EE2 Mathematics

SOME STANDARD LAPLACE TRANSFORMS

Notation: The Laplace transform, $\overline{f}(s)$, of $f(t)$ is defined by

$$\mathcal{L}\left[f(t)\right] \equiv \overline{f}(s) = \int_0^\infty f(t)\exp(-st)dt.$$

1. $f(t) = 1;\quad \overline{f}(s) = 1/s \qquad\qquad s > 0.$
2. $f(t) = \exp(at);\quad \overline{f}(s) = 1/(s-a) \qquad$ Re $(s) > a.$
3. $f(t) = \sin(at);\quad \overline{f}(s) = a/(a^2 + s^2) \qquad\qquad s > 0.$
4. $f(t) = \cos(at);\quad \overline{f}(s) = s/(a^2 + s^2) \qquad\qquad s > 0.$
5. $f(t) = t^n;\quad \overline{f}(s) = n!/s^{n+1} \qquad (n \geq 0) \qquad\qquad s > 0.$
6. $f(t) = H(t - t_0);\quad \overline{f}(s) = \exp(-st_0)/s \qquad\qquad s > 0.$
7. $f(t) = \delta(t - t_0);\quad \overline{f}(s) = \exp(-st_0) \qquad\qquad t_0 \geq 0.$
8. Shift theorem: $\mathcal{L}\ \left[\exp(at)f(t)\right] = \overline{f}(s-a).$
9. Second shift theorem: $\mathcal{L}\ \left[H(t-a)f(t-a)\right] = \exp(-sa)\,\overline{f}(s).$
10. Convolution theorem: $\mathcal{L}\left\{f \star g\right\} = \overline{f}(s)\,\overline{g}(s)$ where

    $$f \star g = \int_0^t f(t-t')g(t')\,dt'.$$

    Note that the convolution integral on the RHS can also be written with the f and g reversed: $\int_0^t f(t')g(t-t')\,dt'$.
11. Integral: $\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{1}{s}\overline{f}(s).$
12. Derivative: $\mathcal{L}\left[\dot{f}(t)\right] = s\overline{f}(s) - f(0).$
13. Second derivative: $\mathcal{L}\left[\ddot{f}(t)\right] = s^2\overline{f}(s) - sf(0) - \dot{f}(0).$

Alternative to dealing with inverse transforms $\frac{1}{(s^2+1)^2}$ & $\frac{s}{(s^2+1)^2}$

The square in the denominators of these expressions look ominous as they are not listed in the Library of Laplace Transforms. In the notes a method was devised for dealing with these using the Convolution Theorem. However, there is an alternative trick that gives the same answer.

Consider first the sine function $\sin at$ where we are going to treat a as a variable independent of t although the Laplace Transform will still be taken over t

$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2} \tag{1}$$

Now we differentiate the last equation (partially) w.r.t. a to give

$$\mathcal{L}\left(t \cos at\right) = \frac{1}{s^2 + a^2} - \frac{2a^2}{\left(s^2 + a^2\right)^2}$$

Divide by $2a$ and use (1) to get

$$\mathcal{L}^{-1}\left(\frac{a}{\left(s^2 + a^2\right)^2}\right) = \frac{1}{2a^2}\sin at - \frac{1}{2a}t \cos at.$$

Now we can set $a = 1$ (or whatever value we like) to get the final result

$$\mathcal{L}^{-1}\left(\frac{1}{\left(s^2 + 1\right)^2}\right) = \frac{1}{2}\sin t - \frac{1}{2}t \cos t.$$

We repeat the process by taking

$$\mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}$$

and again we differentiate w.r.t. a to get

$$\mathcal{L}\left(-t \sin at\right) = -\frac{2as}{\left(s^2 + a^2\right)^2}.$$

Hence

$$\mathcal{L}^{-1}\left(\frac{s}{\left(s^2 + a^2\right)^2}\right) = \frac{1}{2a}t \sin at$$

and we can set $a = 1$ or anything we like.