M45P72 Modular Representation Theory
Progress Test 1

This is an open book test: you may use your lecture notes, but not books/laptops/phones/...

1. Let $G = D_{10}$, the dihedral group of order 10, and let $K = \bar{\mathbb{F}}_2$, the algebraic closure of the field $\mathbb{F}_2$.

(i) Calculate the number of non-isomorphic simple KG-modules.

(ii) By constructing them (or otherwise), find the dimensions of the simple KG-modules.

(iii) Find the dimension of the radical $\text{Rad}(KG)$.

(iv) Let t be an element of order 2 in G, and let H be the cyclic subgroup $\langle t \rangle$. Write down a basis of $\text{Rad}(KH)$.

(v) Show that $\text{Rad}(KH) \not\subseteq \text{Rad}(KG)$.

2. Let G be a finite group, let K be a field, and let N be a *normal* subgroup of G. Prove that

$$\text{Rad}(KN) \subseteq \text{Rad}(KG).$$

Solutions

1. (i) By Thm 5.1, this is the number of 2-regular classes in D_{10}, which is 3. (**1 mark**)

(ii) We are looking for 3 simple KG-modules. One is the trivial module, of dimension 1. Writing $D_{10} = \langle x, y \rangle$ with x of order 5 and y of order 2, these are the 2-dimensional reps ρ_1, ρ_2 given by

$$\rho_1 : x \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad y \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\rho_2 : x \to \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix}, \quad y \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\omega \in K$ is a 5th root of 1. (Other descriptions possible – these just come from the standard 2-dimensional complex rep of D_{10}). (**4 marks**)

(iii) From 4.7 of lecs, we know that if $A = KG$, and $S_1 \ldots, S_r$ are the simple KG-modules, then

$$\dim A/Rad(A) = \sum_1^r n_i^2,$$

where $n_i = \dim S_i$. In our case $\dim A = |G| = 10$, $r = 3$ and the simple modules have dimensions 1,2,2. So Rad(A) has dimension $10 - \sum n_i^2 = 1$. (**4 marks**)

(iv) By Sheet 1, Q7, a basis is $1 - t$. (**2 marks**)

(v) Suppose $1 - t \in \text{Rad}(KG) := R$. Then by (iii), $R = \langle 1 - t \rangle$. However R is a 2-sided ideal, so for example contains $utu^{-1} - 1$, where u is another element of order 2. This is a contradiction. Hence $\text{Rad}(KH) \not\subseteq \text{Rad}(KG)$. (**3 marks**)

2. Recall from Thm 3.2 that

$$\text{Rad}(KG) = \{a \in KG : aS = 0 \text{ for all simple } KG - \text{modules } S\}. \qquad (1)$$

Let $a \in \text{Rad}(KN)$. If S is a simple KG-module, then the restriction $S|_N$ is semisimple, by Clifford's theorem, and hence $aS = 0$ by (1). Hence $a \in \text{Rad}(KG)$, again by (1). Therefore $\text{Rad}(KN) \subseteq \text{Rad}(KG)$, as required. (**6 marks**)