EE2 Maths: Taylor's theorem for multi-variable functions

Reminder: in univariate case

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + ... + \frac{(x - x_0)^{n-1}}{(n-1)!}f^{n-1}(x_0) + ... \quad (1)$$

where we could also have written $x - x_0 = \Delta x$.

Generalizing to two variables:

$$f(x, y) = f(x_0, y_0) + \Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y} + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2\right] + ... \quad (2)$$

Where we define $\Delta x = x - x_0$ and $\Delta y = y - y_0$ and evaluate the derivatives at x_0, y_0. We could write the quadratic term in different notation as $(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})f$ where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ act only on f not on Δx or Δy. The above expansion then becomes:

$$f(x, y) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^n f(x, y)\right]_{x_0, y_0} \quad (3)$$

For more than two dimensions we can write this as

$$f(\mathbf{x}) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[(\Delta\mathbf{x}\cdot\nabla)^n f(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x_0}} \quad (4)$$

or

$$f(\mathbf{x}) = f(\mathbf{x_0}) + \sum_i\frac{\partial f}{\partial x_i}\Delta x_i + \frac{1}{2!}\sum_i\sum_j\frac{\partial^2 f}{\partial x_i \partial x_j}\Delta x_i \Delta x_j + \dots \quad (5)$$

Hessian: If $f(x_1...x_n)$ we can define $H_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}f(x_1, ..., x_n)$. The matrix H is called the Hessian. H_{ij} appears in the equation above (5).