M45P72 Modular Representation Theory
Assessed Coursework
Deadline Monday December 2

Throughout, let G be a finite group and K an algebraically closed field of characteristic p.

1. This is a follow-up to Q2 of Sheet 2. Let G be a subgroup of S_n, let $\Omega = \{1, \dots, n\}$, and denote by $K\Omega$ the KG-module with basis Ω, where the multiplication by $g \in G$ is defined by the permutation action on Ω. Let

$$S = \{\sum_{\omega \in \Omega} \lambda_\omega \omega : \sum \lambda_\omega = 0\}, \quad T = \{\lambda \sum_{\omega \in \Omega} \omega : \lambda \in K\}.$$

Then S and T are KG-submodules of $K\Omega$, and we define the KG-module $V = S/(S \cap T)$. (Sheet 2, Q2 shows that if $G = S_n$ then V is a simple KG-module.)

(i) Suppose $G = A_n$. Show that V is a simple KG-module provided $n \geq 4$ and $(n,p) \neq (4,2)$.

(ii) Now suppose n is prime, and identify Ω with the set of elements of the field $\mathbb{F}_n = \{0, 1, \dots, n-1\}$ of n elements. For $0 \neq a \in \mathbb{F}_n$, $b \in \mathbb{F}_n$ define the permutation $t_{a,b} : \mathbb{F}_n \to \mathbb{F}_n$ by

$$t_{a,b}(x) = ax + b \quad (x \in \mathbb{F}_n),$$

and let $G = \{t_{a,b} : a, b \in \mathbb{F}_n, a \neq 0\}$.

(a) Show that G is a subgroup of S_n.

(b) Show that if $p \neq n$, then V is a simple KG-module. (Recall p is the characteristic of K.)

(c) Show that if $p = n$ and $p \geq 5$, then V is not a simple KG-module.

2. Find the Brauer character table of G in the following cases:

(i) $G = A_4$, $p = 3$.

(ii) $G = A_5$, $p = 2$. (You may assume the isomorphism $A_5 \cong SL_2(4)$.)

(iii) $G = SL_2(3)$, $p = 2$.

(iv) $p = 2$, $G = NH < GL_3(3)$, where $H = \langle h \rangle$ and

$$N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{F}_3 \right\}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(Note that $N \cong C_3^2$, $N \lhd G$ and $H \cong C_8$.)

(v) In each of cases (i)-(iv), find the composition factors (and multiplicities) of $V \otimes V$, where V is a simple KG-module of largest dimension.

3. Let V be a KG-module, and let $V^* = \mathrm{Hom}_K(V, K)$, the dual space.

(a) Show that V^* is a KG-module, with multiplication defined for $g \in G, \phi \in V^*$ by

$$(g\phi)(v) = \phi(g^{-1}v) \quad (v \in V).$$

(b) If W is a submodule of V, show that V^* has a submodule $X \cong (V/W)^*$, and that $V^*/X \cong W^*$.

(c) Deduce that V is semisimple iff V^* is semisimple.

(d) Prove that $\mathrm{Soc}(V^*) \cong (V/\mathrm{Rad}(V))^*$.

(e) Show that $(KG)^* \cong KG$ (isomorphism of KG-modules).

(f) Deduce that a KG-module P is projective iff P^* is projective.

(g) Show finally that if P is a projective indecomposable KG-module, then $\mathrm{Soc}(P)$ is simple.

Note: in fact $\mathrm{Soc}(P) \cong P/\mathrm{Rad}(P)$, *but this is more tricky to prove (and not required).*