Primer: Double integration

Consider the value of a scalar function $\psi(x_i, y_i)$ at the co-ordinate point (x_i, y_i) at the lower left hand corner of the square of area $\delta A_i = \delta x_i \delta y_i$. Then

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \psi(x_i, y_i)\, \delta A_i \to \underbrace{\int\!\!\int_R \psi(x, y)\, dxdy}_{double\ integral} \qquad \text{as} \qquad \delta x \to 0,\ \delta y \to 0\,. \tag{0.1}$$

We say that the RHS is the "double integral of ψ over the region R". **Note: do not confuse this with the area of R itself, which is**

$$\text{Area of R} = \int\!\!\int_R dxdy\,. \tag{0.2}$$

0.1 How to evaluate a double integral

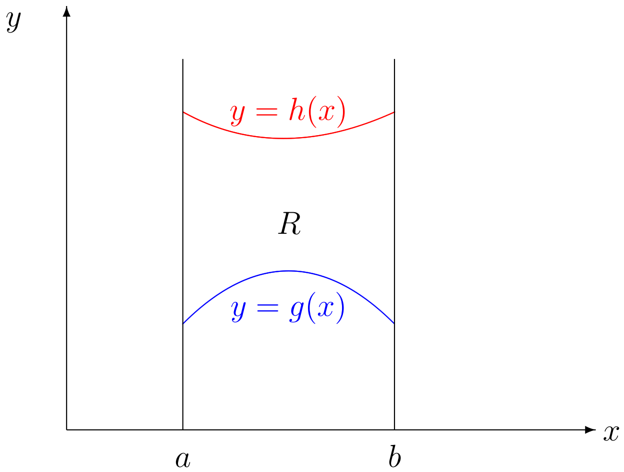


Figure 1.2: *The region R is bounded between the upper curve $y = h(x)$, the lower curve $y = g(x)$ and the vertical lines $x = a$ and $x = b$.*

$$\int\!\!\int_R \psi(x, y)\, dxdy = \int_a^b \left\{ \int_{y=g(x)}^{y=h(x)} \psi(x, y)\, dy \right\} dx \tag{0.3}$$

The inner integral is a partial integral over y holding x constant. Thus the inner integral is a function of x

$$\int_{y=g(x)}^{y=h(x)} \psi(x, y)\, dy = P(x) \tag{0.4}$$

and so

$$\int\!\!\int_R \psi(x, y)\, dxdy = \int_a^b P(x)\, dx\,. \tag{0.5}$$

Moreover the area of R itself is

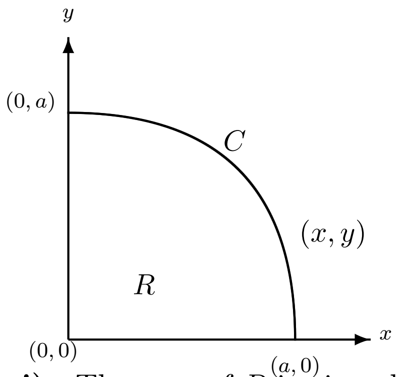
$$\text{Area of R} = \int_a^b \left\{ \int_{y=g(x)}^{y=h(x)} dy \right\} dx = \int_a^b \{h(x) - g(x)\}\, dx \tag{0.6}$$

Area under a curve: For a function of a single variable $y = f(x)$ between limits $x = a$ and $x = b$

$$\text{Area} = \int_a^b \left\{ \int_0^{f(x)} dy \right\} dx = \int_a^b f(x)\, dx\,. \tag{0.7}$$

0.2 Example of multiple integration

Consider the first quadrant of a circle of radius a. Show that :



$$(i) \qquad \text{Area of R} = \pi a^2/4$$

$$(ii) \qquad \int\!\!\int_R xy\, dxdy = a^4/8$$

$$(iii) \qquad \int\!\!\int_R x^2y^2\, dxdy = \pi a^6/96$$

i) : The area of R is given by

$$A = \int_0^a \left\{ \int_0^{\sqrt{a^2-x^2}} dy \right\} dx = \int_0^a \sqrt{a^2 - x^2}\, dx\,. \tag{0.8}$$

Let $x = a\cos\theta$ and $y = a\sin\theta$ then $A = \frac{1}{2}a^2 \int_0^{\pi/2}(1 - \cos 2\theta)\, d\theta = \pi a^2/4$.

ii) :

$$\begin{aligned} \int\!\!\int_R xy\, dxdy &= \int_0^a x \left(\int_0^{\sqrt{a^2-x^2}} y\, dy \right) dx \\ &= \tfrac{1}{2} \int_0^a x(a^2 - x^2)\, dx \\ &= \tfrac{1}{2} \left[\tfrac{1}{2}x^2a^2 - \tfrac{1}{4}x^4 \right]_0^a = a^4/8\,. \end{aligned} \tag{0.9}$$

iii) :

$$\begin{aligned} \int\!\!\int_R x^2y^2\, dxdy &= \int_0^a x^2 \left(\int_0^{\sqrt{a^2-x^2}} y^2\, dy \right) dx \\ &= \tfrac{1}{3} \int_0^a x^2(a^2 - x^2)^{3/2}\, dx \\ &= \tfrac{1}{3}a^6 \int_0^{\pi/2} \cos^2\theta \sin^4\theta\, d\theta \\ &= \tfrac{1}{3}a^6 \big(I_2 - I_3 \big)\,. \end{aligned} \tag{0.10}$$

where $I_n = \int_0^{\pi/2} \sin^{2n}\theta\, d\theta$. Using a Reduction Formula method we find that

$$I_n = \left(\frac{2n-1}{2n} \right) I_{n-1}\,, \qquad I_0 = \pi/2\,. \tag{0.11}$$

Thus $I_2 = 3\pi/16$ and $I_2 = 15\pi/96$ and so from (0.10) the answer is $\pi a^6/96$.