EE2 Mathematics

Solutions to Example Sheet 4: Fourier Transforms

1) Because $f(t) = e^{-|t|} = \left\{ \begin{array}{ll} e^{-t}, & t > 0 \\ e^{t}, & t < 0 \end{array} \right\}$ the Fourier transform of $f(t)$ is

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t - |t|} dt = \int_{-\infty}^{0} e^{t(1-i\omega)} dt + \int_{0}^{\infty} e^{-t(1+i\omega)} dt = \frac{2}{1+\omega^2}$$

2) (i) Designate $\mathcal{F}\{f(t)\} = \overline{f}(\omega)$ with a a real constant of either sign. Then $\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} e^{-i\omega t} f(at) dt$. Define $\tau = at$ so $d\tau = a\, dt$. When $a > 0$ the limits $(-\infty, \infty)$ for τ correspond to those for t, but when $a < 0$ the direction reverses. Thus

$$\mathcal{F}\{f(at)\} = |a|^{-1} \int_{-\infty}^{\infty} e^{-i\left(\frac{\omega}{a}\right)\tau} f(\tau) d\tau = |a|^{-1}\, \overline{f}\left(\frac{\omega}{a}\right)$$

(ii) The 'shift property' (in the formula sheets) $\mathcal{F}\{f(t-a)\} = e^{-i\omega a}\, \overline{f}(\omega)$ can simply be proved by defining $\tau = t - a$ then

$$\mathcal{F}\{f(t-a)\} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t-a) dt = e^{-i\omega a} \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau) d\tau = e^{-i\omega a}\, \overline{f}(\omega)$$

3) To find the Fourier transform of the non-normalized Gaussian $f(t) = e^{-t^2}$ we first complete the square in the exponential

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t - t^2} dt = e^{-\frac{1}{4}\omega^2} \int_{-\infty}^{\infty} e^{-(t+\frac{1}{2}i\omega)^2} dt = \sqrt{\pi} e^{-\frac{1}{4}\omega^2}$$

The normalized auto-correlation function of e^{-t^2} is

$$\gamma(t) = \frac{\int_{-\infty}^{\infty} e^{-u^2} e^{-(t-u)^2} du}{\int_{-\infty}^{\infty} e^{-2u^2} du} = \frac{e^{-\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-2(u-\frac{1}{2}t)^2} du}{\int_{-\infty}^{\infty} e^{-2u^2} du} = e^{-\frac{1}{2}t^2}$$

The integrals in the numerator & denominator cancel because they are equal; the origin of the former is shifted w.r.t. to the latter on the infinite u-axis but its value is not affected.

4) With $f(t) = e^{-at^2}$ and $g(t) = e^{-bt^2}$, a minor re-scaling of the results of Q3 shows that

$$\overline{f}(\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \qquad \overline{g}(\omega) = \sqrt{\frac{\pi}{b}} e^{-\frac{\omega^2}{4b}}$$

The convolution theorem says that $\mathcal{F}\left\{\int_{-\infty}^{\infty} f(t')g(t-t')\, dt'\right\} = \overline{f}(\omega)\, \overline{g}(\omega)$. Therefore, with

$$h(t) = \int_{-\infty}^{\infty} \exp\left[-at'^2\right] \exp\left[-b(t-t')^2\right]\, dt' = f * g$$

we have $\overline{h}(\omega) = \frac{\pi}{\sqrt{ab}}\, e^{-\frac{\omega^2(a+b)}{4ab}}$.

5) With $f(t) = e^{-t^2}$ for which $\overline{f}(\omega) = \sqrt{\pi}e^{-\frac{1}{4}\omega^2}$ and $g(t) = \cos at$ for which

$$\begin{aligned}\overline{g}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} \cos at \, dt = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-i\omega t}(e^{iat} + e^{-iat}) \, dt \\ &= \tfrac{1}{2} \int_{-\infty}^{\infty} \left\{ e^{-it(\omega - a)} + e^{-it(\omega + a)} \right\} dt = \pi \left\{ \delta(\omega - a) + \delta(\omega + a) \right\}\end{aligned}$$

Thus

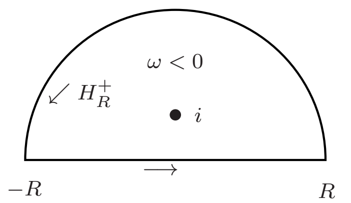
$$\int_{-\infty}^{\infty} f(t)g(t) \, dt = \frac{\pi\sqrt{\pi}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{4}\omega^2} \left\{ \delta(\omega - a) + \delta(\omega + a) \right\} d\omega = \sqrt{\pi} \, e^{-\frac{1}{4}a^2}.$$

6) Write $f(t) = (1 + t^2)^{-1}$ so that

$$\int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)^2} = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 d\omega$$

Hence we want to evaluate $\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 d\omega$. To do this we must first find $\overline{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t} dt}{1+t^2}$. To apply Jordan's Lemma it is necessary to consider the two separate cases $\omega < 0$ and $\omega > 0$.

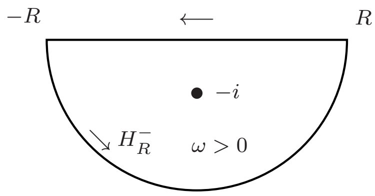
(i) $\omega < 0$: Consider the complex integral $\oint_{C_U} \frac{e^{-i\omega z} dz}{1+z^2}$ with C_U a semi-circle in the *upper* $\frac{1}{2}$-plane in which there is a simple pole at $z = i$. The residue at this pole is $e^{\omega}/2i$ and the integral over $H_R^+ \to 0$ as $R \to \infty$ by Jordan's Lemma.



For the contour C_U in the upper $\frac{1}{2}$-plane ($\omega < 0$):

$$\pi e^{\omega} = \oint_{C_U} \frac{e^{-i\omega z} dz}{1 + z^2} = \int_{-R}^{R} \frac{e^{-i\omega x} dx}{1 + x^2} + \int_{H_R^+} \frac{e^{-i\omega z} dz}{1 + z^2}$$

(ii) $\omega > 0$: Consider the complex integral $\oint_{C_L} \frac{e^{-i\omega z} dz}{1+z^2}$ with C_L a semi-circle in the *lower* $\frac{1}{2}$-plane in which there is a simple pole at $z = -i$. The residue at this pole is $-e^{-\omega}/2i$ & the integral over $H_R^- \to 0$ as $R \to \infty$ by Jordan's Lemma.



For the contour C_L in the lower $\frac{1}{2}$-plane ($\omega > 0$):

$$-\pi e^{-\omega} = \oint_{C_L} \frac{e^{-i\omega z}dz}{1+z^2} = \int_R^{-R} \frac{e^{-i\omega x}dx}{1+x^2} + \int_{H_R^-} \frac{e^{-i\omega z}dz}{1+z^2}$$

Note the reverse order of the limits in the real integral.

Thus, in the limit $R \to \infty$, we have $\overline{f}(\omega) = \left\{ \begin{array}{ll} \pi e^{-\omega}, & \omega > 0 \\ \pi e^{\omega}, & \omega < 0 \end{array} \right\}$. Finally we can now calculate

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 d\omega = \frac{\pi^2}{2\pi}\left\{\int_{-\infty}^{0} e^{2\omega}d\omega + \int_0^{\infty} e^{-2\omega}d\omega\right\} = \tfrac{1}{2}\pi\,.$$