Unipotent class representatives for finite classical groups

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Abstract

We describe explicitly representatives of the conjugacy classes of unipotent elements of the finite classical groups.

1 Introduction

The conjugacy classes of unipotent elements in finite and algebraic groups of Lie type have been studied over a long period. There is a substantial literature on the topic, starting with fundamental papers such as [3, 5, 7, 8], and culminating in the book [6], in which most questions about class representatives and centralizers of unipotent elements are answered. However, perhaps surprisingly, there is a rather natural problem that has not been completely addressed in the literature – namely, finding explicit representatives for the conjugacy classes of unipotent elements in finite classical groups. In this paper we solve this problem completely.

For classical groups over algebraically closed fields, such representatives can be found in [6, Chapters 3 and 6], but such a class can intersect a corresponding finite classical group in many classes. Describing representatives for these classes when the characteristic of the underlying field is good – meaning that it is odd in the case of symplectic and orthogonal groups – is fairly straightforward, one reason being that in the algebraic group the Jordan form of an element determines its conjugacy class. However, this is not true in bad characteristic, and considerably more work is required for this case. We cover both good and bad characteristics; see Propositions 2.1–2.4 and Theorem 3.1, respectively.

Once we have a list of representatives, it is easy to see over which fields they can be realised, and we give a couple of such consequences in Proposition 4.1. This result is used in [1, pp. 274-5]. In Section 5 we present

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tables listing all unipotent class representatives and their centralizer orders for some classical groups in low dimensions.

Apart from their intrinsic interest, our results form part of an ongoing research program of the authors and others to list conjugacy class representatives for finite groups of Lie type, together with algorithms to produce generators for their centralizers and also to test arbitrary elements for conjugacy.

2 Good characteristic

In this section we deal with the finite classical groups in good characteristic – that is, special linear and unitary groups in arbitrary characteristic, and symplectic and orthogonal groups in odd characteristic.

2.1 $SL_n(q)$

Writing down unipotent class representatives in $SL_n(q)$ is elementary, but we include this for completeness. For a vector space V of dimension k over the field $\mathbb{F}_q$ with basis $v_1, \ldots, v_k$, and $\beta \in \mathbb{F}_q^*$, define $J_\beta(k) \in GL(V)$ by

$$\begin{aligned}&v_1 \to v_1 + \beta v_2,\\ &v_i \to v_i + v_{i+1}\ (2 \le i \le k-1),\\ &v_k \to v_k.\end{aligned}$$

By a sum of such elements, for example $J_1(k_1)+J_\beta(k_2)$, we mean the natural direct sum.

Write $J_k = J_1(k)$, a unipotent Jordan block of size k. It is well-known that the conjugacy classes of unipotent elements in $GL_n(q)$ are in bijective correspondence with the Jordan forms $\sum_{i=1}^s (J_{n_i})^{r_i}$ (where $n = \sum r_i n_i$).

For a positive integer t, write $(\mathbb{F}_q^*)^t = \{x^t : x \in \mathbb{F}_q^*\}$.

Proposition 2.1 *The number of unipotent classes in $SL_n(q)$ with Jordan form $\sum_1^s (J_{n_i})^{r_i}$ (where the n_i are distinct and $n = \sum r_i n_i$) is equal to $\gcd(n_1, \ldots, n_s, q-1)$. Representatives of these classes are*

$$J_\beta(n_1) + J_1(n_1)^{r_1 - 1} + \sum_{i=2}^{s} J_1(n_i)^{r_i},$$

where β runs over representatives of $\mathbb{F}_q^*/(\mathbb{F}_q^*)^t$ with $t = \gcd(n_1, \ldots, n_s)$.

Proof. Write $G = GL_n(q)$. Let $u = \sum_1^k J_1(n_i)^{r_i}$. Writing u as $\sum_1^k J_1(n_i) \otimes I_{r_i}$, we see that $C_G(u)$ contains a subgroup $R := \prod_1^k GL_{r_i}(q)$. By [6, Theorem 7.1], $C_G(u) = UR$ where U is a unipotent normal subgroup.

If $r = (g_1, \dots, g_k) \in R$ (where $g_i \in GL_{r_i}(q)$), then $\det(r) = \prod \det(g_i)^{n_i}$, so $\{\det(r) : r \in R\} = (\mathbb{F}_q^*)^t$. Hence the number of $SL_n(q)$-classes in u^G is equal to $|\mathbb{F}_q^*/(\mathbb{F}_q^*)^t|$. Representatives of these classes are given by $d(\beta)ud(\beta)^{-1}$, where $d(\beta) = \mathrm{diag}(\beta, 1, \dots, 1)$ and β runs over representatives of $\mathbb{F}_q^*/(\mathbb{F}_q^*)^t$. ∎

2.2 Notation

We now establish some general notation for the unitary, symplectic and orthogonal groups. Let V be a vector space over a field $\mathbb{F}_{q^u}$, where $u = 2$ in the unitary case and $u = 1$ otherwise, and let $(\ ,\ )$ be a non-degenerate unitary, symplectic or orthogonal form on V. We may take V to have a standard basis of the form

$$\begin{array}{l} e_1, \dots, e_k, f_k, \dots, f_1 \ \ (\dim V = 2k) \\ e_1, \dots, e_k, d, f_k, \dots, f_1 \ \ (\dim V = 2k+1), \end{array}$$

where $(e_i, e_j) = (f_i, f_j) = 0$, $(e_i, f_j) = \delta_{ij}$, $(e_i, d) = (f_i, d) = 0$ and $(d, d) = \epsilon$ (a nonzero scalar to be specified). For $\beta, \gamma, \delta \in \mathbb{F}_{q^u}$, we define the following elements of $GL(V)$. For $\dim V = 2k$,

$$\begin{array}{ll} A_\beta(2k): & e_i \to e_i + \cdots + e_k + \beta f_k \ (1 \le i \le k), \\ & f_i \to f_i - f_{i-1} \ (2 \le i \le k), \\ & f_1 \to f_1, \end{array}$$

and for $\dim V = 2k+1$,

$$\begin{array}{ll} A_{\gamma,\delta}(2k+1): & e_i \to e_i + \cdots + e_k + d + \gamma f_k \ (1 \le i \le k), \\ & d \to d + \delta f_k, \\ & f_i \to f_i - f_{i-1} \ (2 \le i \le k), \\ & f_1 \to f_1. \end{array}$$

Note that if $\beta, \gamma, \delta \neq 0$ then $A_\beta(2k)$ and $A_{\gamma,\delta}(2k+1)$ have Jordan forms J_{2k} and J_{2k+1} respectively; and $A_0(2k)$ has Jordan form J_k^2.

2.3 $SU_n(q)$

For the unitary case we take $(d, d) = 1$ in the above standard basis when the dimension is odd. Let $\lambda \to \bar{\lambda}$ be the involutory automorphism of $\mathbb{F}_{q^2}$, and pick $\beta, \gamma \in \mathbb{F}_{q^2}^*$ satisfying $\beta + \bar{\beta} = 0$, $\gamma + \bar{\gamma} = -1$. For $l \in \{2k, 2k+1\}$ define

$$U(l) = \begin{cases} A_\beta(2k), \ l = 2k \\ A_{\gamma,-1}(2k+1), \ l = 2k+1 \end{cases}$$

Observe that $U(l) \in SU_l(q)$. By a sum of such elements, for example $U(l_1) + U(l_2)$, we mean the natural perpendicular sum. For a unitary group of given dimension n, we can find, via a suitable change of basis, a unipotent element corresponding to any perpendicular sum of such elements of total dimension n.

Define $Z = \{\alpha \in \mathbb{F}_{q^2} : \alpha\bar{\alpha} = 1\}$, and for $\alpha \in Z$ let $h(\alpha) \in GU_n(q)$ have determinant α; for example, we could take $h(\alpha)$ to send $e_1 \to \lambda e_1$, $f_1 \to \bar{\lambda}^{-1} f_1$ and fix all other basis vectors, where $\alpha = \lambda\bar{\lambda}^{-1}$. As in the case of GL_n, it is well-known that the unipotent classes of $GU_n(q)$ correspond bijectively with the possible Jordan forms.

Proposition 2.2 *The number of unipotent classes in $SU_n(q)$ with Jordan form $\sum_1^s (J_{n_i})^{r_i}$ (where the n_i are distinct and $n = \sum r_i n_i$) is equal to* $\gcd(n_1, \ldots, n_s, q+1)$. *Representatives of these classes are*

$$\sum_{i=1}^{s} U(n_i)^{r_i},$$

together with conjugates of these elements by $h(\alpha)$, where α runs over representatives of Z/Z^t with $t = \gcd(n_1, \ldots, n_s)$.

The proof is very similar to that of Proposition 2.1.

2.4 $Sp_{2m}(q)$, q odd

In this section let $G = Sp_{2m}(q)$ with q odd. Adopting the notation of Section 2.2, for $\beta \in \mathbb{F}_q^*$ and $k \geq 1$, $l \geq 0$, define

$$\begin{aligned} V_\beta(2k) &= A_\beta(2k), \\ W(2l+1) &= A_0(4l+2). \end{aligned}$$

Observe that $V_\beta(2k) \in Sp_{2k}(q)$ and $W(2l+1) \in Sp_{4l+2}(q)$. As before, given a symplectic group in dimension $2m$, we can find, via a suitable change of basis, a unipotent element corresponding to any perpendicular sum of such elements of total dimension $2m$.

For unipotent elements of G, all odd Jordan block sizes occur with even multiplicity (see [6, Cor. 3.6]), so the Jordan form is

$$\sum_{i=1}^{r} (J_{2k_i})^{a_i} + \sum_{i=1}^{s} (J_{2l_i+1})^{2b_i}, \tag{1}$$

where the k_i are distinct, the l_i are distinct, and $2(\sum k_i a_i + \sum (2l_i+1) b_i) = 2m$.

Let α be a fixed non-square in the field $\mathbb{F}_q$.

Proposition 2.3 *There are 2^r unipotent classes in $Sp_{2m}(q)$ (q odd) having Jordan form* (1)*. Representatives are*

$$\sum_{i=1}^{r}(V_{\beta_i}(2k_i)+V_1(2k_i)^{a_i-1})+\sum_{i=1}^{s}W(2l_i+1)^{b_i} \tag{2}$$

where each $\beta_i \in \{1,\alpha\}$.

Proof. The fact that there are 2^r classes having Jordan form (1) is given by [6, Theorem 7.1].

Consider a single Jordan block $u = V_1(2k) \in Sp_{2k}(q)$. Let $C = CSp_{2k}(q)$, the conformal symplectic group (that is, the group preserving the symplectic form up to scalar multiples). Then $C_G(u) = U \times \langle -1 \rangle$, while $C_C(u) = U \times \mathbb{F}_q^*$, where U is a unipotent group. Since $|C : G| = q-1$ it follows that u^C splits into two G-classes, and these are represented by u and $u^{d(\alpha)}$, where $d(\alpha)$ is an element of C multiplying the form by the non-square α^{-1}. Taking $d(\alpha)$ to send $e_i \to \alpha^{-1}e_i$ and $f_i \to f_i$ for all i, we can take these representatives to be $V_1(2k)$ and $V_1(2k)^{d(\alpha)} = V_\alpha(2k)$. This proves the result for a single Jordan block.

Now consider elements with Jordan form J_{2k}^2 in $Sp_{4k}(q)$, of which there are two classes (since the parameter $r = 1$ for this Jordan form). Let $V_2^\epsilon(\epsilon = \pm)$ be a 2-space over $\mathbb{F}_q$ equipped with a non-degenerate orthogonal form of type ϵ, and let V_{2k} be a symplectic $2k$-space. Let e_i, f_i be a standard symplectic basis for V_{2k}, and let v_1, v_2 be a basis of V_2^ϵ with respect to which the orthogonal form has Gram matrix $\mathrm{diag}(1,\beta)$ for some $\beta \in \mathbb{F}_q^*$. Then the product form on $V_{2k} \otimes V_2^\epsilon$ is symplectic, and has standard basis

$$e_i \otimes v_1,\ e_i \otimes v_2,\ f_i \otimes v_1,\ f_i \otimes \beta^{-1}v_2\ \ (1 \le i \le k).$$

The two classes in $Sp_{4k}(q)$ are those corresponding to $V_1(2k) \otimes I(V_2^\epsilon)$ for $\epsilon = +$ and $\epsilon = -$ (as these have centralizers having reductive parts $O(V_2^+)$ and $O(V_2^-)$ respectively). Since the matrix of $V_1(2k) \otimes I(V_2^\epsilon)$ with respect to the above basis is that of $V_1(2k)+V_\beta(2k)$, the two classes are represented by $V_1(2k)+V_1(2k)$ and $V_1(2k)+V_\alpha(2k)$.

It follows from the previous two paragraphs that elements with Jordan form J_{2k}^a fall into two classes in $Sp_{2ka}(q)$, with representatives $V_\beta(2k) + V_1(2k)^{a-1}$ for $\beta \in \{1,\alpha\}$. Also elements with Jordan form J_{2l+1}^{2b} are all conjugate in $Sp_{2b(2l+1)}(q)$ (as the parameter $r = 0$ for this Jordan form). Hence every element with Jordan form (1) is conjugate in G to one of the representatives (2). The proof is complete. ■

2.5 $SO_n^\epsilon(q)$, q odd

In this section let $G = O_n^\epsilon(q)$ with q odd and $\epsilon = \pm$ (for n odd we set $O_n^+(q) = O_n^-(q) = O_n(q)$). Let $\beta \in \mathbb{F}_q^*$ and take $(d,d) = 2\beta$ in the standard basis when the dimension is odd. Adopting the notation of Section 2.2, for $k \geq 0, l \geq 1$, define

$$\begin{aligned} V_\beta(2k+1) &= A_{-\beta,-2\beta}(2k+1), \\ W(2l) &= A_0(4l). \end{aligned}$$

Observe that $W(2l) \in SO_{4l}^+(q)$ and $V_\beta(2k+1) \in SO_{2k+1}(q)$, the latter fixing an orthogonal form of discriminant $2\beta \cdot (-1)^k$. A perpendicular sum $\sum V_{\beta_i}(2k_i+1) + \sum W(2l_j)$ lies in an orthogonal group that fixes a form of discriminant $\prod(2\beta_i \cdot (-1)^{k_i})$.

For unipotent elements of G, all even Jordan block sizes occur with even multiplicity (see [6, Cor. 3.6]), so the Jordan form is

$$\sum_{i=1}^{r}(J_{2k_i+1})^{a_i} + \sum_{j=1}^{s}(J_{2l_j})^{2b_j}, \tag{3}$$

where the k_i are distinct, the l_j are distinct, and $\sum a_i(2k_i+1)+4\sum b_j l_j = n$.

Let α be a fixed non-square in the field $\mathbb{F}_q$.

Proposition 2.4 *Suppose q is odd.*

(i) *If n is even, the unipotent elements with Jordan form (3) fall into 2^{r-1} classes in each of $O_n^+(q)$ and $O_n^-(q)$, with the exception that if $r = 0$, it is 1 class in $O_n^+(q)$ and none in $O_n^-(q)$. Representatives are*

$$\sum_{i=1}^{r}(V_{\beta_i}(2k_i+1) + V_1(2k_i+1)^{a_i-1}) + \sum_{j=1}^{s} W(2l_j)^{b_j} \tag{4}$$

where each $\beta_i \in \{1, \alpha\}$; for $r > 0$, half of these lie in $O_n^+(q)$ and half in $O_n^-(q)$. If u is such a representative, then u^G splits into two $SO_n^\epsilon(q)$-classes (with representatives u and u^t for t a reflection) if and only if $\epsilon = +$ and $r = 0$.

(ii) *If n is odd, there are 2^{r-1} classes in $SO_n^\epsilon(q)$ with Jordan form (3); representatives are as in (4) – half of these 2^r representatives fix an orthogonal form of square discriminant, and half fix a form of non-square discriminant.*

(iii) *An $SO_n^\epsilon(q)$-class with representative u as in (4) splits into two $\Omega_n^\epsilon(q)$-classes if and only if either $r = 0$, or $r \geq 1$ and the following hold:*

(a) *$a_i = 1$ for all i, and*

(b) *$\beta_i \equiv (-1)^{k_1+k_i}\beta_1$ mod $(\mathbb{F}_q^*)^2$ for all i.*

In case of splitting, representatives of the $\Omega_n^\epsilon(q)$-classes are u and u^s, where $s \in SO_n^\epsilon(q) \setminus \Omega_n^\epsilon(q)$.

Proof. The numbers of classes in (i) and (ii) are given by [6, Theorem 7.1]. All elements with Jordan form J_{2l}^{2b} are conjugate in $O_{4bl}^{+}(q)$. And for any a, the classes with Jordan form J_{2k+1}^{a} in $O_{a(2k+1)}^{\epsilon}(q)$ are represented by $V_{\beta}(2k+1) + V_1(2k+1)^{a-1}$ for $\beta \in \{1, \alpha\}$, since one of these fixes a form of square discriminant, the other a form of non-square discriminant. Hence every element with Jordan form (3) is conjugate in G to one of the representatives (4). The splitting statement in the last sentence of (i) is given by [6, Lemma 3.11]. Parts (i) and (ii) follow.

Now consider part (iii). Note that an $SO_n^{\epsilon}(q)$-class with representative u as in (4) splits into two $\Omega_n^{\epsilon}(q)$-classes if and only if $C_{SO_n^{\epsilon}(q)}(u) \leq \Omega_n^{\epsilon}(q)$. Suppose first that either $r = 0$, or $r \geq 1$ and (a) and (b) hold. Write $V_i = V_{\beta_i}(2k_i+1)$. For distinct i, j the orthogonal form restricted to the subspace $V_i + V_j$ has discriminant $4\beta_i\beta_j \cdot (-1)^{k_i+k_j}$, which by (b) is a square; so by [4, Prop. 2.5.13], $-1 \in \Omega(V_i + V_j)$. Hence it follows from [6, Theorem 7.1] that $C_{SO_n^{\epsilon}(q)}(u) \leq \Omega_n^{\epsilon}(q)$ and the class splits. Conversely, if the class splits then [6, Prop. 7.2] implies that either $r = 0$, or $r \geq 1$ and $a_i = 1$ for all i; and if (b) does not hold then there exists i such that $-1 \notin \Omega(V_1 + V_i)$, so $C_{SO_n^{\epsilon}(q)}(u) \not\leq \Omega_n^{\epsilon}(q)$, a contradiction. ■

Remark Part (iii) is an improvement of [6, Prop. 7.2]. It implies the following: of the classes with representative as in (4) with $r \geq 1$ and $a_i = 1$ for all i, exactly one splits in $\Omega_n(q)$ if n is odd; and if n is even, exactly two such classes split, both of which lie in $SO_n^{\epsilon}(q)$ with $\epsilon = (-1)^{n(q-1)/4}$.

3 Bad characteristic

In this section we deal with the symplectic and orthogonal groups in characteristic 2. This requires considerably more work than the good characteristic case.

In order to state our results, we need to define various indecomposable unipotent elements in symplectic and orthogonal groups in characteristic 2. The following definitions appear in [6, Section 6.1] for these groups over algebraically closed fields; we now adapt them to finite fields. Let $q = 2^a$, let $F = \mathbb{F}_q$ and let V be a vector space of dimension $2k$ over F, where k is a positive integer. Let $(\,,\,)$ be a non-degenerate symplectic form on V, with a standard basis $e_1, \ldots, e_k, f_k, \ldots, f_1$ such that $(e_i, e_j) = (f_i, f_j) = 0$, $(e_i, f_j) = \delta_{ij}$ for all i, j. Denote by $Sp(V)$ the symplectic group on V preserving this form.

For $k \geq 2$ and $\beta \in F$, define $V_{\beta}(2k)$ to be the element of $GL(V)$ acting as follows:

$$\begin{aligned}
&e_i \to e_i + \cdots + e_k + f_k + \beta f_{k-1} \text{ for } 1 \leq i \leq k-1,\\
&e_k \to e_k + f_k,\\
&f_i \to f_i + f_{i-1} \text{ for } 2 \leq i \leq k,\\
&f_1 \to f_1.
\end{aligned}$$

For $k = 1$, define $V_\beta(2)$ to send $e_1 \to e_1 + f_1$, $f_1 \to f_1$ (this definition does not depend on β, but we will define corresponding elements of orthogonal groups which do). It is easily checked that $V_\beta(2k)$ lies in $Sp(V)$ and has a single Jordan block. Moreover, if we define a quadratic form Q_β on V associated to (,) by setting

$$Q_\beta(e_k) = \beta,\ Q_\beta(f_k) = 1 \text{ and } Q_\beta(e_i) = Q_\beta(f_i) = 0 \text{ for } i \neq k,$$

then $V_\beta(2k)$ lies in the orthogonal group $O(V)$ preserving this form. Choose a fixed $\alpha \in F$ such that the polynomial $x^2+x+\alpha$ is irreducible over F. Now Q_0 is a quadratic form of plus type, and Q_α is of minus type (see Lemma 3.3). We shall write just $V(2k)$ instead of $V_0(2k)$. Thus

$$V(2k) \in O^+_{2k}(q),\ V_\alpha(2k) \in O^-_{2k}(q).$$

For $k \geq 1$ define $W(k)$ to be the element of $GL(V)$ acting as follows:

$$\begin{aligned}
&e_i \to e_i + e_{i+1} \text{ for } 1 \leq i \leq k-1,\\
&e_k \to e_k,\\
&f_i \to f_i + f_{i-1} + \cdots + f_1 \text{ for } 1 \leq i \leq k,
\end{aligned}$$

and let Q be the quadratic form associated to (,) such that $Q(e_i) = Q(f_i) = 0$ for all i. Note that $W(1) = I_2$, the 2-dimensional identity. Observe that $W(k)$ lies in $Sp(V)$, and also in $\Omega^+(V)$, the orthogonal group of plus type preserving Q; that it lies in $\Omega^+(V)$ rather than just $SO^+(V)$ follows from the fact that it fixes a maximal totally singular subspace – see [4, p. 30, Description 4]. When k is even, define $W(k)'$ to be the conjugate of $W(k)$ by the reflection in $e_k + f_k$; thus $W(k)'$ sends

$$\begin{aligned}
&e_i \to e_i + e_{i+1} \text{ for } 1 \leq i \leq k-2,\\
&e_{k-1} \to e_{k-1} + f_k,\\
&e_k \to e_k + f_{k-1} + \cdots + f_1,\\
&f_k \to f_k,\\
&f_i \to f_i + f_{i-1} + \cdots + f_1 \text{ for } 1 \leq i \leq k-1.
\end{aligned}$$

So for $k = 2l$

$$W(2l),\ W(2l)' \in \Omega^+_{4l}(q).$$

For $k = 2l+1 \geq 3$ odd and $\beta \in F$, define $W_\beta(2l+1)$ to be the element of $GL(V)$ acting as follows:

$$\begin{aligned}
&e_i \to e_i + e_{i+1} + \beta f_l \text{ for } 1 \leq i \leq l,\\
&e_i \to e_i + e_{i+1} \text{ for } l+1 \leq i \leq k-1,\\
&e_k \to e_k,\\
&f_i \to f_i + f_{i-1} + \cdots + f_1 \text{ for } 1 \leq i \leq l+1\\
&f_i \to f_i + f_{i-1} + \cdots + f_1 + \beta e_{l+2} \text{ for } l+2 \leq i \leq k.
\end{aligned}$$

For $k = 1$, define $W_\beta(1) = I_2$, the identity element of $GL_2(q)$.

Let Q'_β be the quadratic form associated to $(\, , \,)$ such that $Q'_\beta(e_l) = Q'_\beta(e_{l+1}) = Q'_\beta(f_{l+1}) = \beta$ and $Q'_\beta(e_i) = Q'_\beta(f_i) = 0$ for all other values of i. Observe $W_\beta(2l+1)$ lies in $Sp(V)$ and preserves the quadratic form Q'_β, which is of plus type if $\beta = 0$ and of minus type if $\beta = \alpha$ (see Lemma 3.3). Writing just $W(2l+1)$ instead of $W_0(2l+1)$, for $l \geq 0$

$$W(2l+1) \in \Omega^+_{4l+2}(q),\ W_\alpha(2l+1) \in \Omega^-_{4l+2}(q).$$

As above, these lie in Ω^+, Ω^- rather than just SO^+, SO^- since they fix maximal totally singular subspaces (over $\mathbb{F}_{q^2}$ in the latter case). Each of $W(k)$ and $W_\alpha(k)$ has two Jordan blocks of size k. Note also that $W(1) = I_2 \in \Omega_2^+(q)$, while $W_\alpha(1) = I_2 \in \Omega_2^-(q)$.

We have now defined indecomposable elements $V(2k)$, $V_\alpha(2k)$, $W(k)$, $W(k)'$ (k even) and $W_\alpha(k)$ (k odd). By a sum of such elements, for example $V(2k) + W_\alpha(l)$, we mean the natural perpendicular sum. For a symplectic or orthogonal group in a given dimension n, we can find, via a suitable change of basis, a unipotent element corresponding to any perpendicular sum of such elements of total dimension n.

We can now state the main result.

Theorem 3.1 *Let $q = 2^a$ and let $\alpha \in \mathbb{F}_q$ be such that $x^2 + x + \alpha$ is irreducible over $\mathbb{F}_q$. Let G be a symplectic group $Sp_{2m}(q)$ or an orthogonal group $O^\pm_{2m}(q)$, where $m \geq 1$. Every unipotent element of G is G-conjugate to exactly one element of the form*

$$\sum_i W(m_i)^{a_i} + \sum_j V(2k_j)^{b_j} + \sum_r W_\alpha(m'_r) + \sum_s V_\alpha(2k'_s) \tag{5}$$

satisfying the following conditions:

(i) $\sum a_i m_i + \sum b_j k_j + \sum m'_r + \sum k'_s = m$;

(ii) the m'_r are odd and distinct, and the k'_s are distinct;

(iii) $b_j \leq 2$, *and* $b_j \leq 1$ *if there exist j, s such that $k_j = k'_s$;*

(iv) there exist no j, s such that $k'_s - k_j = 1$ or $k'_s - k'_j = 1$;

(v) there exist no j, r such that $m'_r = 2k_j \pm 1$ or $m'_r = 2k'_j \pm 1$;

(vi) for $G = Sp_{2m}(q)$, each $m'_r \geq 3$ and each $k'_s \geq 2$.

In the orthogonal case, an element of the form (5) lies in $O^{\epsilon}_{2m}(q)$ where $\epsilon = (-1)^t$ and t is the total number of W_α- and V_α-blocks; the element lies in $G' = \Omega^{\pm}_{2m}(q)$ if and only if the total number of V- and V_α-blocks is even; moreover, the only G-classes which split into two G'-classes are those of the form $\sum W(m_i)^{a_i}$ with all m_i even, and for these a second class representative can be obtained by replacing one summand $W(m_i)$ by $W(m_i)'$.

We prove the theorem in Section 3.2.

3.1 Preliminaries

Here we summarise the unipotent class representatives in symplectic and orthogonal groups over algebraically closed fields of characteristic 2, and discuss how they split in the corresponding finite groups. Let $\bar{F}$ be such a field, and define elements $V(2k)$ and $W(k)$ of the groups $Sp_{2k}(\bar{F})$, $O_{2k}(\bar{F})$ as above.

Let $\bar{G}$ be $Sp_{2m}(\bar{F})$ or $O_{2m}(\bar{F})$. In [6, Lemma 6.2] it is shown that every unipotent element of $\bar{G}$ is conjugate to a unique element of the form

$$\sum_i W(m_i)^{c_i} + \sum_j V(2k_j)^{d_j}, \tag{6}$$

where all $d_j \leq 2$ and $\sum c_i m_i + \sum d_j k_j = m$. List the k_js in increasing order, and as in [6, Theorem 6.21] define integers s, t, δ as follows:

$s =$ number of distinct odd m_i such that $m_i \neq 2k_j \pm 1$ for all j, and also $m_i \neq 1$ when $\bar{G}$ is symplectic;

$t =$ number of values of j such that $k_{j+1} - k_j \geq 2$;

$\delta = 0$ if there are no V-blocks in (6), or if $\bar{G}$ is symplectic and $k_j = 1$ for some j; and $\delta = 1$ otherwise.

Note that if there is no more than one distinct value of k_j in (6), we set $t = 0$.

Proposition 3.2 ([6, Theorem 7.3]) *Let $\bar{G}$ be $Sp_{2m}(\bar{F})$ or $O_{2m}(\bar{F})$. Let σ be a Frobenius morphism of $\bar{G}$ with fixed point group $\bar{G}_\sigma = Sp_{2m}(q)$ or $O^{\epsilon}_{2m}(q)$ where $\epsilon = \pm$. Let u be a unipotent element of $\bar{G}$ of the form (6).*

(i) *If $\bar{G}_\sigma = Sp_{2m}(q)$, then $u^{\bar{G}} \cap \bar{G}_\sigma$ splits into $2^{s+t+\delta}$ $\bar{G}_\sigma$-classes.*

(ii) *Let $\bar{G}_\sigma = O^\epsilon_{2m}(q)$. If $u = \sum W(m_i)^{a_i}$ with all m_i even, then $u^{\bar{G}} \cap \bar{G}_\sigma$ is empty for $\epsilon = -$, and is a single $\bar{G}_\sigma$-class for $\epsilon = +$ which splits into two $\bar{G}'_\sigma$-classes. Otherwise, $u^{\bar{G}} \cap \bar{G}_\sigma$ splits into $2^{s+t+\delta-1}$ $\bar{G}_\sigma$-classes, each of which is also a $\bar{G}'_\sigma$-class.*

This is a slight generalisation of [6, Theorem 7.3]: part (ii) covers all classes in $O^\epsilon_{2m}(q)$, whereas [6, 7.3(iii)] covers only classes in $\Omega^\epsilon_{2m}(q)$. For the proof we apply [6, 7.3(iii)], together with the observation that each $\bar{G}_\sigma$-class (apart from $\sum W(m_i)^{a_i}$ with all m_i even) is also a $\bar{G}'_\sigma$-class.

3.2 Proof of Theorem 3.1

We begin with some basic facts about indecomposable unipotent elements.

Lemma 3.3 *Let $k \geq 1$ and let $V(2k)$, $V_\alpha(2k)$, $W(k)$ and $W_\alpha(k)$ be the indecomposable unipotent elements of $Sp_{2k}(q)$ and $O^\epsilon_{2k}(q)$ defined above, where $\alpha \in \mathbb{F}_q$ is such that $x^2+x+\alpha$ is irreducible over $\mathbb{F}_q$.*

(i)

$$\begin{array}{l} V(2k) \in O^+_{2k}(q) \setminus \Omega^+_{2k}(q), \\ V_\alpha(2k) \in O^-_{2k}(q) \setminus \Omega^-_{2k}(q), \\ W(k) \in \Omega^+_{2k}(q), \text{ and} \\ W_\alpha(k) \in \Omega^-_{2k}(q) \text{ for } k \text{ odd.} \end{array}$$

(ii) *Let $G = Sp_{2k}(q)$ or $O^\epsilon_{2k}(q)$, and let $u \in G$ be $V_\beta(2k)\,(k \geq 2)$ or $W_\beta(k)\,(k \geq 3)$, where $\beta \in \mathbb{F}_q$. Write V for the natural G-module $V_{2k}(q)$. If $C = C_V(u)$, the fixed point space of u, then C is totally singular, and u acts on $C^\perp/C$ as $V_\beta(2k-2)$ or $W_\beta(k-2)$, respectively.*

(iii) *For $k \geq 2$, $V(2k)$ and $V_\alpha(2k)$ are not conjugate in $Sp_{2k}(q)$, but they are $SO_{2k}(\bar{\mathbb{F}}_q)$-conjugate. Also $V(2)$ and $V_\alpha(2)$ are conjugate in $Sp_2(q)$.*

(iv) *For k odd and $k \neq 1$, $W(k)$ and $W_\alpha(k)$ are not conjugate in $Sp_{2k}(q)$, but they are $SO_{2k}(\bar{\mathbb{F}}_q)$-conjugate.*

(v) *Every element of $Sp_{2k}(q)$ that is $Sp_{2k}(\bar{\mathbb{F}}_q)$-conjugate to $V(2k)$ is $Sp_{2k}(q)$-conjugate to either $V(2k)$ or $V_\alpha(2k)$; and every element of $Sp_{2k}(q)$ that is $Sp_{2k}(\bar{\mathbb{F}}_q)$-conjugate to $W(k)$ is $Sp_{2k}(q)$-conjugate to $W(k)$ if k is even, and to either $W(k)$ or $W_\alpha(k)$ if k is odd.*

Proof. (i) If $\beta \in \mathbb{F}_q$, then $V_\beta(2k)$ preserves the quadratic form Q_β defined above. Since $Q_\beta(e_k) = \beta$, $Q_\beta(f_k) = 1$, the 2-space $\langle e_k, f_k \rangle$ is of plus type if $\beta = 0$, and of minus type if $\beta = \alpha$, and the remaining standard basis vectors $e_i, f_i\,(i \neq k)$ span a space of plus type. Thus $V(2k) \in O^+_{2k}(q)$ and $V_\alpha(2k) \in O^-_{2k}(q)$. Being single Jordan blocks, these elements are $SO_{2k}(\bar{\mathbb{F}}_q)$-conjugate to $V(2k)$; such elements do not lie in $\Omega^\pm_{2k}(q)$, as shown in [6, p. 91].

Analogously, the quadratic form Q'_β, defined above and preserved by $W_\beta(k)$, is of plus type if $\beta = 0$ and of minus type if $\beta = \alpha$. Hence these elements lie in $SO^\pm_{2k}(q)$. As observed in the preamble to Theorem 3.1, they lie in $\Omega^\pm_{2k}(q)$ since they fix maximal totally singular subspaces (over $\mathbb{F}_{q^2}$ in the Ω^- case).

(ii) It is clear from the definition that $C = C_V(u)$ is totally singular. The second assertion follows from inspection of the matrix representation of u relative to a suitable basis of V containing bases of C and $C^\perp$.

(iii) Suppose $V(2k)$ and $V_\alpha(2k)$ are conjugate for some $k \geq 2$, say $V(2k)^g = V_\alpha(2k)$ with $g \in Sp_{2k}(q) = Sp(V)$. Then g sends the fixed point space of $V(2k)$ to that of $V_\alpha(2k)$; hence by (ii), it induces conjugation of $V(2k-2)$ to $V_\alpha(2k-2)$. Repeating, we see that $V(4)$ must be conjugate to $V_\alpha(4)$ in $Sp_4(q)$. Working relative to a standard basis e_1, e_2, f_2, f_1, the equation $V(4)X = XV_\alpha(4)$ (for $X \in SL_4(q)$) implies that X sends

$$\begin{aligned} &e_1 \to e_1 + be_2 + (\alpha + b + c)f_2 + df_1, \\ &e_2 \to e_2 + bf_2 + cf_1 \end{aligned}$$

for some $a, b, c, d \in \mathbb{F}_q$. But $X \in Sp_4(q)$ so $(e_1X, e_2X) = 0$, which implies that $b^2 + b + \alpha = 0$. However α was chosen so that $x^2 + x + \alpha$ is irreducible over $\mathbb{F}_q$, so this is a contradiction. Finally, $V(2k)$ and $V_\alpha(2k)$ are conjugate in $SO_{2k}(\bar{\mathbb{F}}_q)$, by [6, Lemma 6.2].

The last sentence of (iii) is trivial, since by their definition the elements $V(2)$ and $V_\alpha(2)$ of $Sp_2(q)$ are equal.

(iv) Suppose $W(k)$ and $W_\alpha(k)$ are conjugate in $Sp_{2k}(q)$. Arguing as in (iii), this implies that $W(3)$ and $W_\alpha(3)$ are conjugate in $Sp_6(q)$. Working relative to the reordered basis $e_1, f_3, e_2, f_2, e_3, f_1$, the equation $W(3)X = XW_\alpha(3)$ (for $X \in GL_6(q)$) implies that X has the following form:

$$\begin{pmatrix} x_1 & x_2 & y_1 + \alpha x_2 & y_2 + x_2 + \alpha x_1 & s & t \\ x_3 & x_4 & y_3 + x_3 + \alpha x_4 & y_4 + \alpha x_3 & u & v \\ & & x_1 & x_2 & y_1 & y_2 \\ & & x_3 & x_4 & y_3 & y_4 \\ & & & & x_1 & x_2 \\ & & & & x_3 & x_4 \end{pmatrix}$$

The equation $(e_1X, e_2X) = 0$ implies that $\alpha x_1^2 + \alpha x_2^2 + x_1x_2 = 0$. As $\det(X) \neq 0$, x_1 and x_2 cannot both be zero, so both are nonzero. If $z = \frac{\alpha x_1}{x_2}$,

Table 1: Conjugate unipotent elements of $Sp_{2n}(q)$

u	v	Conditions	n
$V(2k)+V(2k)$	$V_\alpha(2k)+V_\alpha(2k)$	$k \geq 2$	$2k$
$V(2k+2)+V(2k)$	$V_\alpha(2k+2)+V_\alpha(2k)$	$k \geq 2$	$2k+1$
$V_\alpha(2k+2)+V(2k)$	$V(2k+2)+V_\alpha(2k)$	$k \geq 2$	$2k+1$
$V_\alpha(4)+V(2)$	$V(4)+V(2)$		3
$V(2k)^3$	$W(2k)+V(2k)$		$3k$
$V(2k)^2+V_\alpha(2k)$	$W(2k)+V_\alpha(2k)$	$k \geq 2$	$3k$
$W(m)+W(m)$	$W_\alpha(m)+W_\alpha(m)$	$m \geq 3, m$ odd	$2m$
$W(m)+V(m+1)$	$W_\alpha(m)+V_\alpha(m+1)$	$m \geq 3, m$ odd	$\frac{1}{2}(3m+1)$
$W(m)+V_\alpha(m+1)$	$W_\alpha(m)+V(m+1)$	$m \geq 3, m$ odd	$\frac{1}{2}(3m+1)$
$W(m)+V(m-1)$	$W_\alpha(m)+V_\alpha(m-1)$	$m \geq 5, m$ odd	$\frac{1}{2}(3m-1)$
$W(m)+V_\alpha(m-1)$	$W_\alpha(m)+V(m-1)$	$m \geq 5, m$ odd	$\frac{1}{2}(3m-1)$
$W_\alpha(3)+V(2)$	$W(3)+V(2)$		4

then $z^2+z+\alpha^2=0$. Since $z=y^2$ for some $y \in \mathbb{F}_q$, this implies that $y^2+y+\alpha=0$, contrary to the choice of α. Finally, $W(k)$ and $W_\alpha(k)$ are conjugate in $SO_{2k}(\bar{\mathbb{F}}_q)$, by [6, Lemma 6.2].

(v) Let $G=Sp_{2k}(q)$, $\bar{G}=Sp_{2k}(\bar{\mathbb{F}}_q)$. If $u \in G$ is $\bar{G}$-conjugate to $V(2k)$ or $W(k)$, then Proposition 3.2(i) shows that $u^{\bar{G}} \cap G$ splits into two G-classes. Parts (iii) and (iv) show that these are represented by $V(2k)$ and $V_\alpha(2k)$, or $W(k)$ and $W_\alpha(k)$, respectively. ∎

Lemma 3.4 *Let $G=Sp_{2n}(q)$. If u and v are unipotent elements of G as in Table 1, then u and v are conjugate in G.*

Proof. Consider u,v as in the first row of the table. Observe that both lie in a subgroup $G_+=O^+_{2n}(q)$ of G. Let $\bar{G}_+=O_{2n}(\bar{\mathbb{F}}_q)$ and $\bar{G}=Sp_{2n}(\bar{\mathbb{F}}_q)$. By Proposition 3.2(ii), $u^{\bar{G}_+} \cap G_+$ consists of a single G_+-class. Since $v \in u^{\bar{G}_+} \cap G_+$, it follows that u and v are G_+-conjugate, hence G-conjugate, as required.

The above argument applies (in some cases replacing G_+ by $G_- = O^-_{2n}(q)$) to all cases apart from those in rows $4,5,6$ and 12 of the table. Rows 4 and 12 are straightforward, since in these cases u and v lie in $u^{\bar{G}} \cap G$, which by Proposition 3.2(i) consists of a single G-class. Consider row 5. Here $u=V(2k)^3$ and $v=W(2k)+V(2k)$ both lie in G_+, and $v^{\bar{G}_+} \cap G_+$ consists of a single G_+-class, by Proposition 3.2. The conclusion follows, since $u \in v^{\bar{G}_+}$ (see [6, Prop. 5.1]). The argument for row 6 is similar, using G_- instead of G_+. ∎

Proof of Theorem 3.1 for symplectic groups

Let $G = Sp_{2m}(q)$ and let $\alpha \in \mathbb{F}_q$ be such that $x^2 + x + \alpha$ is irreducible over $\mathbb{F}_q$. Write $\bar{G} = Sp_{2m}(\bar{\mathbb{F}}_q)$.

Let u be a unipotent element of G. Then u is $\bar{G}$-conjugate to an element of $\bar{G}$ of the form $\sum_i W(m_i)^{c_i} + \sum_j V(2k_j)^{d_j}$, where all $d_j \leq 2$ and $\sum c_i m_i + \sum d_j k_j = m$, as in (6). Define s, t and δ as in the preamble to Proposition 3.2.

By Lemma 3.3(v), u is G-conjugate to a perpendicular sum of elements of the form $W(m_i)$, $W_\alpha(m_i)$ (m_i odd), $V(2k_j)$ or $V_\alpha(2k_j)$. This sum can be written down in a number of different ways, given the conjugacies of Lemma 3.4. We claim that one of these ways is as in (5), satisfying conditions (i)-(vi) of Theorem 3.1, and prove this by induction on the number of summands. It is trivial if there is just one summand, so suppose there are at least two.

Assume first that u can be written as $X + Y$, where $X = W(m)$ or $W_\alpha(m)$ for some m; by Lemma 3.3(v) we may assume that m is odd in the latter case. By induction, Y is conjugate to a sum of the form (5) satisfying (i)-(vi) of Theorem 3.1. If $X = W(m)$ then $u = W(m) + Y$ also satisfies (i)-(vi). If $X = W_\alpha(m)$ and the sum $W_\alpha(m) + Y$ does not satisfy (i)-(vi), then Y must have a summand $Z = W_\alpha(m)$, $V(m \pm 1)$ or $V_\alpha(m \pm 1)$. But then $X + Y$ has a summand $W_\alpha(m) + Z$, which by Lemma 3.4 is conjugate to a term with a summand $W(m)$, so we may replace X by $W(m)$ and argue as before.

Thus we may suppose that u is a sum of terms of the form $V(2k)$ or $V_\alpha(2k)$, so (using Lemma 3.4) for each k there are at most two Jordan blocks of size $2k$. If there are no terms $V_\alpha(2k)$, then the sum satisfies (i)-(vi), so suppose there is such a term, and consider sums for u with the minimal possible number of V_α-terms. Among such sums, choose one with a summand $X = V_\alpha(2k)$ with k minimal (and $k \geq 2$, since there is no $V_\alpha(2)$ term for symplectic groups – see Lemma 3.3(iii)), and write the sum as $u = X + Y$. We show that this sum satisfies (i)-(vi). If not, then Y must have a summand $Z = V_\alpha(2k)$ or $V(2k-2)$; but then by Lemma 3.4, $X + Z$ is conjugate to a sum with either fewer V_α-terms, or with a term $V_\alpha(2k-2)$, both of which are contradictions.

This proves our claim that u is conjugate to a sum of the form (5), satisfying (i)-(vi) of Theorem 3.1. Given the $\bar{G}$-class of u, there are $2^{s+t+\delta}$ such sums. Hence Proposition 3.2 shows that every unipotent element of G is conjugate to precisely one such sum. ■

Proof of Theorem 3.1 for orthogonal groups

Let $H = Sp_{2m}(q)$, and for $\epsilon = \pm$ let $G_\epsilon = O^\epsilon_{2m}(q) < H$. Consider a unipotent class in $\bar{H} = Sp_{2m}(\bar{\mathbb{F}}_q)$, corresponding to a sum $\sum_i W(m_i)^{c_i} +$

$\sum_j V(2k_j)^{d_j}$ as in (6). If this is $\sum W(m_i)^{c_i}$ with all m_i even, this class corresponds to a single class in G_+ by Proposition 3.2(ii), so the conclusion of Theorem 3.1 holds for this class. Otherwise, by Theorem 3.1 for H, there are $2^{s+t+\delta}$ corresponding classes in H, each conjugate to a sum of the form (5) satisfying conditions (i)-(vi) of the theorem (for the symplectic case). By Proposition 3.2(ii), this number is equal to the sum of the numbers of corresponding classes in G_+ and G_-, except in the case where there is a summand $W(1)$ or $V(2)$. In the latter case there may be more corresponding classes in G_+ and G_- than in H, and the extra classes are accounted for precisely by allowing for terms $W_\alpha(1)$ and $V_\alpha(2)$ in decompositions (5). Finally, the assertions about G'_ϵ-conjugacy in Theorem 3.1 follow from Proposition 3.2(ii). ∎

4 A consequence

From our lists of class representatives, we can deduce the following result, which holds in all characteristics. This is used in [1, pp. 274-5].

Proposition 4.1 *Let q be a prime power, and let $Cl_n(q)$ denote a symplectic or orthogonal group $Sp_n(q)$ or $O_n^\epsilon(q)$.*

(i) *If k is odd, then all classes of unipotent elements in $Cl_n(q^k)$ are represented in the subgroup $Cl_n(q)$.*

(ii) *Any two unipotent elements in $Cl_n(q)$ that are conjugate in the algebraic group $Cl_n(\bar{\mathbb{F}}_q)$ are conjugate in $Cl_n(q^2)$, where we take $Cl_n(q^2) = O_n^+(q^2)$ in the even-dimensional orthogonal case. In particular, if q is odd, then all unipotent elements in $Cl_n(q)$ of a given Jordan form are conjugate in $Cl_n(q^2)$.*

Proof. (i) The class representatives for $Cl_n(q^k)$ given in Propositions 2.3, 2.4 and Theorem 3.1, and the form defining the group, have all their nonzero coefficients in $\{\pm1, \pm\alpha, \pm2\alpha\}$, where $\alpha \in \mathbb{F}_{q^k}$ is a non-square for q odd, and $x^2 + x + \alpha$ is irreducible over $\mathbb{F}_{q^k}$ for q even. Since k is odd, α can be chosen to lie in $\mathbb{F}_q$. Hence all the classes are represented in $Cl_n(q)$. (Note that when n is even and $Cl_n(q^k) = O_n^-(q^k)$, the orthogonal form over $\mathbb{F}_{q^k}$ is also of minus type over $\mathbb{F}_q$, so $Cl_n(q) = O_n^-(q)$ in this case.)

(ii) Suppose first that q is odd. Then two unipotent elements are conjugate in $Cl_n(\bar{\mathbb{F}}_q)$ if and only if they have the same Jordan form. The element α used in Propositions 2.3 and 2.4 to define the class representatives in $Cl_n(q)$ is a square in $\mathbb{F}_{q^2}$. For $Sp_n(q)$, the second paragraph of the proof of Proposition 2.3 shows that $V_1(2k)$ and $V_\alpha(2k)$ are conjugate in $Sp_n(q^2)$, and the result follows in this case. Similar observations apply to $O_n^\epsilon(q)$, noting that for n even and $\epsilon = -$, the orthogonal form fixed by the group is of type $+$ over $\mathbb{F}_{q^2}$.

Now suppose q is even. The unipotent classes in $Cl_n(\bar{\mathbb{F}}_q)$ are described in Section 3.1 (see (6)). If $\alpha \in \mathbb{F}_q$ is used to define the class representatives in Theorem 3.1, then $x^2 + x + \alpha$ is reducible over $\mathbb{F}_{q^2}$, and so $V_\alpha(2k), W_\alpha(k) \in O^+_{2k}(q^2)$. It follows that $V(2k), V_\alpha(2k)$ are conjugate in $Cl_{2k}(q^2)$, as are $W(k), W_\alpha(k)$. The conclusion now follows from Theorem 3.1. ■

5 Low-dimensional cases

We illustrate our results with Tables 2–7, which list the unipotent class representatives and centralizer orders for the 8-dimensional symplectic and orthogonal groups over all finite fields and for the 7-dimensional orthogonal groups over finite fields of odd order. These are refinements of [6, Tables 8.3a, 8.4a, 8.5a]. The structures of the centralizers are given by [6, Theorem 7.3].

Notation in the tables is as in previous sections. In particular, when q is odd, $\alpha \in \mathbb{F}_q$ is a fixed non-square; and when q is even, $\alpha \in \mathbb{F}_q$ is a fixed element such that $x^2 + x + \alpha$ is irreducible in $\mathbb{F}_q[x]$.

There is some extra notation in Table 5 for $G = SO^\epsilon_8(q)$ with q odd. Here the discriminant of the orthogonal form fixed by G is a square in $\mathbb{F}_q$ if and only if $\epsilon = +$ (see [4, Prop. 2.5.13]); we use the notation Δ for the value of the discriminant modulo $(\mathbb{F}_q^*)^2$. Also column 3 of the table gives the conditions for determining the sign $\epsilon = \pm$ of the group $SO^\epsilon_8(q)$ containing the element u.

Tables 2 and 5 for orthogonal groups in odd characteristic have a column indicating which unipotent classes split in the corresponding group $\Omega_7(q)$ or $\Omega^\epsilon_8(q)$; this information is given by Proposition 2.4(iii).

We have implemented procedures in MAGMA [2] which, given the dimension and field size, use Propositions 2.1 - 2.4 and Theorem 3.1 to list explicitly the representatives in the relevant classical group.

Table 2: Unipotent class representatives in $G = SO_7(q)$ (q odd)

Rep. u	$\lvert C_G(u)\rvert$	class splits in $\Omega_7(q)$?
$V_1(1)^7$	$\lvert G\rvert$	no
$W(2) + V_1(1)^3$	$q^7\lvert Sp_2(q)\rvert\,\lvert SO_3(q)\rvert$	no
$W(2) + V_1(3)$	$q^6\lvert Sp_2(q)\rvert$	yes
$V_\beta(3) + V_\beta(1) + V_1(1)^3$ ($\beta \in \{1, \alpha\}$)	$q^5\lvert O_4^\pm(q)\rvert$	no
$V_\beta(3) + V_1(3) + V_\beta(1)$ ($\beta \in \{1, \alpha\}$)	$2q^6(q \pm 1)$	no
$V_\beta(5) + V_\beta(1) + V_1(1)$ ($\beta \in \{1, \alpha\}$)	$2q^4(q \pm 1)$	no
$V_1(7)$	q^3	yes

Table 3: Unipotent class representatives in $G = Sp_8(q)$ (q odd)

Rep. u	$\lvert C_G(u)\rvert$
$W(1)^4$	$\lvert G\rvert$
$W(3) + W(1)$	$q^8\lvert Sp_2(q)\rvert^2$
$W(3) + V_\beta(2)\,(\beta \in \{1,\alpha\})$	$2q^9\lvert Sp_2(q)\rvert$
$W(1)^3 + V_b(2)\,(\beta \in \{1,\alpha\})$	$2q^7\lvert Sp_6(q)\rvert$
$W(1)^2 + V_\beta(2) + V_1(2)\,(\beta \in \{1,\alpha\})$	$2q^{11}\lvert Sp_4(q)\rvert(q \pm 1)$
$W(1)^2 + V_\beta(4)\,(\beta \in \{1,\alpha\})$	$2q^6\lvert Sp_4(q)\rvert$
$W(1) + V_b(2) + V_1(2)^2\,(\beta \in \{1,\alpha\})$	$q^{12}\lvert Sp_2(q)\rvert\,\lvert O_3(q)\rvert$
$W(1) + V_\beta(4) + V_\gamma(2)\,(\beta,\gamma \in \{1,\alpha\})$	$4q^9\lvert Sp_2(q)\rvert$
$W(1) + V_\beta(6)\,(\beta \in \{1,\alpha\})$	$2q^5\lvert Sp_2(q)\rvert$
$V_\beta(2) + V_1(2)^3\,(\beta \in \{1,\alpha\})$	$q^{10}\lvert O_4^\pm(q)\rvert$
$V_\beta(4) + V_\gamma(2) + V_1(2)\,(\beta,\gamma \in \{1,\alpha\})$	$4q^9(q \pm 1)$ (2 of each size)
$V_\beta(4) + V_1(4)\,(\beta \in \{1,\alpha\})$	$2q^7(q \pm 1)$
$V_\beta(6) + V_\gamma(2)\,(\beta,\gamma \in \{1,\alpha\})$	$4q^6$
$V_\beta(8)\,(\beta \in \{1,\alpha\})$	$2q^4$

Table 4: Unipotent class representatives in $G = Sp_8(q)$ (q even)

Rep. u	$\lvert C_G(u)\rvert$
$W(1)^4$	$\lvert G\rvert$
$W(4)$	$q^7\lvert Sp_2(q)\rvert$
$W(3) + W(1)$	$q^{10}\lvert Sp_2(q)\rvert\,\lvert O_2^+(q)\rvert$
$W_\alpha(3) + W(1)$	$q^{10}\lvert Sp_2(q)\rvert\,\lvert O_2^-(q)\rvert$
$W(3) + V(2)$	$q^9\lvert Sp_2(q)\rvert$
$W(2)^2$	$q^{10}\lvert Sp_4(q)\rvert$
$W(2) + W(1) + V(2)$	$q^{12}\lvert Sp_2(q)\rvert^2$
$W(2) + V(2)^2$	$q^{13}\lvert Sp_2(q)\rvert$
$W(2) + V_\beta(4)\,(\beta \in \{1,\alpha\})$	$2q^9\lvert Sp_2(q)\rvert$
$W(1)^3 + V(2)$	$q^7\lvert Sp_6(q)\rvert$
$W(1)^2 + W(2)$	$q^{11}\lvert Sp_2(q)\rvert\,\lvert Sp_4(q)\rvert$
$W(1)^2 + V(2)^2$	$q^{12}\lvert Sp_4(q)\rvert$
$W(1)^2 + V_\beta(4)\,(\beta \in \{1,\alpha\})$	$2q^6\lvert Sp_4(q)\rvert$
$W(1) + V(4) + V(2)$	$q^9\lvert Sp_2(q)\rvert$
$W(1) + V_\beta(6)\,(\beta \in \{1,\alpha\})$	$2q^5\lvert Sp_2(q)\rvert$
$V(4) + V(2)^2$	q^{10}
$V(4) + V_\beta(4)\,(\beta \in \{1,\alpha\})$	$2q^8$
$V_\beta(6) + V(2)\,(\beta \in \{1,\alpha\})$	$2q^6$
$V_\beta(8)\,(\beta \in \{1,\alpha\})$	$2q^4$

Table 5: Unipotent class representatives in $G = SO_8^\epsilon(q)$ (q odd)
Notation: Δ = discriminant mod $(\mathbb{F}_q^*)^2$; and $\beta, \beta_i \in \{1, \alpha\}$

Rep. u	$\lvert C_G(u)\rvert$	condition	class splits in $\Omega_8^\epsilon(q)$?
$V_\beta(1) + V_1(1)^7$	$\lvert G\rvert$	$\beta \equiv \Delta$ mod $(\mathbb{F}_q^*)^2$	no
$W(4)$, $W(4)^t$ (t reflection)	$q^5\lvert Sp_2(q)\rvert$	$\epsilon = +$	yes
$W(2)^2$, $W(2)^t + W(2)$	$q^6\lvert Sp_4(q)\rvert$	$\epsilon = +$	yes
$W(2) + V_\beta(1) + V_1(1)^3$	$q^9\lvert Sp_2(q)\rvert\,\lvert SO_4^\epsilon(q)\rvert$	$\beta \equiv \Delta$	no
$W(2) + V_{\beta_1}(3) + V_{\beta_2}(1)$	$2q^9\lvert Sp_2(q)\rvert$	$-\beta_1\beta_2 \equiv \Delta$	yes, $\epsilon = +$ no, $\epsilon = -$
$V_{\beta_1}(3) + V_{\beta_2}(1) + V_1(1)^4$	$2q^6\lvert SO_5(q)\rvert$	$-\beta_1\beta_2 \equiv \Delta$	no
$V_{\beta_1}(3) + V_1(3) + V_{\beta_2}(1) + V_1(1)$	$2q^8(q \pm 1)^2$, $\epsilon = +$ $2q^8(q^2 - 1)$, $\epsilon = -$	$-\beta_1\beta_2 \equiv \Delta$	no
$V_{\beta_1}(5) + V_{\beta_2}(1) + V_1(1)^2$	$2q^5\lvert SO_3(q)\rvert$	$\beta_1\beta_2 \equiv \Delta$	no
$V_{\beta_1}(5) + V_{\beta_2}(3)$	$2q^6$	$-\beta_1\beta_2 \equiv \Delta$	yes, $\epsilon = +$ no, $\epsilon = -$
$V_{\beta_1}(7) + V_{\beta_2}(1)$	$2q^4$	$-\beta_1\beta_2 \equiv \Delta$	yes, $\epsilon = +$ no, $\epsilon = -$

Table 6: Unipotent class representatives in $G = \Omega_8^+(q)$ (q even)

Rep. u	$\lvert C_G(u)\rvert$
$W(1)^4$	$\lvert G\rvert$
$W(4)$, $W(4)'$	$q^5\lvert Sp_2(q)\rvert$
$W(2)^2$, $W(2) + W(2)'$	$q^6\lvert Sp_4(q)\rvert$
$W(3) + W(1)$	$2q^8(q-1)^2$
$W_\alpha(3) + W_\alpha(1)$	$2q^8(q+1)^2$
$W(2) + W(1)^2$	$q^9\lvert Sp_2(q)\rvert\,\lvert\Omega_4^+(q)\rvert$
$W(2) + V(2)^2$	$q^9\lvert Sp_2(q)\rvert$
$W(1)^2 + V(2)^2$	$q^6\lvert Sp_4(q)\rvert$
$W(1) + V(4) + V(2)$	$q^5\lvert Sp_2(q)\rvert$
$V(4)^2$	q^6
$V(6) + V(2)$	$2q^4$
$V_\alpha(6) + V_\alpha(2)$	$2q^4$

Table 7: Unipotent class representatives in $G = \Omega_8^-(q)$ (q even)

Rep. u	$\lvert C_G(u)\rvert$
$W(1)^3 + W_\alpha(1)$	$\lvert G\rvert$
$W_\alpha(3) + W(1)$	$2q^8(q^2-1)$
$W(3) + W_\alpha(1)$	$2q^8(q^2-1)$
$W(2) + W_\alpha(1) + W(1)$	$q^9\lvert Sp_2(q)\rvert\,\lvert\Omega_4^-(q)\rvert$
$W(2) + V_\alpha(2) + V(2)$	$q^9\lvert Sp_2(q)\rvert$
$W(1)^2 + V_\alpha(2) + V(2)$	$q^6\lvert Sp_4(q)\rvert$
$W(1) + V(4) + V_\alpha(2)$	$q^5\lvert Sp_2(q)\rvert$
$V_\alpha(4) + V(4)$	q^6
$V_\alpha(6) + V(2)$	$2q^4$
$V(6) + V_\alpha(2)$	$2q^4$

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