EE2 Mathematics

Hints and Solutions 1: Functions of Multiple Variables

1) Consider $f(p,q) = 2p(q^2 + 2q - 1)$. Find the components of ∇f, i.e. $(\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q})$, at the points i) (-1,0), ii) (1,0), iii) (-1,1) and iv) (1,1). Produce a sketch showing ∇f, the gradient vector, at each of these points.

Pick your answers from:
a) (0,-4) b) (4,8) c) (2,-4) d) (0,4) e) (-2,-4) f) (4,-8) g) (4,4) h) (-2,4)

actual answers: i) (-2,-4) - e ii) (-2,4) - h iii) (4,-8) - f iv) (4,8) - b

2) I am an indifferent experimentalist and can be slightly inaccurate in my measurements of time intervals and distances. Recall that a pendulum has period P and length l and these are related by $P = 2\pi(\frac{l}{g})^{\frac{1}{2}}$. What is the percentage error in my estimate of g if
i) I measure P perfectly but make a 0.2% error in estimating l.
ii) I measure l perfectly but make a 0.3% error in estimating P.
It might help you to rearrange the above expression in P.

Pick your answers from:
a) 0.1 b) 0.2 c) 0.4 d) 0.3 e) 0.6 f) 0.8 g) 0.5

actual answers: i) 0.2 - b ii) 0.6 - e

Find g as a function of l and P. Find an expression for $\Delta g(l,P)$ in terms of $\frac{\partial g}{\partial P}$ and $\frac{\partial g}{\partial l}$ (approximate version of the total differential). What we want to find is $\Delta g/g$ and for part i) we have $\Delta P/P = 0$ and $\Delta l/l = 0.002$ and for part ii) $\Delta P/P = 0.003$ and $\Delta l/l = 0$.

3) Consider $f(x,y) = exp(-2xy)$. Find: i) $\frac{\partial f}{\partial x}|_y$ and ii) $\frac{\partial f}{\partial y}|_x$
Check that $\frac{\partial}{\partial y}|_x \frac{\partial f}{\partial x}|_y = \frac{\partial}{\partial x}|_y \frac{\partial f}{\partial y}|_x$. Points (x,y) can be re-expressed in polar co-ordinates (r,θ). Use a *change of variables* (via the chain rule) to obtain expressions for: iii) $\frac{\partial f}{\partial r}|_\theta$ and iv) $\frac{\partial f}{\partial \theta}|_r$. Check that you have obtained the correct answers by first re-expressing $f(x,y)$ as $f(r,\theta)$.

Pick your answers from:
a) $-2xy\exp(-2xy)$ b) $-2x\exp(-2xy)$ c) $-2y\exp(-2xy)$ d) $-2\exp(-2xy)$ e) $-2\exp(-2xy)/r$ f) $2(y^2-x^2)\exp(-2xy)$ g) $-xy\exp(-2xy)/r$ h) $-4xy\exp(-2xy)/r$

actual answers: i) $-2y\exp(-2xy)$ - c ii) $-2x\exp(-2xy)$ - b iii) $-4xy\exp(-2xy)/r$ - h iv) $2(y^2-x^2)\exp(-2xy)$ - f

An example of the change of variables expression you need to construct is $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$. These can be obtained from the total differential of $f(x,y)$. Since $x = r\cos\theta$ and $y = r\sin\theta$ we can calculate expressions like $\frac{\partial x}{\partial r}$ and $\frac{\partial y}{\partial \theta}$ and then find the answers to iii) and iv). We can check directly by using $x = r\cos\theta$ and $y = r\sin\theta$ to eliminate x and y from $f(x,y) = exp(-2xy)$. From this new expression, $f(r,\theta)$, iii) $\frac{\partial f}{\partial r}|_\theta$ and iv) $\frac{\partial f}{\partial \theta}|_r$ can be evaluated directly.

4) Consider the equation $xyz + x^3 + y^4 + z^6 = 0$. What is the value of the product below?

$$\frac{\partial x}{\partial y}|_z \cdot \frac{\partial y}{\partial z}|_x \cdot \frac{\partial z}{\partial x}|_y \tag{1}$$

Pick your answer from:
a) 1 b) 0 c) -1 d) ∞

[harder] Show that this holds in general for $f(x, y, z) = 0$. It might help to consider $x = x(y, z)$, $y = y(x, z)$ and the total differentials dx and dy.

actual answer: -1 - c

The following notes should help:

Note Implicit expressions have cropped up already e.g. $f(x, y, z) = 0$. First of all it might be useful to remember how, in the univariate case you have been taught to do implicit differentation. E.g. Remind yourself how you would find $\frac{dy}{dx}$ when $y^2x + x \log y + 3 = 0$ (hint: you don't do it by rearranging for an expression $y = h(x)$). So what about the multivariate case if I want to find e.g. $\frac{\partial z}{\partial x}\Big|_y$? Example: for $y = z^3 + xz$. Differentiate with respect to x holding y constant (and noting the obvious, but perhaps helpful to some, that $[\frac{\partial}{\partial x}\Big|_y]z = \frac{\partial z}{\partial x}\Big|_y$). It follows from the expression for y that $0 = 3z^2 \frac{\partial z}{\partial x}\Big|_y + \frac{\partial x}{\partial x}\Big|_y z + x \frac{\partial z}{\partial x}\Big|_y$. We can then rearrange and solve for $\frac{\partial z}{\partial x}\Big|_y = \frac{-z}{3z^2+x}$.

Note There are two identities which hold for $f(x, y, z) = 0$ which you may have encountered previously. These are called the reciprocity relation and the cyclic relation.

$$\frac{\partial x}{\partial y}\Big|_z = \left[\frac{\partial y}{\partial x}\Big|_z\right]^{-1} \tag{2}$$

$$\frac{\partial x}{\partial y}|_z \cdot \frac{\partial y}{\partial z}|_x \cdot \frac{\partial z}{\partial x}|_y = -1 \tag{3}$$

These can be obtained by writing out the total differentials for $dx(y, z)$ and $dy(x, z)$ and using the expression for dy to eliminate dy from the expression for the total differential in $dx(y, z)$. Then, by considering the scenarios first when z is constant ($dz = 0$) and then when x is constant ($dx = 0$) you can obtain the above.

5) Expressions for $f(x, y)$ can be re-expressed in polar co-ordinates as $f(r, \theta)$. i) Does the following expression have the correct dimensions? ii) And is it correct?

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \tag{4}$$

Pick your answers from:
a) yes! b) no.

actual answers: i) yes - a ii) yes - a

Given $x = r\cos\theta$ and $y = r\sin\theta$ one can find $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$ one can find $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial \theta}{\partial x} = -\sin\theta/r$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial y} = \cos\theta/r$.

How does one find the double partial differentials? $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = [\frac{\partial}{\partial x}][\frac{\partial}{\partial x}]f$. So

$$\frac{\partial^2 f}{\partial x^2} = \left[\frac{x}{r}\frac{\partial}{\partial r}\bigg|_\theta + \frac{-y}{r^2}\frac{\partial}{\partial \theta}\bigg|_r\right]\left[\frac{x}{r}\frac{\partial}{\partial r}\bigg|_\theta + \frac{-y}{r^2}\frac{\partial}{\partial \theta}\bigg|_r\right] f \tag{5}$$

.

How should I interpret expressions like this? The first term when I expand the above:

$$\frac{x}{r}\frac{\partial}{\partial r}\left[\frac{x}{r}\frac{\partial}{\partial r}\right] f = \frac{x}{r}\frac{\partial}{\partial r}\left[\frac{x}{r}\frac{\partial f}{\partial r}\right] \tag{6}$$

$$= \frac{x}{r}\left[\frac{\partial (x/r)}{\partial r}\frac{\partial f}{\partial r} + \frac{x}{r}\frac{\partial^2 f}{\partial r^2}\right] \tag{7}$$

Using this insight you should be able to obtain

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial f}{\partial \theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} \tag{8}$$

and similarly

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\theta\frac{\partial^2 f}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} \tag{9}$$

this allows one to check the identity in the question.

6) For:
- $(x^2 + y^2 + 1)^{-1}$ find i) all stationary points and ii) their characters.
- $\sin x \sin y$ with $(0 < x < \pi, 0 < y < \pi)$ find iii) all stationary points and iv) their characters.
- $x^4 + y^4$ find v) all stationary points and vi) their characters. [tricky to do properly]

Calculating the Hessian in each case might be useful.

Pick your answers from:
a) $(1,1)$ b) $(1,0)$ c) $(0,1)$ d) $(0,0)$ e) $(\pi/4, \pi/2)$ f) $(\pi/2, \pi/2)$ g) max h) min i) saddle

actual answers: i) (0,0) - d ii) max - g iii) $(\pi/2, \pi/2)$ - f iv) max - g v) (0,0) - d

In each case solve for $\nabla f = \underline{0}$. For each solution to this equation one checks the signs of f_{xx} and $f_{xy}^2 - f_{xx}f_{yy}$.

The last question. Sketching this reveals that there is clearly a minimum at $(0,0)$ but how can one show this? One soon finds that at $(0,0)$ not only is $\nabla f = \underline{0}$ but also $f_{xx} = 0$ and $f_{xy}^2 - f_{xx}f_{yy} = 0$. The sufficient conditions for establishing whether a stationary point is a maximum or a minimum that I gave you in the lectures are not enough to characterize this stationary point. Recall that minima are such that all small changes away from that minimum will increase the value of the function. In order to probe this we need to revisit the Taylor's expansion:

$$
\begin{aligned}
\Delta f &= f(x,y) - f(x_0,y_0) \\
&\simeq 0 + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2\right] .... \qquad (10)
\end{aligned}
$$

Our problem is that because $x^4 + y^4$ contains terms to the fourth power all of the terms in this expansion up to cubic will be zero when we evaluate them at $(0,0)$ (all terms in a Taylor expansion to 3rd order will be of the form $x^n y^m$ with $n, m \geq 1$ so all terms are zero at $(0,0)$). Some thought reveals that to fourth order the only two terms in the Taylor expansion around $(0,0)$ will be non-zero $\Delta f = f(x,y) - f(0,0) = \frac{1}{4!}[\frac{\partial^4 f}{\partial x^4}(\Delta x)^4 + \frac{\partial^4 f}{\partial y^4}(\Delta y)^4] = (\Delta x)^4 + (\Delta y)^4$ which is positive for all small changes (so all small changes from $(0,0)$ lead to an increase in f) so we have a minimum.