M45P72 Modular Representation Theory Problem Sheet 3

1. Let G be a finite group and let K be a splitting field for G.

(i) Let S be a simple KG-module with projective cover P_S. Prove that for any simple KG-module T,

$$\dim \mathrm{Hom}_{KG}(P_S, T) = \begin{cases} 1, & \text{if } T \cong S \\ 0, & \text{if } T \ncong S \end{cases}$$

(ii) Let $0 \to V \to^{\phi} W \to^{\psi} X \to 0$ be an exact sequence of KG-modules (i.e. ϕ is injective, ψ is surjective and $\mathrm{Im}\phi = \mathrm{Ker}\psi$). Prove that if P is a projective KG-module, there is a corresponding exact sequence

$$0 \to \mathrm{Hom}_{KG}(P, V) \to \mathrm{Hom}_{KG}(P, W) \to \mathrm{Hom}_{KG}(P, X) \to 0.$$

(iii) Let U be a KG-module, and S a simple KG-module. Prove that the multiplicity of S as a composition factor of U is $\dim \mathrm{Hom}_{KG}(P_S, U)$. (Hint: consider a composition series $0 = U_0 \subset \cdots \subset U_n = U$. Use induction on n and the previous parts of the question.)

2. Let G be a non-abelian finite simple group of even order, let K be a field of odd characteristic, and let V be a nontrivial simple KG-module. Prove that $\dim V > 2$. (Hint: let $t \in G$ be an element of order 2. Consider the image of t under the corresponding matrix representation.)

3. Find the decomposition matrix D and the Cartan matrix C for the following groups and characteristics:

(i) $S_3 \times S_3$, $p = 2$

(ii) D_{2n} (n odd), $p = 2$

(iii) S_4, $p = 2$ or 3

(iv) A_5, $p = 2, 3$ or 5

(v) S_5, $p = 2$

(vi) $PSL_2(11)$, $p = 11$

(vii) the subgroup $\langle (12345), (2354) \rangle$ of S_5, with $p = 2$.

4. Let G be a finite group, and K an algebraically closed field of characteristic p, where p does not divide $|G|$. Prove that if $\mathrm{IBr}(G)$ is the set of Brauer characters of the simple KG-modules, then $\mathrm{IBr}(G) = \mathrm{Irr}(G)$, the set of characters of the simple $\mathbb{C}G$-modules.

5. Let G be a finite group, and K an algebraically closed field of characteristic p. Suppose that H is a subgroup of G of order coprime to p. Let P_1 denote the projective cover of the trivial KG-module. Prove that $\dim P_1 \le |G : H|$. (Hint: as $H \subseteq G_{p'}$, for $\chi \in \mathrm{Irr}(G)$ we have $\chi_H = \sum_{\phi \in IBr(G)} d_{\chi\phi} \phi_H$. Now consider the induced character 1_H^G and use Frobenius reicprocity.)

Find some examples where equality holds (i.e. $\dim P_1 = |G : H|$).