EE2 Mathematics
Example Sheet 4: Fourier Transforms

Recall that a function $f(t)$ and its Fourier Transform $\mathcal{F}\{f(t)\} = \overline{f}(\omega)$ are related by

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt \qquad\qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\overline{f}(\omega)\, d\omega$$

and that the Convolution Theorem is $\mathcal{F}(f * g) = \overline{f}(\omega)\,\overline{g}(\omega)$ where $f * g = \int_{-\infty}^{\infty} f(t')g(t-t')\,dt'$.

1. What is the Fourier transform of $f(t) = e^{-|t|}$?

   Pick you answer from: a) $\overline{f}(\omega) = \frac{2}{1-\omega^2}$ b) $\overline{f}(\omega) = \frac{2}{1+\omega^2}$ c) $\overline{f}(\omega) = \frac{2}{\omega^2}$

2. (i) Show that if $f(t)$ has Fourier transform $\overline{f}(\omega)$ and a is a real constant of either sign, then $f(at)$ has Fourier transform

   $$|a|^{-1}\,\overline{f}\left(\frac{\omega}{a}\right) .$$

   (ii) Demonstrate the shift property (in the formula sheets), namely that for a constant a

   $$\mathcal{F}\{f(t-a)\} = e^{-i\omega a}\,\overline{f}(\omega)$$

3. Is it the case that when $f(t) = e^{-t^2}$ then $\overline{f}(\omega) = \sqrt{\pi}e^{-\frac{1}{4}\omega^2}$? Show also that the normalized auto-correlation function associated with $f(t)$ is given by $\gamma(t) = e^{-t^2/2}$. *Note: You may use the result $\int_{-\infty}^{\infty} e^{-(t+i\alpha)^2} dt = \sqrt{\pi}$ despite the complex argument in the exponential.*

   Pick your answer from: a) yes b) no

4. Use the convolution theorem to evaluate the Fourier transform of

   $$h(t) = \int_{-\infty}^{\infty} e^{-at'^2} e^{-b(t-t')^2} dt'$$

   Pick your answer from:

   a) $\overline{h}(\omega) = \frac{\pi}{\sqrt{ab}}e^{-\frac{\omega^2(a+b)}{4ab}}$ b) $\overline{h}(\omega) = \pi e^{-\frac{\omega^2(a+b)}{4ab}}$ c) $\overline{h}(\omega) = \frac{\pi}{\sqrt{ab}}e^{-\frac{\omega^2(a+b)}{2ab}}$

5. Use Plancherel's formula $\int_{-\infty}^{\infty} f(t)g^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)\overline{g}^*(\omega)\,d\omega$ to evaluate

   $$\int_{-\infty}^{\infty} e^{-t^2} \cos at\, dt.$$

   Pick your answer from: a) $\sqrt{\pi}\,e^{-a}$ b) $e^{-a^2/4}$ c) $\sqrt{\pi}\,e^{-a^2/4}$

6. Use Parseval's theorem $\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2\,d\omega$ to evaluate

   $$\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^2}.$$

   You will need to evaluate a pair of contour integrals to find $\overline{f}(\omega)$; one in the upper $\frac{1}{2}$-plane for $\omega < 0$ and one in the lower $\frac{1}{2}$-plane for $\omega > 0$.

   Pick your answer from: a) $\frac{\pi}{4}$ b) $\frac{1}{2}\pi$ c) $-\pi$