Power sets and soluble subgroups

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Abstract

We prove that for certain positive integers k, such as 12, a normal subgroup of a finite group which consists of k^{th} powers is necessarily soluble. This gives rise to new solubility criteria, and solves an open problem from [2].

1 Introduction

For a group G and a positive integer k, denote by $G^{[k]}$ the set $\{x^k : x \in G\}$ of k^{th} powers in G. Define a positive integer k to be *nice* if k is a multiple of one of the following numbers:

$2^a p$, where $a > 1$ and p is a prime divisor of $2^{2^{a+1}} - 1$

$2 \cdot 3^a p$, where $a \geq 1$ and p is an odd prime divisor of $3^{3^a} \pm 1$

$3^a \cdot 5p$, where $a \geq 1$ and p is an odd prime divisor of $3^{3^a} \pm 1$.

Note that the smallest few nice numbers are multiples of 12, 20, 42, 68, 78, 105.

Theorem 1 *Let G be a finite group, and suppose N is a normal subgroup of G contained in $G^{[k]}$ for some nice integer k. Then N is soluble.*

Corollary 2 *Let k be a nice integer, and suppose G is a finite group such that $G^{[k]}$ contains a subgroup H of index c in G. Then G has a soluble normal subgroup of index dividing c!*

Indeed, the core of H is the required normal subgroup, by Theorem 1. Corollary 2 is in the same spirit as [1], where certain infinite groups G with the property that $G^{[k]}$ contains a subgroup of finite index in G are studied.

Corollary 3 *Suppose k is a nice integer and G is a finite group such that $\mathrm{Aut}(G)^{[k]}$ contains $\mathrm{Inn}(G)$. Then G is soluble.*

This follows immediately from Theorem 1: since $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$ the theorem implies that $\mathrm{Inn}(G)$ is soluble, hence so is G.

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In [2], some solubility criteria for finite groups are established, and Theorem 1 implies some of them. For example if $G^{[12]}$ is a subgroup of G, then Theorem 1 implies that $G^{[12]}$ is soluble, and since $G/G^{[12]}$ is also soluble (by Burnside's $p^a q^b$ theorem), G is also soluble, which is [2, Theorem 1].

Theorem 1 is proved via the following two results.

Proposition 4 *Let k be a positive integer, and suppose that there is no non-abelian finite simple group T such that $(\mathrm{Aut}(T))^{[k]}$ contains T. Then any normal subgroup of a finite group which consists of k^{th} powers is soluble.*

Proposition 5 *A number k is nice if and only if there is no non-abelian finite simple group T such that $(\mathrm{Aut}(T))^{[k]}$ contains T.*

A main problem posed in [2, Section 4] is the characterization of positive integers k with the property that finite groups G for which $G^{[k]}$ is a subgroup are all soluble. Our last result solves this problem.

Theorem 6 *For a positive integer k, the following two conditions are equivalent.*

(i) *Every finite group G such that $G^{[k]}$ is a subgroup is soluble.*

(ii) *Either*

(a) *k is an odd number of the form $3^a 5^b m$, where $a, b \geq 1$ and $(m, 3^{2\cdot 3^a} - 1) \neq 1$, or*

(b) *k is an even number which is not one of the following:*

(α) a multiple of $\exp(T)$, the exponent of some finite non-abelian simple group T

(β) $2 \cdot 3^a \cdot m$, where $a \geq 0$ and $(m, 3(3^{2\cdot 3^a} - 1)) = 1$

(γ) $2^a \cdot m$, where $a \geq 2$ and $(m, 2(2^{2^{a+1}} - 1)) = 1$.

2 Proof of Theorem 1

Proof of Proposition 4

Let k be as in the statement of the proposition, and suppose G is a minimal counterexample. So G has an insoluble normal subgroup N consisting of k^{th} powers. Let M be a minimal normal subgroup of G. Then NM/M is soluble. If M_1 is another minimal normal subgroup of G, then G embeds in $G/M \times G/M_1$ and so N is soluble, a contradiction. Hence M is the unique minimal normal subgroup of G and $M \leq N$. Moreover M is non-abelian (otherwise N would be soluble), so $C_G(M) = 1$, $M = T^r$ for some non-abelian simple group T, and G embeds in $\mathrm{Aut}(M) = \mathrm{Aut}(T) \wr S_r$.

By the choice of k, $(\mathrm{Aut}(T))^{[k]}$ does not contain T, and so there exists $t \in T \setminus \mathrm{Aut}(T)^{[k]}$. We claim that the element $n = (t, 1, \ldots, 1) \in T^r = M$ is not a k^{th} power in G. To see this, suppose $n = x^k$ where $x = (x_1, \ldots, x_r)\sigma$ with each $x_i \in \mathrm{Aut}(T)$ and $\sigma \in S_r$. Then $\sigma^k = 1$. If $\sigma(1) = 1$ then $t = x_1^k$, contradicting the fact that t is not a k^{th} power in $\mathrm{Aut}(T)$. So σ has a cycle $(1\, i_2 \cdots i_s)$ with $s \geq 1$. Calculating the coordinates of x^k in positions 1 and i_s, we get $t = x_1 x_{i_2} \cdots x_{i_s}$ and $1 = x_{i_s} x_1 \cdots x_{i_{s-1}}$, a contradiction.

It follows from the claim that that $G^{[k]}$ does not contain M, which is a contradiction since $M \leq N \subseteq G^{[k]}$. This completes the proof.

■

Proof of Proposition 5

The main tool for this proof is the following result from [2].

Theorem 7 ([2, Propositions 5,6 and Theorem 7]) *Let T be a finite simple group, and let $m > 1$ be a positive integer dividing $|T|$. Suppose* $\mathrm{Aut}(T)^{[m]}$ *contains T. Then $m = p^r$ or $2p^r$ for some prime p. Further, if $m = 2$ then $T = L_2(q)$ (q odd), $L_2(q^2)$ (q even) or $L_3(4)$; and if $m = p^r > 2$ or $m = 2p^r$ (p odd), then $T = L_2(p^{ml})$ or $L_2(p^{ml/2})$, respectively. Conversely,* $\mathrm{Aut}(T)^{[m]}$ *contains T for all such T and m.*

We embark on the proof of Proposition 5.

Suppose $k > 1$ is an integer which is nice. Assume for a contradiction that there exists a non-abelian simple group T such that $\mathrm{Aut}(T)^{[k]}$ contains T. Then $\mathrm{Aut}(T)^{[m]}$ contains T for any divisor m of k, so we may assume that k is one of the numbers $2^a p$, $2 \cdot 3^a p$, $3^a \cdot 5p$ as in the definition of nice numbers.

Consider $k = 2^a p$ with $a > 1$ and p a prime divisor of $2^{2^{a+1}} - 1$. Certainly 4 divides $|T|$, so Theorem 7 implies that $T = L_2(2^{4l})$ for some l. This is then divisible by 2^3, so if $a \geq 3$, Theorem 7 gives $T = L_2(2^{2^3 l'})$ for some l'. Repeating this argument, we see that $T = L_2(2^{2^a l''})$ for some l''. But then p divides $|T|$, and $\mathrm{Aut}(T)^{[p]}$ contains T, which is a contradiction by Theorem 7.

Now consider $k = 2 \cdot 3^a p$, where $a \geq 1$ and p is an odd prime divisor of $3^{3^a} \pm 1$. Since $\mathrm{Aut}(T)^{[2]}$ contains T, Theorem 7 gives $T = L_2(q)$ or $L_3(4)$. In particular 3 divides $|T|$, so again by Theorem 7, $T = L_2(3^{3l})$, and arguing as before, $T = L_2(3^{3^a l'})$. Then p divides $|T|$, giving a contradiction by Theorem 7.

Finally, consider $k = 3^a \cdot 5p$, where $a \geq 1$ and p is an odd prime divisor of $3^{3^a} \pm 1$. If 3 does not divide $|T|$ then T is a Suzuki group; but then 5 divides $|T|$ and $\mathrm{Aut}(T)^{[5]}$ contains T, contrrary to Theorem 7. Hence 3 divides $|T|$ and so $T = L_2(3^{3l})$. Now argue as in the previous paragraph. This proves one implication in Proposition 5, and already establishes Theorem 1.

For the converse implication of Proposition 5, assume that $k > 1$ is not nice. We need to find a non-abelian simple group T such that $\mathrm{Aut}(T)^{[k]}$ contains T.

First consider the case where k is odd. If $(k, 3) = 1$, one can see using Dirichlet's theorem on primes in arithmetic progession that there is a prime $p > 3$ such that $T = L_2(p)$ has order coprime to k, hence $T^{[k]} = T$. And if $(k, 5) = 1$ then there is a large prime p such that the Suzuki group $T = {}^2B_2(2^p)$ has order coprime to k, giving the same conclusion. Hence we may assume that 15 divides k. Let $k = 3^a 5^b m$ with m coprime to 15. As k is not nice we have $(m, 3^{3^a} \pm 1) = 1$. Then the group $T = L_2(3^{3^a})$ has order coprime to $5^b m$ and hence satisfies $T \subseteq \mathrm{Aut}(T)^{[k]}$ by Theorem 7.

Now assume k is even and divisible by 4, and write $k = 2^a m$ with m odd. As k is not nice, $(m, 2^{2^{a+1}} - 1) = 1$. Then by Theorem 7, we have $T \subseteq \mathrm{Aut}(T)^{[k]}$ for $T = L_2(2^{2^a})$.

Finally, assume $k = 2l$ with l odd. If $(k, 3) = 1$ then we can find a prime $p > 3$ such that l is coprime to the order of $T = L_2(p)$, and then $T \subseteq \mathrm{Aut}(T)^{[k]}$ by Theorem 7. So assume 3 divides k and write $k = 2 \cdot 3^a m$ with m coprime to 6. As k is not nice, $(m, 3^{3^a} \pm 1) = 1$. But then $T \subseteq \mathrm{Aut}(T)^{[k]}$ for $T = L_2(3^{3^a})$. This completes the proof of Proposition 5. ■

3 Proof of Theorem 6

As in [2], define a positive integer k to be *good* if it satisfies condition (i) of Theorem 6, and *bad* otherwise.

First let k be an odd integer. If k is coprime to 3 or 5, then as above, there is a simple group $T = L_2(p)$ or ${}^2B_2(q)$ of order coprime to k, and then $T^{[k]} = T$, showing that k is bad. So assume $k = 3^a 5^b m$ with $a, b \geq 1$ and m coprime to 15. If $(m, 3^{3^a} \pm 1) = 1$ then $T = L_2(3^{3^a})$ has order coprime to $5^b m$, and also by [2, Proposition 6], $G = T\langle\sigma\rangle$ satisfies $G^{[k]} = T$ for a field automorphism σ; hence k is bad. On the other hand, if $(m, 3^{3^a} \pm 1) \neq 1$ then we claim that k is good. For suppose G is a finite group such that $G^{[k]}$ is a subgroup, and suppose G has a non-abelian composition factor T. By [2, Theorem 4], we have $T \subseteq \mathrm{Aut}(T)^{[k]}$. Hence we can use Theorem 7 as before to see that T must be $L_2(3^{3^a l})$ for some l. But if p is a prime divisor of $(m, 3^{3^a} \pm 1)$, then p divides $|T|$, so $T \subseteq \mathrm{Aut}(T)^{[p]}$, which is a contradiction by Theorem 7. Hence G is soluble and so k is good, proving the claim.

We have now shown that the odd good numbers are precisely those in (a) of Theorem 6.

Now let k be even. Of course if k is a multiple of the exponent $\exp(T)$ of a simple group T, then k is bad.

Assume that 4 divides k, and write $k = 2^a m$ with $a \geq 2$ and m odd. If $(m, 2^{2^{a+1}} - 1) = 1$ then for $T = L_2(2^{2^a})$, the group $G = T\langle\sigma\rangle$, where σ is a field automorphism of order 2^a, satisfies $G^{[k]} = T$ (see [2, Proposition 6]), so k is bad. On the other hand, if $(m, 2^{2^{a+1}} - 1) \neq 1$, then the argument given above for odd numbers shows that k is good.

Finally, assume that $k = 2l$ with l odd. If l is coprime to 3 then there is a prime $p > 3$ such that $L_2(p)$ has order coprime to l, and $G = PGL_2(p)$ satisfies $G^{[k]} = L_2(p)$ (see [2, Proposition 5]), so k is bad. Now assume 3 divides l, and write $k = 2 \cdot 3^a \cdot m$ with m coprime to 6. If $(m, 3^{2 \cdot 3^a} - 1) = 1$, then the group $G = L_2(3^{3^a})\langle\sigma\rangle$ satisfies $G^{[k]} = T$; and otherwise, the usual argument shows that k is good. This completes the proof of Theorem 6. ■

References

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