm{d}t} = -\frac{1}{\epsilon}\nabla_q V(q;\theta) - \frac{1}{\epsilon^2}p + \sqrt{\frac{2\beta^{-1}}{\epsilon^2}}\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{5.2}$$

In the notation of the general homogenization set-up we have $(x, y) = (q, p)$ and

$$f_0 = p, \qquad f_1 = 0, \qquad \alpha_0 = 0, \qquad \alpha_1 = 0$$

and

$$g_0 = -p, \qquad g_1 = -\nabla_q V(q), \qquad \beta \mapsto \sqrt{2\beta^{-1}}I.$$

The fast process is simply an Ornstein–Uhlenbeck process with generator

$$\mathcal{L}_0 = -p \cdot \nabla_p + \beta^{-1}\Delta_p.$$

Note that p has mean zero in the invariant measure of this process, and hence Assumption 2.4 is satisfied. The unique square integrable (with respect to the invariant measure of the OU process) solution of the Poisson equation (2.10) is $\Phi = p$. Therefore,

$$F_0 = -\nabla_q V(q;\theta), \qquad F_1 = 0, \qquad A_1 = \sqrt{2\beta^{-1}}I.$$

Hence the homogenized equation is[7]

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\nabla V(X;\theta) + \sqrt{2\beta^{-1}}\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{5.3}$$

Consider now the parameter estimation problem for “full dynamics” (5.1) and the “coarse-grained” model (5.3): We are given data from (5.1) and we want to fit it to Eq. (5.3). Theorem 3.12 implies that for this problem the maximum likelihood estimator is asymptotically biased.[8] In fact, in this case we can compute the term E_∞, responsible for the bias and given in Eq. (3.11). We have the following result.

Proposition 5.1. *Assume that the potential $V(q;\theta) \in C^\infty(\mathbb{R}^d)$ is such that $\mathrm{e}^{-\beta V(q;\theta)} \in L^1(\mathbb{R}^d)$ for every $\beta > 0$ and all $\theta \in \Theta$. Then error term E_∞, Eq. (3.11) for the SDE (5.1) is given by the*

[5] We have rescaled the equation in such a way that we actually consider the small mass, rather than the high friction limit. In the case where the mass and the friction are scalar quantities the two scaling limits are equivalent.

[6] A standard example is that of a quadratic potential $V(q;\theta) = \frac{1}{2}q\theta q'$ where the parameters to be estimated from time series are the elements of the stiffness matrix θ.

[7] In this case we can actually prove strong convergence of $q(t)$ to $X(t)$ [24,35].

[8] Subsampling, at the rate given in Theorem 4.1, is necessary for the correct estimation of the parameters in the drift of the homogenized equation (5.3).

formula

$$E_\infty(\theta) = -Z_V^{-1}\frac{\beta}{2}\int_{\mathbb{R}^d} |\nabla_q V(q;\theta)|^2 \mathrm{e}^{-\beta V(q;\theta)}\,\mathrm{d}q, \tag{5.4}$$

where $Z_V = \int_{\mathbb{R}^d} \mathrm{e}^{-\beta V(q;\theta)}\,\mathrm{d}q$. *In particular,* $E_\infty < 0$.

Proof. We have that

$$\mathcal{L}_1 = p\cdot\nabla_q - \nabla_q V\cdot\nabla_p.$$

The invariant measure of the process is ϵ-independent and we write it as

$$\rho(q,p;\theta)\,\mathrm{d}q\mathrm{d}p = Z^{-1}\mathrm{e}^{-\beta H(p,q;\theta)}\,\mathrm{d}q\mathrm{d}p.$$

Furthermore, since the homogenized diffusion matrix is $\sqrt{2\beta^{-1}}I$,

$$\langle\cdot,\cdot\rangle_{a(z)} = \frac{\beta}{2}\langle\cdot,\cdot\rangle,$$

where $\langle\cdot,\cdot\rangle$ stands for the standard Euclidean inner product. We readily check that

$$\frac{2}{\beta}\mathcal{L}_1\Gamma = \mathcal{L}_1\langle-\nabla_q V, p\rangle = -p\otimes p : D_q^2 V(q;\theta) + |\nabla_q V(q;\theta)|^2$$

and

$$\frac{2}{\beta}\langle F, \mathcal{L}_1\Phi\rangle_a = \langle -\nabla_q V, \mathcal{L}_1 p\rangle = |\nabla_q V(q;\theta)|^2.$$

Thus,

$$\begin{aligned} E_\infty(\theta) &= -\frac{\beta}{2}\int_{\mathbb{R}^{2d}} p\otimes p : D_q^2 V(q;\theta) Z^{-1}\mathrm{e}^{-\beta H(p,q;\theta)}\,\mathrm{d}q dp \\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\Delta_q V(q;\theta) Z_V^{-1}\mathrm{e}^{-\beta V(q;\theta)}\,\mathrm{d}q = -\frac{\beta}{2}\int_{\mathbb{R}^d}|\nabla_q V(q;\theta)|^2 Z_V^{-1}\mathrm{e}^{-\beta V(q;\theta)}\,\mathrm{d}q, \end{aligned}$$

which is precisely (5.4). $\square$

5.2. *Motion in a multiscale potential*

Consider the equation [36]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla V^\epsilon(x) + \sqrt{2\beta^{-1}}\frac{\mathrm{d}W}{\mathrm{d}t} \tag{5.5}$$

where

$$V^\epsilon(x) = V(x) + p(x/\epsilon),$$

where the fluctuating part of the potential $p(\cdot)$ is taken to be a smooth 1-periodic function.

Setting $y = x/\epsilon$ we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\Big(\nabla V(x) + \frac{1}{\epsilon}\nabla p(y)\Big) + \sqrt{2\beta^{-1}}\frac{\mathrm{d}W}{\mathrm{d}t} \tag{5.6a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{\epsilon}\Big(\nabla V(x) + \frac{1}{\epsilon}\nabla p(y)\Big) + \frac{1}{\epsilon}\sqrt{2\beta^{-1}}\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{5.6b}$$

In the notation of the general homogenization set-up we have

$$f_0 = g_0 = -\nabla_y p(y), \qquad f_1 = g_1 = -\nabla V(x)$$

and

$$\alpha_0 = 0, \qquad \alpha_1 = \beta = \sqrt{2\beta^{-1}}.$$

The fast process has generator

$$\mathcal{L}_0 = -\nabla_y p(y) \cdot \nabla_y + \beta^{-1} \Delta_y.$$

The invariant density is $\rho(y) = Z_p^{-1} \exp(-\beta p(y))$ with $Z_p = \int_{\mathbb{T}^d} \exp(-\beta p(y)) \mathrm{d}y$. Because f_0 is the gradient of p it follows, by periodicity, that Assumption 2.4 is satisfied. The Poisson equation for Φ is

$$\mathcal{L}_0 \Phi(y) = \nabla_y p(y).$$

Notice that Φ is a function of y only. The homogenized equation is

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -K \nabla V(X) + \sqrt{2\beta^{-1} K} \frac{\mathrm{d}W}{\mathrm{d}t} \tag{5.7}$$

where

$$K = \int_{\mathbb{T}^d} (I + \nabla_y \Phi(y))(I + \nabla_y \Phi(y))' \rho(y) \, \mathrm{d}y.$$

Suppose now that the potential contains parameters, $V = V(x, \theta)$, $\theta \in \Theta \subset \mathbb{R}^\ell$. We want to estimate the parameter θ, given data from (5.5) and using the homogenized equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -K \nabla V(X; \theta) + \sqrt{2\beta^{-1} K} \frac{\mathrm{d}W}{\mathrm{d}t}.$$

Theorem 3.12 implies that, for this problem, the maximum likelihood estimator is asymptotically biased and that subsampling at the appropriate rate is necessary for the accurate estimation of the parameter θ. As in the example presented in the previous section, we can calculate explicitly the error term E_∞. For simplicity we will consider the problem in one dimension.

Proposition 5.2. *Assume that the potential $V(x; \theta) \in C^\infty(\mathbb{R})$ is such that $\mathrm{e}^{-\beta V(x;\theta)} \in L^1(\mathbb{R})$ for every $\beta > 0$ and all $\theta \in \Theta$. Then error term E_∞, Eq. (3.11) for the SDE (5.5) is given by the formula*

$$E_\infty(\theta) = \left(-1 + \widehat{Z}_p^{-1} Z_p^{-1}\right) \frac{\beta Z_V^{-1}}{2} \int_{\mathbb{R}} |\partial_x V|^2 \mathrm{e}^{-\beta V(x;\theta)} \, \mathrm{d}x \tag{5.8}$$

where $Z_V = \int_{\mathbb{R}} \mathrm{e}^{-\beta V(q;\theta)} \, \mathrm{d}q$, $Z_p = \int_0^1 \mathrm{e}^{-\beta p(y)} \mathrm{d}y$, $\widehat{Z}_p = \int_0^1 \mathrm{e}^{\beta p(y)} \mathrm{d}y$. In particular, $E_\infty < 0$.

Proof. Eqs. (5.6) in one dimension become

$$\dot{x} = -\partial_x V(x; \theta) - \frac{1}{\epsilon} \partial_y p(y) + \sqrt{2\beta^{-1}} \dot{W}, \tag{5.9a}$$

$$\dot{y} = -\frac{1}{\epsilon} \partial_x V(x; \theta) - \frac{1}{\epsilon^2} \partial_y p(y) + \frac{2\beta^{-1}}{\epsilon^2} \dot{W}. \tag{5.9b}$$

The invariant measure of this system is (notice that it is independent of ϵ)

$$\rho(y, x; \theta)\, \mathrm{d}x\mathrm{d}y = Z_V^{-1}(\theta) Z_p^{-1} \mathrm{e}^{-\beta V(x;\theta)-\beta p(y)}\, \mathrm{d}x\mathrm{d}y.$$

The homogenized equation is

$$\dot{X} = -K \partial_x V(x; \theta) + \sqrt{2\beta^{-1} K}\, \dot{W}.$$

The cell problem is

$$\mathcal{L}_0 \phi = \partial_y p$$

and the homogenized coefficient is

$$K = Z_p^{-1} \int_0^1 (1 + \partial_y \phi)^2 \mathrm{e}^{-p(y)/\sigma} \mathrm{d}y.$$

We have that

$$\langle p, q\rangle_{\alpha(x)} = \frac{\beta}{2K} pq.$$

The error in the likelihood is

$$E_\infty(\theta) = \int_{-\infty}^{\infty} \int_0^1 \Big(\mathcal{L}_1 \Gamma(x, y) - \langle F, \mathcal{L}_1 \phi\rangle_{\alpha(x)} \Big) \rho(x, y) \mathrm{d}y\mathrm{d}x,$$

where

$$\Gamma = \langle F, \phi\rangle_{\alpha(x)},$$
$$F = -K \partial_x V.$$

We have that

$$\Gamma(x, y) = \frac{\beta}{2K}(-K \partial_x V \phi) = -\frac{\beta}{2} \partial_x V \phi.$$

Furthermore

$$\mathcal{L}_1 = -\partial_x V \partial_y - \partial_y p \partial_x + 2\beta^{-1} \partial_x \partial_y.$$

Consequently

$$\mathcal{L}_1 \Gamma(x, y) = \frac{\beta}{2} \Big(|\partial_x V|^2 \partial_y \phi + \partial_y p \partial_x^2 V \phi - 2\beta^{-1} \partial_x^2 V \partial_y \phi \Big).$$

In addition,

$$\langle F, \mathcal{L}_1 \phi\rangle_{\alpha(x)} = \frac{\beta}{2} |\partial_x V|^2 \partial_y \phi.$$

The error in the likelihood is

$$\begin{aligned} E_\infty(\theta) &= \frac{\beta}{2} \int_{\mathbb{R}} \int_0^1 \Big(-\partial_y p \partial_x^2 V \phi + 2\beta^{-1} \partial_x^2 \partial_y \phi \Big) Z_V^{-1} Z_p^{-1} \mathrm{e}^{-\beta V(x;\theta)-\beta p(y)}\, \mathrm{d}x\mathrm{d}y \\ &= -\frac{Z_V^{-1} Z_p^{-1}}{2} \int_{\mathbb{R}} \partial_x^2 V \mathrm{e}^{-\beta V(x;\theta)}\, \mathrm{d}x \int_0^1 \partial_y \phi \mathrm{e}^{-\beta p(y)} \mathrm{d}y \\ &\quad + Z_V^{-1} Z_p^{-1} \int_{\mathbb{R}} \partial_x^2 V \mathrm{e}^{-\beta V(x;\theta)}\, \mathrm{d}x \int_0^1 \partial_y \phi \mathrm{e}^{-\beta p(y)} \mathrm{d}y \end{aligned}$$

$$= \frac{Z_V^{-1} Z_p^{-1}}{2} \int_{\mathbb{R}} \partial_x^2 V \mathrm{e}^{-\beta V(x;\theta)} \, \mathrm{d}x \int_0^1 \partial_y \phi \mathrm{e}^{-\beta p(y)} \mathrm{d}y$$

$$= \frac{\beta Z_V^{-1}}{2} \int_{\mathbb{R}} |\partial_x V|^2 \mathrm{e}^{-\beta V(x;\theta)} \, \mathrm{d}x \left(-1 + \widehat{Z}_p^{-1} Z_p^{-1}\right).$$

In the above derivation we used various integrations by parts, together with the formula for the derivative of the solution of the Poisson equation $\partial_y \phi = -1 + \widehat{Z}_p^{-1} \mathrm{e}^{\beta p(y)}$, [37, p. 213]. The fact that E_∞ is non-positive follows from the inequality $Z_p^{-1} \widehat{Z}_p^{-1} < 1$ (for $p(y)$ not identically equal to 0), which follows from the Cauchy–Schwarz inequality. □

Remark 5.3. An application of Laplace's method shows that, for $\beta \gg 1$, $Z_p^{-1} \widehat{Z}_p^{-1} \sim \mathrm{e}^{-2\beta}$.

6. Conclusions

The problem of parameter estimation for fast/slow systems of SDEs which admit a coarse-grained description in terms of an SDE for the slow variable was studied in this paper. It was shown that, when applied to the averaging problem, the maximum likelihood estimator (MLE) is asymptotically unbiased and we can use it to estimate accurately the parameters in the drift coefficient of the coarse-grained model using data from the slow variable in the fast/slow system. On the contrary, the MLE is asymptotically biased when applied to the homogenization problem and a systematic asymptotic error appears in the log-likelihood function, in the long time/infinite scale separation limit. However the MLE *can* lead to the correct estimation of the parameters in the drift coefficient of the homogenized equation provided that we subsample the data from the fast/slow system at the appropriate sampling rate.

The averaging/homogenization systems of SDEs that we consider in this paper are of quite general form and have been studied quite extensively in the last several decades since they appear in various applications, e.g. molecular dynamics, chemical kinetics, mathematical finance, atmosphere/ocean science — see the references in [37], for example. Thus, we believe that our results show that great care has to be taken when using maximum likelihood estimation in order to infer information about parameters in stochastic systems with multiple characteristic time scales. Similar caution would apply also to Bayesian methods.

In this paper we have only considered the case where the state space of the fast/slow process is compact and that the generator of the fast process is a uniformly elliptic operator. Under these assumptions Theorem 2.2 and, consequently, the averaging and homogenization theorems, follows from standard elliptic PDE theory. Similar results can also be proved without the compactness assumption and, under additional assumptions, even for non-uniformly elliptic fast processes [32–34]. In particular, the regularity of solutions to a Poisson equation of the form (2.5) with respect to the parameter ξ, together with estimates on the derivatives, when $\mathcal{Y} = \mathbb{R}^{d-\ell}$ and under the assumption of uniform ellipticity was proved in [33]. Alternatively, one could use a stopping time argument; see [21] for details.

Similarly, in the non-compact case $\mathcal{X} = \mathbb{R}^\ell$, $\mathcal{Y} = \mathbb{R}^{\ell-d}$ more work is needed in order to prove Theorem 3.8 and, in particular, that the invariant measure satisfies Poincaré's inequality with an ϵ-independent constant. The proof of Poincaré's inequality, essentially, requires to prove that the generator of the fast/slow system has an ϵ-independent spectral gap. In the case where the fast/slow system has a gradient structure with a smooth potential $V(x, y)$, then simple criteria on the potential have been derived that facilitate determination of whether or not the invariant measure satisfies the Poincaré inequality. We refer to [41,3] and the references therein for more details.

There are various problems, both of theoretical and of applied interest, that remain open and that we plan to address in future work. We list some of them below.

- Bayesian techniques for parameter estimation of multiscale diffusion processes.
- The development of efficient algorithms for estimating the parameters in the coarse-grained model of a fast/slow stochastic system. Based on the work that has been done to similar models in the context of econometrics [25,2] one expects that such an algorithm would involve the estimation of an appropriate measure of scale separation ϵ, and of the optimal sampling rate, averaging over all the available data and a bias reduction step.
- Study classical problems from parameter estimation, generalizing to the framework here. For example relaxing ergodicity assumptions, studying asymptotic normality of the estimators, and so forth.
- Investigate whether there is any advantage in using random sampling rates.
- Investigate similar issues for deterministic fast/slow systems of differential equations.

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Appendix

A.1. An ergodic theorem with convergence rates

Consider the SDE

$$\frac{\mathrm{d}z}{\mathrm{d}t} = h(z) + \gamma(z)\frac{\mathrm{d}W}{\mathrm{d}t}, \tag{A.1}$$

with $z \in \mathcal{Z}$, where $\mathcal{Z}$ is either $\mathbb{R}^k$ or $\mathbb{T}^k$, $h : \mathcal{Z} \to \mathbb{R}^k$, $\gamma : \mathcal{Z} \to \mathbb{R}^{k\times p}$ and $w \in \mathbb{R}^p$ a standard Brownian motion. Assume that h, γ are C^∞ with bounded derivatives. Let $\psi : \mathcal{Z} \to \mathbb{R}$ be bounded, and $\phi : \mathcal{Z} \to \mathbb{R}$ be bounded. We denote the generator of the Markov process (A.1) by $\mathcal{A}$.

Assumptions A.1. Eq. (A.1) is ergodic with invariant measure $\nu(z)\mathrm{d}z$. Let

$$\overline{\phi} = \int_{\mathcal{Z}} \phi(z)\nu(z)\mathrm{d}z.$$

Then the equation

$$-\mathcal{A}\Phi = \phi - \overline{\phi}, \qquad \int_{\mathcal{Z}} \Phi(z)\nu(z)\mathrm{d}z = 0$$

has a unique solution $\Phi : \mathcal{Z} \to \mathbb{R}$, with Φ and $\nabla\Phi$ bounded.

Lemma A.2. *Let*

$$I = \frac{1}{\sqrt{T}}\int_0^{\mathrm{T}} \psi(z(t))\mathrm{d}W(t).$$

Then there exists a constant $C > 0$: $\mathbb{E}|I|^2 \le C$ for all $T > 0$.

Proof. Use the Itô isometry and invoke the boundedness of ψ. $\square$

Lemma A.3. *Time averages converge to their mean value almost surely. Furthermore there is a constant $C > 0$:*

$$\mathbb{E}\left|\frac{1}{T}\int_0^{\mathrm{T}} \phi(z(t))\mathrm{d}t - \overline{\phi}\right|^2 \leq \frac{C}{T}.$$

Proof. By applying the Itô formula to Φ we obtain

$$-\int_0^{\mathrm{T}} \mathcal{A}\Phi(z(t))\mathrm{d}t = \Phi(z(0)) - \Phi(z(T)) + \int_0^{\mathrm{T}} (\nabla\Phi\gamma)\,(z(t))\mathrm{d}W(t).$$

Thus

$$\int_0^{\mathrm{T}} \phi(z(t))\mathrm{d}t = \overline{\phi} + \frac{1}{T}\left(\Phi(z(0)) - \Phi(z(T))\right) + \frac{1}{\sqrt{T}}I,$$

$$I = \frac{1}{\sqrt{T}}\int_0^{\mathrm{T}} (\nabla\Phi\gamma)\,(z(t))\mathrm{d}W(t).$$

The result concerning $L^2(\Omega)$ convergence follows from boundedness of Φ, $\nabla\Phi$ and γ, together with Lemma A.2. Almost sure convergence follows from the ergodic theorem. □

A.2. Consistency of the estimators

Lemma A.4. *Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a probability space and $g_\epsilon : \tilde{\Omega} \times \Theta \to \mathbb{R}$, $g_0 : \Theta \to \mathbb{R}$ be such that*

$$\forall\theta \in \Theta, \quad g_\epsilon \to g_0 \quad \textit{in probability, as } \epsilon \to 0 \tag{A.2}$$

and

$$\forall\delta, \kappa > 0 : \mathbb{P}\left\{\omega : \sup_{|u|>\delta}\left(g_\epsilon(\omega, \hat{\theta}_0 + u) - g_0(\hat{\theta}_0 + u)\right) > \kappa\right\} \to 0, \quad \text{as } \epsilon \to 0, \tag{A.3}$$

where

$$\hat{\theta}_0 = \arg\sup_{\theta\in\Theta} g_0(\theta).$$

Moreover, we assume that

$$\forall\delta > 0, \quad \sup_{|u|>\delta}\left(g_0(\hat{\theta}_0 + u) - g_0(\hat{\theta}_0)\right) \leq -\kappa(\delta) < 0. \tag{A.4}$$

If

$$\hat{\theta}_\epsilon(\omega) = \arg\sup_{\theta\in\Theta} g_\epsilon(\omega, \theta)$$

then

$$\hat{\theta}_\epsilon \to \hat{\theta}_0 \quad \textit{in probability}.$$

Proof. First note that $\forall\delta > 0$

$$\tilde{\mathbb{P}}\left\{|\hat{\theta}_\epsilon - \hat{\theta}_0| > \delta\right\} \leq \tilde{\mathbb{P}}\left\{\sup_{|u|>\delta}\left(g_\epsilon(\omega, \hat{\theta}_0 + u) - g_\epsilon(\omega, \hat{\theta}_0)\right) \geq 0\right\}. \tag{A.5}$$

We define

$$G_\epsilon(\omega;\theta,u) := g_\epsilon(\omega,\theta+u) - g_\epsilon(\omega,\theta) \text{ and } G_0(\theta,u) := g_0(\theta+u) - g_0(\theta).$$

Clearly,

$$\sup_{|u|>\delta} G_\epsilon(\omega;\hat{\theta}_0,u) \le \sup_{|u|>\delta}\left(G_\epsilon(\omega;\hat{\theta}_0,u) - G_0(\hat{\theta}_0,u)\right) + \sup_{|u|>\delta} G_0(\hat{\theta}_0,u)$$

and thus

$$\begin{aligned}
&\tilde{\mathbb{P}}\left\{\sup_{|u|>\delta} G_\epsilon(\omega;\hat{\theta}_0,u) \ge 0\right\}\\
&\quad\le \tilde{\mathbb{P}}\left\{\sup_{|u|>\delta}\left(G_\epsilon(\omega;\hat{\theta}_0,u) - G_0(\hat{\theta}_0,u)\right) \ge -\sup_{|u|>\delta} G_0(\hat{\theta}_0,u)\right\}\\
&\quad\le \tilde{\mathbb{P}}\left\{\sup_{|u|>\delta}\left(G_\epsilon(\omega;\hat{\theta}_0,u) - G_0(\hat{\theta}_0,u)\right) \ge \kappa(\delta) > 0\right\}
\end{aligned} \tag{A.6}$$

by Assumption (A.4). Note that

$$G_\epsilon(\omega;\hat{\theta}_0,u) - G_0(\hat{\theta}_0,u) = \left(g_\epsilon(\omega;\hat{\theta}_0+u) - g_0(\hat{\theta}_0+u)\right) - \left(g_\epsilon(\omega;\hat{\theta}_0) - g_0(\hat{\theta}_0)\right).$$

So, by conditioning on $\left\{\omega\colon |\, g_\epsilon(\omega;\hat{\theta}_0) - g_0(\hat{\theta}_0)| \ge \frac{1}{2}\kappa(\delta)\right\}$ and (A.5) and (A.6), we get that

$$\begin{aligned}
\tilde{\mathbb{P}}\left\{|\hat{\theta}_\epsilon - \hat{\theta}_0| > \delta\right\} &\le \tilde{\mathbb{P}}\left\{\sup_{|u|>\delta}\left(g_\epsilon(\omega;\hat{\theta}_0+u) - g_0(\hat{\theta}_0+u)\right) \ge \frac{1}{2}\kappa(\delta) > 0\right\}\\
&\quad + \tilde{\mathbb{P}}\left\{|g_\epsilon(\omega;\hat{\theta}_0) - g_0(\hat{\theta}_0)| \ge \frac{1}{2}\kappa(\delta) > 0\right\}
\end{aligned}$$

Both probabilities on the right-hand side go to zero as $\epsilon \to 0$, by Assumptions (A.3) and (A.2) respectively. We conclude that $\hat{\theta}_\epsilon \to \hat{\theta}_0$ in probability. □

A.3. Proof of Propositions 4.2 and 4.3

In this section we present the proofs of Propositions 4.2 and 4.3 which we repeat here, for the reader's convenience.

Proposition A.5. *Let $(x(t), y(t))$ be the solution of (2.4) and assume that Assumptions 2.1 and 2.4 hold. Then, for ϵ, δ sufficiently small, the increment of the process $x(t)$ can be written in the form*

$$x_{n+1} - x_n = F(x_n;\theta_0)\,\delta + M_n + R(\epsilon,\delta),$$

where M_n denotes the martingale term

$$M_n = \int_{n\delta}^{(n+1)\delta} \left(\nabla_y \Phi\beta + \alpha_0\right)(x(s), y(s))\mathrm{d}V(s) + \int_{n\delta}^{(n+1)\delta} \alpha_1(x(s), y(s))\mathrm{d}U(s)$$

with $\|M_n\|_p \le C\sqrt{\delta}$ and

$$\|R(\epsilon,\delta)\|_p \le C(\delta^{3/2} + \epsilon\delta^{\frac{1}{2}} + \epsilon).$$

Proposition A.6. *Let $g \in C^1(\mathcal{X})$ and let Assumption 3.7 hold. Assume that ϵ and N are related as in Theorem 4.1. Then*

$$\lim_{\epsilon \to 0} \frac{1}{N} \sum_{n=0}^{N-1} g(x_n) = \mathbb{E}^{\pi} g,$$

where the convergence is in L^2 with respect to the measure on initial conditions with density $\pi(x)\rho(y; x)$.

For the proofs of Propositions A.5 and A.6, both used in the proof of Theorem 4.1, we will need the following two technical lemmas. We start with a rough estimate on the increments of the process $x(t)$.

Lemma A.7. *Let $(x(t), y(t))$ be the solution of (2.4) and assume that Assumptions 2.1 and 2.4 hold. Let $s \in [n\delta, (n+1)\delta]$. Then, for ϵ, δ sufficiently small, the following estimate holds:*

$$\|x(s) - x_n\|_p \le C(\epsilon + \delta^{\frac{1}{2}}). \tag{A.7}$$

Proof. We apply Itô's formula to Φ, the solution of the Poisson equation (2.10), to obtain

$$\begin{aligned} x(s) - x_n = {} & -\epsilon(\Phi(x(s), y(s)) - \Phi(x_n, y_n)) + \int_{n\delta}^{s} ((\mathcal{L}_1 \Phi + f_1))(x(s), y(s))\mathrm{d}s \\ & + \int_{n\delta}^{s} \left(\nabla_y \Phi \beta + \alpha_0\right)(x(s), y(s))\mathrm{d}V(s) + \int_{n\delta}^{s} \alpha_1(x(s), y(s))\mathrm{d}U(s) \\ & + \epsilon \int_{n\delta}^{s} (\mathcal{L}_2 \Phi)(x(s), y(s))\mathrm{d}s + \epsilon \int_{n\delta}^{s} \left(\nabla_y \Phi \alpha_0\right)(x(s), y(s))\,\mathrm{d}U(s) \\ & + \epsilon \int_{n\delta}^{s} (\nabla_x \Phi \alpha_1)(x(s), y(s))\mathrm{d}V(s) \\ =: {} & J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \end{aligned}$$

Our assumptions on $\Phi(x, y)$, together with standard inequalities, imply that

$$\begin{aligned} & \|J_1\|_p \le C\epsilon, \qquad \|J_2\|_p \le C\delta, \qquad \|J_3\|_p \le C\delta^{\frac{1}{2}}, \\ & \|J_4\|_p \le C\delta^{\frac{1}{2}}, \qquad \|J_5\|_p \le C\epsilon\delta, \qquad \|J_6\|_p \le C\epsilon\delta^{1/2}, \qquad \|J_7\|_p \le C\epsilon\delta^{1/2}. \end{aligned}$$

Estimate (A.7) follows from these estimates. □

Using this lemma we can prove the following estimate.

Lemma A.8. *Let $h(x, y)$ be a smooth, bounded function, let $(x(t), y(t))$ be the solution of (2.4) and assume that Assumptions 2.1 hold. Define*

$$H(x) := \int_{\mathcal{Y}} h(x, y)\, \rho(y; x)\mathrm{d}y.$$

Then, for ϵ, δ sufficiently small, the following estimate holds:

$$\int_{n\delta}^{(n+1)\delta} h(x(s), y(s))\mathrm{d}s = H(x_n)\,\delta + R(\epsilon, \delta) \tag{A.8}$$

where

$$\|R(\epsilon, \delta)\|_p \le C(\epsilon^2 + \delta^{3/2} + \epsilon\delta^{1/2}).$$

Proof. Let ϕ be the mean zero solution of the equation

$$-\mathcal{L}_0\phi = h(x, y) - H(x). \tag{A.9}$$

By Theorem 2.2 this solution is smooth in both x, y and it is unique and bounded. We apply Itô's formula to obtain

$$\begin{aligned}
\int_{n\delta}^{(n+1)\delta} (h(x(s), y(s)) - H(x(s)))\,\mathrm{d}s &= -\epsilon^2(\phi(x_{n+1}, y_{n+1}) - \phi(x_n, y_n)) \\
&\quad + \epsilon \int_{n\delta}^{(n+1)\delta} \mathcal{L}_1\phi(x(s), y(s))\mathrm{d}s + \epsilon^2 \int_{n\delta}^{(n+1)\delta} \mathcal{L}_2\phi(x(s), y(s))\mathrm{d}s \\
&\quad + \epsilon^2 \int_{n\delta}^{(n+1)\delta} (\nabla_x\phi\alpha_0)(x(s), y(s))\mathrm{d}U(s) \\
&\quad + \epsilon \int_{n\delta}^{(n+1)\delta} (\nabla_y\phi\beta + \epsilon\nabla_x\phi\alpha_1)(x(s), y(s))\mathrm{d}V(s) \\
&=: J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Our assumptions on the solution ϕ of the Poisson equation (A.9), together with standard estimates for the moments of stochastic integrals and Hölder's inequality give the estimates

$$\begin{aligned}
&\|J_1\|_p \le C\epsilon^2, \qquad \|J_2\|_p \le C\epsilon\delta, \qquad \|J_3\|_p \le C\epsilon^2\delta, \\
&\|J_4\|_p \le C\epsilon^2\delta^{1/2}, \qquad \|J_5\|_p \le C\epsilon\delta^{1/2}.
\end{aligned}$$

The above estimates imply that

$$\int_{n\delta}^{(n+1)\delta} h(x(s), y(s))\mathrm{d}s = \int_{n\delta}^{(n+1)\delta} H(x(s))\mathrm{d}s + R_1(\epsilon, \delta)$$

with

$$\|R_1(\epsilon, \delta)\|_p \le C\left(\epsilon\delta^{1/2} + \epsilon^2\right).$$

We use the Hölder inequality and the Lipschitz continuity of $H(x)$ to estimate:

$$\begin{aligned}
\left\| \int_{n\delta}^{(n+1)\delta} H(x(s))\mathrm{d}s - H(x_n)\,\delta \right\|_p^p &= \left\| \int_{n\delta}^{(n+1)\delta} (H(x(s)) - H(x_n))\,\mathrm{d}s \right\|_p^p \\
&\le \delta^{p-1} \int_{n\delta}^{(n+1)\delta} \|H(x(s)) - H(x_n)\|_p^p\,\mathrm{d}s \\
&\le C\delta^{p-1} \int_{n\delta}^{(n+1)\delta} \|x(s) - x_n\|_p^p\,\mathrm{d}s \\
&\le C\delta^p \left(\delta^{1/2} + \epsilon\right)^p = R_2(\epsilon, \delta)^p,
\end{aligned}$$

where Lemma A.7 was used and $R_2(\epsilon, \delta) = (\epsilon\delta + \delta^{3/2})$. We combine the above estimates to obtain

$$\begin{aligned}
\int_{n\delta}^{(n+1)\delta} h(x(s), y(s))\mathrm{d}s &= \int_{n\delta}^{(n+1)\delta} H(x(s))\mathrm{d}s + R_1(\epsilon, \delta) \\
&= H(x_n)\,\delta + R_1(\epsilon, \delta) + R_2(\epsilon, \delta),
\end{aligned}$$

from which (A.8) follows. □

Proof of Proposition 4.2 (Proposition A.5). This follows from the first line of the proof of Lemma A.7, the estimates therein concerning all the J_i with the exception of J_2, and the use of Lemma A.8 to estimate J_2 in terms of $\delta F(x_n;\theta_0)$. $\square$

Proof of Proposition 4.3 (Proposition A.6). We have

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1} g(x_n) &= \frac{1}{N\delta}\sum_{n=0}^{N-1}\int_{n\delta}^{(n+1)\delta} g(x_n)\mathrm{d}s\\ &= \frac{1}{N\delta}\sum_{n=0}^{N-1}\int_{n\delta}^{(n+1)\delta} g(x(s))\mathrm{d}s + \frac{1}{N\delta}\sum_{n=0}^{N-1}\int_{n\delta}^{(n+1)\delta}(g(x_n)-g(x(s)))\,\mathrm{d}s\\ &= \frac{1}{N\delta}\int_0^{N\delta} g(x(s))\mathrm{d}s + \frac{1}{N\delta}\sum_{n=0}^{N-1}\int_{n\delta}^{(n+1)\delta}(g(x_n)-g(x(s)))\,\mathrm{d}s\\ &=: I_1 + R_1.\end{aligned}$$

We introduce the notation

$$f_n := \int_{n\delta}^{(n+1)\delta}\big(g(x_n)-g(x(s))\big)\mathrm{d}s.$$

By Lemma A.7 we have that $x(s)-x_n = \mathcal{O}(\epsilon+\delta^{\frac{1}{2}})$ in $L^p(\Omega')$. We use this, together with the Lipschitz continuity of g and Hölder's inequality, to estimate:

$$\begin{aligned}\|f_n\|_p^p &\le \delta^{p/q}\int_{n\delta}^{(n+1)\delta}\mathbb{E}\,|g(x_n)-g(x(s))|^p\,\mathrm{d}s\\ &\le C\delta^{1+p/q}\big(\epsilon^p+\delta^{p/2}\big).\end{aligned}$$

Here $p^{-1}+q^{-1}=1$. Using this we can estimate R_1 using:

$$\|R_1\|_p \le \frac{1}{N\delta}\sum_{n=0}^{N-1}\|f_n\|_p \le C\frac{1}{N\delta}N\delta^{(1/p+1/q)}\big(\epsilon+\delta^{1/2}\big) = C\big(\epsilon+\delta^{1/2}\big)\to 0,$$

as $\epsilon\to 0$.

Thus it remains to estimate I_1. Let $T=N\delta$. Let ψ^ϵ solve

$$-\mathcal{L}_{hom}\psi^\epsilon(x,y) = \hat{g}(x) := g(x)-\mathbb{E}^{\rho^\epsilon} g. \tag{A.10}$$

Apply Itô's formula. This gives

$$\begin{aligned}\frac{1}{T}\int_0^{\mathrm{T}} g(x(s))\mathrm{d}s - \mathbb{E}^{\rho^\epsilon} g &= -\frac{1}{T}\Big(\psi^\epsilon\big(x(T),y(T)\big)-\psi^\epsilon\big(x(0),y(0)\big)\Big)\\ &\quad+\frac{1}{\epsilon T}\int_0^{\mathrm{T}}(\nabla_y\psi^\epsilon\beta)(x(s),y(s))\,\mathrm{d}V(s) + \frac{1}{T}\int_0^{\mathrm{T}}(\nabla_x\psi^\epsilon\alpha)(x(s),y(s))\,\mathrm{d}U'(s),\\ &=: J_1+J_2\end{aligned}$$

where J_2 denotes the two stochastic integrals and we write $\alpha\mathrm{d}U' = \alpha_0\mathrm{d}U+\alpha_1\mathrm{d}V$, in law. Note that

$$\mathbb{E}^{\rho^\epsilon} g \to \mathbb{E}^\pi g$$

as $\epsilon \to 0$ by Assumption 3.7. Thus the theorem will be proved if we can show that $J_1 + J_2$ tends to zero in the required topology on the initial conditions. Note that

$$\mathbb{E}^{\rho^\epsilon}|J_1|^2 \le \frac{4}{T^2}\mathbb{E}^{\rho^\epsilon}|\psi^\epsilon|^2,$$
$$\mathbb{E}^{\rho^\epsilon}|J_2|^2 \le \frac{1}{T}\mathbb{E}^{\rho^\epsilon}\langle\nabla\psi^\epsilon, \Sigma\nabla\psi^\epsilon\rangle.$$

Here Σ is defined in Assumption 3.7 and ∇ is the gradient with respect to $(x', y')'$. We note that, by stationarity, we have that

$$\mathbb{E}^{\rho^\epsilon}|\psi^\epsilon|^2 = \|\psi^\epsilon\|, \qquad \mathbb{E}^{\rho^\epsilon}\langle\nabla\psi^\epsilon, \Sigma\nabla\psi^\epsilon\rangle = (\nabla\psi^\epsilon, \Sigma\nabla\psi^\epsilon), \tag{A.11}$$

where $\|\cdot\|$ and $(\cdot,\cdot)$ denote the $L^2(\mathcal{X}\times\mathcal{Y}; \mu^\epsilon(\mathrm{d}x\mathrm{d}y))$ norm and inner product, respectively.

Use of the Dirichlet form (see Theorem 6.12 in [37]) shows that

$$\begin{aligned}\left(\nabla\psi^\epsilon, \Sigma\nabla\psi^\epsilon\right) &\le 2\int \hat{g}(x)\psi^\epsilon(x,y)\rho^\epsilon(x,y)\mathrm{d}x\mathrm{d}y \\ &\le a\|\hat{g}\|^2 + a^{-1}\|\psi^\epsilon\|^2,\end{aligned}$$

for any $a > 0$. Using the Poincaré inequality (3.9), together with Assumption 3.7 and Theorem 3.8, gives

$$\|\psi^\epsilon\|^2 \le C_p^2\|\nabla\psi^\epsilon\|^2 \le aC_\gamma^{-1}C_p^2\|\hat{g}\|^2 + a^{-1}C_\gamma^{-1}C_p^2\|\psi^\epsilon\|^2.$$

Choosing a so that $a^{-1}C_\gamma^{-1}C_p^2 = \frac{1}{2}$ gives

$$\|\psi^\epsilon\|^2 \le C\mathbb{E}^{\rho^\epsilon}|\hat{g}|^2.$$

Hence

$$\left(\psi^\epsilon, \Gamma\nabla\psi^\epsilon\right) \le C\mathbb{E}^{\rho^\epsilon}|\hat{g}|^2,$$

where the notation introduced in (A.11) was used. The constant C in the above inequalities is independent of ϵ. Thus

$$\mathbb{E}^{\rho^\epsilon}|J_1|^2 + \mathbb{E}^{\rho^\epsilon}|J_2|^2 \le \frac{1}{T}C\mathbb{E}^{\rho^\epsilon}|\hat{g}|^2. \tag{A.12}$$

Since the measure with density ρ^ϵ converges to the measure with density $\pi(x)\rho(y;x)$ the desired result follows. □

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