EE2 Mathematics

Solutions to Example Sheet 3: Complex Integration

1a) $F(z) = (z^2 - 2z)^{-1} = \frac{1}{z(z-2)}$ has simple poles at $z = 0$ and $z = 2$ Only $z = 0$ lies in the unit circle $|z| = 1$. The residue is

$$\lim_{z\to 0}\left[\frac{z}{z(z-2)}\right] = -\tfrac{1}{2}$$

Using the residue theorem, $\oint_C F(z)dz = 2\pi i \times -\frac{1}{2} = -\pi i$.

1b) $F(z) = \frac{z+1}{4z^3-z} = \frac{z+1}{z(2z+1)(2z-1)}$ has simple poles at $z = 0,\ \pm\frac{1}{2}$. All of these count as they lie inside $|z| = 1$.

i) Residue at $z = 0$ is $\lim_{z\to 0}\left[\frac{z(z+1)}{z(2z+1)(2z-1)}\right] = -1$

ii) Residue at $z = -\frac{1}{2}$ is $\lim_{z\to -\frac{1}{2}}\left[\frac{(z+\frac{1}{2})(z+1)}{z(2z+1)(2z-1)}\right] = \frac{1}{4}$

iii) Residue at $z = \frac{1}{2}$ is $\lim_{z\to \frac{1}{2}}\left[\frac{(z-\frac{1}{2})(z+1)}{z(2z+1)(2z-1)}\right] = \frac{3}{4}$

The sum of the residues is $-1 + \frac{1}{4} + \frac{3}{4} = 0$. Hence the value of the integral is $2\pi i \times 0 = 0$.

1c) $F(z) = \frac{z}{1+9z^2} = \frac{z}{(3z+i)(3z-i)}$ has simple poles at $\pm i/3$. Both count as they lie inside $|z| = 1$.

i) Residue at $z = i/3$ is $\lim_{z\to i/3}\left[\frac{(z-i/3)z}{9(z-i/3)(z+i/3)}\right] = 1/18$

ii) Residue at $z = -i/3$ is $\lim_{z\to -i/3}\left[\frac{(z+i/3)z}{9(z-i/3)(z+i/3)}\right] = 1/18$

The sum of the residues is $1/18+1/18 = 1/9$. Hence the value of the integral is $2\pi i\times 1/9 = 2\pi i/9$.

2) $F(z) = \frac{z}{(z-i)^2}$ has a double pole at $z = i$ lying inside the contour C, which is the rectangle with vertices at $\pm\frac{1}{2} + 2i$ and $\pm\frac{1}{2} - 2i$.

$$\text{Residue at the double pole } z = i \text{ is:}\quad \lim_{z\to i}\left[\frac{d}{dz}\left\{\frac{(z-i)^2 z}{(z-i)^2}\right\}\right] = 1$$

Hence the integral takes the value $2\pi i$.

3) From the lectures we know that

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2} = 2\pi i \times \left\{\text{Sum of residues in upper } \tfrac{1}{2}\text{-plane of } F(z) = \frac{1}{(1+z^2)^2}\right\}$$

$F(z) = \frac{1}{(1+z^2)^2}$ has double poles at $z = i$ and at $z = -i$: count only the double pole at $z = i$.

$$\text{Residue at the pole } z = i \text{ is:}\quad \lim_{z\to i}\left[\frac{d}{dz}\left\{\frac{(z-i)^2}{(1+z^2)^2}\right\}\right] = -\frac{2}{(2i)^3} = -\tfrac{1}{4}i$$

The Residue Theorem then gives $2\pi i \times (-\frac{1}{4}i) = \frac{1}{2}\pi$ as the answer.

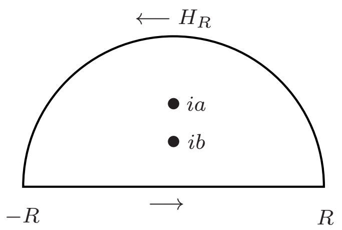
4) With $z = e^{i\theta}$ we use the fact that $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}(z + z^{-1})$ and $dz = iz\,d\theta$. Take C as the unit circle $|z| = 1$ with $\theta:\ 0 \to 2\pi$. Then

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2} = \frac{1}{i}\oint_C \frac{dz}{z\left(1 - p(z + z^{-1}) + p^2\right)} = \frac{i}{p}\oint_C \frac{dz}{(z-p)(z-p^{-1})}$$

This has simple poles at $z = p$ and $z = p^{-1}$. When $|p| < 1$ the pole at $z = p$ lies inside C while $z = p^{-1}$ lies outside and doesn't count. The reverse is true when $|p| > 1$.

(i) When $|p| < 1$ the residue of the last integral at $z = p$ is $\frac{p}{p^2-1}$. Thus $I = 2\pi i \times \frac{i}{p^2-1} = -\frac{2\pi}{p^2-1}$.
(ii) When $|p| > 1$ the residue of the last integral at $z = p^{-1}$ is $\frac{p}{1-p^2}$, so $I = 2\pi i \times \frac{i}{1-p^2} = \frac{2\pi}{p^2-1}$.

5)



The closed contour C is comprised of the semicircular contour $H_R:\quad z = Re^{i\theta}$ for $0 \le \theta \le \pi$ in the upper $\frac{1}{2}$-plane plus that part of the real axis from $x = -R$ to $x = R$.

$$F(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

Now we know that

$$\oint_C e^{iz}F(z)dz = \int_{-R}^{R} e^{ix}F(x)dx + \int_{H_R} e^{iz}F(z)dz$$

where H_R is the semi-circle. We know that $(z^2+a^2)^{-1}(z^2+b^2)^{-1}$ decays as $R \to \infty$ in such a way that Jordan's lemma is satisfied; thus

$$\lim_{R\to\infty}\int_{H_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = 0$$

Now consider the full closed contour integral $\oint_C e^{iz}F(z)dz$:

$$\text{Residue at the simple pole at}\quad z = ia \quad \text{is} \quad \frac{e^{-a}}{2ia(b^2-a^2)}$$

$$\text{Residue at the simple pole at}\quad z = ib \quad \text{is} \quad \frac{e^{-b}}{2ib(a^2-b^2)}.$$

Hence

$$\oint_C e^{iz}F(z)dz = 2\pi i \times \frac{1}{2i(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

and

$$\int_{-\infty}^{\infty} e^{ix}F(x)dx = \int_{-\infty}^{\infty} \frac{e^{ix}dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$$

The imaginary part $i \sin x$ of e^{ix} within the integral is not present because this has cancelled over the two halves of the domain $(-\infty, \infty)$. Thus we have the answer

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{(a^2-b^2)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$