O-minimality, pseudo-o-minimality

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and here:

Definition

Let $\mathcal{R} = (R, <, \dots)$ be an expansion of a dense linear order without endpoints (DLO). Then $\mathcal{R}$ is *o-minimal* if any definable subset of R is a finite union of intervals and points.

Examples

- Any dense linear order without endpoints (like $\langle \mathbb{Q}, < \rangle$).
- Any ordered divisible Abelian group (like $\langle \mathbb{Q}, +, < \rangle$).
- Any ordered real closed field (like $\langle \mathbb{R}, +, -, \cdot, 0, 1, < \rangle$).
- $\mathbb{R}_{an}$: $\langle \mathbb{R}, +, -, \cdot, 0, 1, < \rangle$ expanded by all analytic functions restricted to $[0, 1]^n$.
- $\mathbb{R}_{\mathrm{exp}}$: $\langle \mathbb{R}, +, -, \cdot, 0, 1, < \rangle$ expanded by the exponential function.
- *o*-minimality is preserved under elementary equivalence.

Theorem (Monotonicity Theorem, Knight, Pillay and Steinhorn 1986)

Let $f : I \to R$ be a definable function on some open interval $I = (a, b)$ (where the endpoints could be $\pm\infty$). Then there are $a_0 = a < a1 < \cdots < a_n = b$ such that on each (a_i, a_{i+1}) the function f is either constant or strictly monotone and continuous.

Alexandre Grothendieck, Esquisse d'un programme:

" My approach toward possible foundations for a tame topology has been an axiomatic one. Rather than declaring [...] that the desired "tame spaces" are [...] semianalytic spaces, and then developing [...] constructions and notions which are familiar from topology, [...] I preferred to work on extracting which [...] geometrical properties of the semianalytic sets in a space $\mathbb{R}^n$, make [...] a notion of "tame space" [...] and what (hopefully!) makes this [...] the fundamental notion for a "tame topology" which would express with ease the topological intuition of shapes. "

Definition (Miller, 2002)

Let $\mathcal{R} = (R, <, \dots)$ be an expansion of a DLO. Then $\mathcal{R}$ has the *intermediate value property* (IVP) if for all $a, b \in R$, every continuous definable function $f : [a, b] \to R$ takes on all values in R between $f(a)$ and $f(b)$.

Examples

- Any expansion of $(\mathbb{R}, <)$
- Any *o*-minimal expansion of $(R, <)$.
- $(\mathbb{Q}, <, +, \mathbb{Z})$.
- IVP is preserved under elementary equivalence.
- Let G be a divisible subgroup of $\mathbb{R}$ with $1 \in G$. Then $(G, <, +, \mathbb{Z})$ has IVP.

Basic results (Miller 2002)

Proposition

Assume $\mathcal{R} = (R, <, \dots)$ has IVP. Let $A \subseteq R$ be definable. Then both $\inf(A)$ *and* $\sup(A)$ *exist in $R \cup \{\pm\infty\}$*

Proof.

Let A be bounded from below. To show $\inf(A) \in R$:
Let $A' := \{b \in R : \exists a \in A(a \leq b)\}$.
Then $\inf(A) = \inf(A')$.
If either $\min(A')$ or $\max(R \setminus A')$ exist, we are done.
Otherwise, they are both open.
Choose some $a < b \in R$.
By defining $f : R \to \{a, b\}$ by $f(x) := \begin{cases} a & \text{if } x \notin A' \\ b & \text{if } x \in A' \end{cases}$
we contradict IVP. $\square$

Basic results (Miller 2002)

Definition

Let $\mathcal{R} = (R, <, \dots)$. A set $A \subseteq R^n$ is definably connected if it is definable and for all disjoint open definable sets $U, V \subseteq R^n$, if $A = (A \cap U) \cup (A \cap V)$, then either $A \cap U = \emptyset$ or $A \cap V = \emptyset$.

Proposition

Assume $\mathcal{R} = (R, <, \dots)$ has IVP. Then for any definable $A \subseteq R$, TFAE:

1. *A is definably connected.*
2. *A is an interval.*
3. *A is convex.*

Basic results (Miller 2002)

Definition

An expansion $\mathcal{R} = (R, <, \dots)$ of a DLO is *definably complete* if $\sup(R)$ and $\inf(R)$ exist for every bounded non-empty definable $A \subset R$.

Proposition

If $\mathcal{R} = (R, <, \dots)$ expands a DLO, then TFAE:

1. *$\mathcal{R}$ has IVP.*
2. *$\mathcal{R}$ is definably complete.*
3. *Intervals in R are definably connected.*
4. *R is definably connected.*

Proposition (Miller, 2002)

Let $f : A \to R^n$ be definable and continuous, with $A \subset R^m$ closed and bounded.

1. *$f(A)$ is closed, bounded, and definable.*
2. *If f is injective, then f maps A homeomorphically onto $f(A)$.*
3. *Let $B \subset f(A)$ be definable. Then B is closed if and only if $f(B)$ is closed.*
4. *Let $g : f(A) \to R^l$ be definable. Then g is continuous if and only if $g \circ f$ is.*

Theorem (Fornasiero-Hieronymi, 2015)

Let $\mathcal{K} = (K, <, +, \cdot, ...)$ be a definably complete expansion of an ordered field. Then either

1. *$f(D)$ is nowhere dense for every definable discrete set $D \subseteq K^n$ and every definable function $f : K^n \to K$, or*
2. *$\mathcal{K}$ defines a discrete subring.*

local o-minimality

Definition

Let K be a definably complete, locally o-minimal (expansion of an ordered field) An expansion of a DLO $\mathcal{M} = \langle M; <, \dots \rangle$ is *locally o-minimal* if for any definable subset $A \subseteq M$ and any $a \in M$ there is some interval $a \in I \subset M$ such that $I \cap A$ is a finite union of intervals and points.

Theorem (Fornasiero, 2013)

Let $f : (a, b) \to R$ be a definable function. Then, there exists a closed, bounded and discrete definable set $A \subseteq (a, b)$, such that on each interval $x, s(x)$, with $x \in A \cup \{a\}$, the function is either constant, or strictly monotone and continuous.

Type Completeness

Definition

An expansion of a DLO $\mathcal{M} = \langle M; <, \ldots \rangle$ is *type complete* if it is locally *o*-minimal and, in addition, for any definable subset $A \subseteq M$ there are $c_1, c_2 \in M$ such that if $I = (-\infty, c_1)$ or $(c_2, +\infty)$, then either $I \subset A$ or $I \cap A = \emptyset$.

Theorem (Schoutens, 2014)

Let $\mathcal{M} = \langle M; <, \ldots \rangle$ be a definably complete and type complete structure and let $f : M \to M$ be definable. Then there is a definable discrete $D \subset M$ such that between any two consecutive points in $D \cup \{\pm\infty\}$, the map f is continuous and either strictly monotone or constant.

Theorem (Schoutens, 2014)

Let R be an ordered structure. Then TFAE:

- *R is definably complete and type complete.*
- *Every definable subset of R is a disjoint union of open intervals and a single discrete set.*

"Thus, once this necessary foundational work has been completed, there will appear not one "tame theory", but a vast infinity, ranging from the strictest of all, [...] to the one which appears (whether rightly or not) to be likely to be the vastest of all"

Definition (Schoutens, 2014)

Given a language $\mathcal{L}$ with an order relation, let $T^{omin} := T^{omin}(L)$ be the intersection of the theories of all o-minimal $\mathcal{L}$-structures. Models of T^{omin} will be called *pseudo-o-minimal*.

pseudo-o-minimal $\implies$ DC, TC, locally o-minimal

Definition

Let $\mathcal{M} = \langle M; <, \ldots \rangle$ be a definably complete structure. A definable subset $A \subset M$ is *pseudo-finite* if it is closed, bounded, and discrete.

Definition

Let $\mathcal{M} = \langle M; <, \ldots \rangle$ be a pseudo-o-minimal structure. A definable set $X \subseteq M$ is *pseudo-o-finite* if $(\mathcal{M}, X)$ satisfies the common theory of o-minimal structures expanded by a unary predicate for a distinguished finite subset.

Proposition (Schoutens, 2014)

- *Let $\mathcal{M} = \langle M; <, \ldots \rangle$ be a* definably complete type complete *structure. A definable set $A \subset M$ is pseudo-finite if and only if it is* discrete.
- *Let $\mathcal{M} = \langle M; <, \ldots \rangle$ be a* pseudo-o-minimal *structure. A definable set $A \subset M$ is pseudo-finite if and only if it is* pseudo-o-finite.

Definition

An expansion of a DLO $\mathcal{R} = (R, <, \dots)$ has *the pigeon-hole principle* if whenever $D \subset R$ is pseudo-finite and $f : D \to D$ is definable and injective, then it is also surjective, i.e., every pseudo-finite set is definably Dedekind finite.

Question (Schoutens, 2014)

Does definable completeness with type completeness imply the pigeon-hole principle?

Conjecture (Fornasiero, 2013)

If $\mathcal{R}$ is a locally o-minimal definably complete expansion of a field, then it has the pigeon-hole principle.

Question (Schoutens, 2014)

Can we axiomatize pseudo-o-minimality by first-order conditions (on one-variable formulae only)?

Theorem (Rennet, 2014)

There is no recursive axiomatization of T^{omin}.

Cyclic orders

Definition

A *cyclic order* on a set A is a ternary relation C satisfying the following axioms:

1. Cyclicity: If $C(a,b,c)$, then $C(b,c,a)$.
2. Asymmetry: If $C(a,b,c)$, then not $C(c,b,a)$.
3. Transitivity: If $C(a,b,c)$ and $C(a,c,d)$, then $C(a,b,d)$.
4. Totality: If a,b,c are distinct, then either $C(a,b,c)$ or $C(c,b,a)$.

Fact

If $\langle A,<\rangle$ is a linearly ordered set, then the relation defined by

$$C_<(a,b,c) \iff (a<b<c)\vee(b<c<a)\vee(c<a<b)$$

is a cyclic order on A.
We call $C_<$ the cyclic order induced by $<$.

Let $\mathcal{L}_1 := \langle <, Z \rangle \cup \{ f \}$.

Let T_1 be the $\mathcal{L}_1$-theory consisting of the following:

1. $T^{omin}_{\mathcal{L}_1}$
2. $<$ is DLO.
3. Z is closed, bounded and discrete.
4. f is an automorphism of the cyclic order on Z.
5. f is the identity outside Z.
6. $C_<(f^m(z), f^n(z), z)$ for all $z \in Z$, $m > n > 0$.
7. For every $z \in Z$ and $n > 0$, there are infinitely many elements in Z between z and $f^n(z)$.

Proposition

T_1 is consistent.

Let $\mathcal{L}_1 := \langle <, Z \rangle \cup \{ f, g \}$.

Let T_1 be the $\mathcal{L}_1$-theory consisting of the following:

1. ☑ $T^{omin}_{\mathcal{L}_1}$
2. ☑ $<$ is DLO.
3. ☑ Z is closed, bounded and discrete.
4. ☑ f is an automorphism of the cyclic order on Z.
5. ☑ f is the identity outside Z.
6. ☑ $C_<(f^m(z), f^n(z), z)$ for all $z \in Z, k > m > n > 0$.
7. ☑ For every $z \in Z$ and $n >$, there are at least k many elements in Z between z and $f^n(z)$.

Proposition

T_1 is consistent. ☑

Theorem (M.)

There are two ordered structures in the same language $\mathcal{M}, \mathcal{N}$ on the same universe, admitting the same order and the same definable subsets with $\mathcal{M}$ being pseudo-o-minimal and $\mathcal{N}$ not.

In particular, there is no axiomatization of pseudo-o-minimality by first-order conditions on one-variable formulae only.

Furthermore, there is no axiomatization of pseudo-o-minimality by any second order theory in the language $\mathcal{L}_{\mathrm{Def}} := \{ <, \mathrm{Def} \}$ where Def *is interpreted as the definable one-variable subsets.*

Theorem (M.)

There is a definably complete type complete structure without the pigeonhole property.

Question

Can this be done by expanding a real closed field?

Thank you !

“ My approach toward possible foundations for a tame topology has been an axiomatic one. Rather than declaring (which would indeed be a perfectly sensible thing to do) that the desired “tame spaces” are no other than (say) Hironaka’s semianalytic spaces, and then developing in this context the toolbox of constructions and notions which are familiar from topology, supplemented with those which had not been developed up to now, for that very reason, I preferred to work on extracting which exactly, among the geometrical properties of the semianalytic sets in a space $\mathbb{R}^n$, make it possible to use these as local “models” for a notion of “tame space” (here semianalytic), and what (hopefully!) makes this notion flexible enough to use it effectively as the fundamental notion for a “tame topology” which would express with ease the topological intuition of shapes.
”

“Thus, once this necessary foundational work has been completed, there will appear not one “tame theory”, but a vast infinity, ranging from the strictest of all, the one which deals with “piecewise $\overline{\mathbb{Q}}_r$-algebraic spaces” (with $\overline{\mathbb{Q}}_r = \overline{\mathbb{Q}} \cap \mathbb{R}$), to the one which appears (whether rightly or not) to be likely to be the vastest of all, namely using “piecewise real analytic spaces” (or semianalytic using Hironaka's terminology).

⋮

What is again lacking is not the technical virtuosity of the mathematicians, which is sometimes impressive, but the audacity (or simply innocence...) to free oneself from a familiar context accepted by a flawless consensus...
The advantages of an axiomatic approach towards the foundations of tame topology seem to me to be obvious enough.
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