EE2 Mathematics
Example Sheet 2: Functions of a complex variable

Recall that for a complex function $f(z) = u(x, y) + iv(x, y)$ the Cauchy-Riemann equations are $u_x = v_y$ and $u_y = -v_x$.

1. Do the following satisfy the Cauchy-Riemann equations:

    a) $u = x;$ $v = y$,

    b) $u = e^x \cos y;$ $v = -e^x \sin y$,

    c) $u = x^3 - 3xy^2;$ $v = 3x^2y - y^3$

    Pick your answers from i) Yes and ii) No.

2. Show that the following functions $u(x, y)$ each satisfy Laplace's equation and then use the Cauchy-Riemann equations to determine the conjugate function v. Find also $f(z) = u + iv$.

    $$a) \ \ u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1, \qquad\qquad b) \ \ u = xy.$$

    Pick your answers from: i) $3x^2y - y^3 + 6xy + c$; ii) $3x^2y + y^3 + 6xy + c$; iii) $3x^2y + 3y^3 + 6xy + c$; iv) $\frac{1}{2}(y^2 + x^2) + c$; v) $\frac{1}{2}(y^2 - x^2) + c$; vi) $f(z) = z^3 + 3z^2 + c$ .vii) $f(z) = z^3 + 2z^2 + c$ viii) $f(z) = \frac{i}{2}z + c$ iix) $f(z) = -\frac{i}{2}z^2 + c$.

3. Find an analytic function of $z = x + iy$ with an imaginary part that is $(y \cos y + x \sin y) \exp x$

    Pick your answer from:

    i) $f(z) = e^z + c$ ii) $f(z) = z + c$ iii) $f(z) = z e^{-z} + c$ iv) $f(z) = z e^z + c$

4. Consider the mapping $w = \frac{1}{z-1}$ from the z-plane to the w-plane.

    a) Show that in the z-plane, the circle

    $$(x - 1)^2 + y^2 = 4$$

    maps to a circle in the w-plane. What is the radius of this circle and where is its centre?

    b) To what curve does the line $x = 0$ in the z-plane map in the w-plane? Write out its equation.

    Pick your answers from:

    i) $\frac{1}{4}$ ii) $\frac{1}{2}$ iii) $\frac{1}{8}$ iv) $(-\frac{1}{4}, -\frac{1}{4})$ v) $(-\frac{1}{2}, -\frac{1}{4})$ vi) $(-\frac{1}{2}, 0)$ vii) $(0, 0)$ viii) Circle iix) Line ix) Archimedean Spiral x) Euler Spiral

5. a) Fixed points of a map $w = f(z)$ occur when $w = z$. Do the fixed points of $w = \frac{4z-2}{z+1}$ occur at $z = 1$ and $z = 2$?

    b) For $w = u + iv = \frac{4z-2}{z+1}$ show that the image in the w-plane of the line $x = 0$ is the circle $(u - 1)^2 + v^2 = 9$. What is the image in the w-plane of the unit circle $|z| = 1$? Write out its equation.

    Pick your answers from:

    i) Yes ii) No iii) Circle iv) Line v) Archimedean Spiral vi) Euler Spiral vii) Fermat's Spiral