M45P72 Modular Representation Theory   Problem Sheet 2

Throughout the problem sheet, K denotes an algebraically closed field of characteristic p, and A a (finite-dimensional) algebra over K.

1. (a) Let M be an A-module. Prove that $M/\mathrm{Rad}(M)$ is simple iff M has a unique maximal submodule.

(b) Prove that every A-module is a homomorphic image of a free A-module.

(c) Prove that any direct summand of a projective A-module is projective.

2. Let $G = S_n$, let $\Omega = \{1, \ldots, n\}$, and denote by $K\Omega$ the KG-module with basis Ω, where the multiplication by $g \in G$ is defined by the permutation action on Ω. Let

$$S = \{\sum_{\omega \in \Omega} \lambda_\omega \omega : \sum \lambda_\omega = 0\}, \ \ T = \{\lambda \sum_{\omega \in \Omega} \omega : \lambda \in K\}.$$

Show that S and T are KG-submodules of $K\Omega$. Show also that $S/(S \cap T)$ is a simple KG-module, and find its dimension (in terms of n and $p = \mathrm{char}(K)$).

3. Compute the dimensions of all the simple KG-modules and also the dimension of $\mathrm{Rad}(KG)$ in the following cases:

(i) $G = SL_2(p)$ (ii) $G = C_n$, a cyclic group, where $n = p^a m$ with $p \nmid m$

(iii) $G = D_{2n}$, a dihedral group with n odd, $p = 2$ (iv) $G = S_4$, $p = 2$

(v) $G = S_4$, $p = 3$ (vi) $G = S_5$, $p = 2$.

4. Let $G = SL_2(p)$. As in lectures, define $X = (1, 0)^T, Y = (0, 1)^T$ and for $n \geq 0$ let V_{n+1} be the KG-module consisting of homogeneous polynomials in $K[X, Y]$ of degree n.

(a) Show that $\langle X^p, Y^p \rangle$ is a submodule of V_{p+1}. (Hence V_{p+1} is not simple.)

(b) For any $k \geq 1$, find a proper nonzero submodule of V_{p+k}.

5. Let G be a finite group, and V a KG-module with corresponding representation $\rho : G \to GL(V)$. Prove that V is simple iff the linear span of the image $\rho(G)$ is the whole matrix algebra $\mathrm{End}_K(V)$. (Hint: use Theorem 4.5 of lectures.)

6. Let $G = \langle a, b \rangle \cong C_p \times C_p$, and let V_{2n} be the KG-module of dimension $2n$ defined in lectures, corresponding to the matrix representation sending

$$a \to \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix}, \ b \to \begin{pmatrix} I_n & 0 \\ N & I_n \end{pmatrix},$$

where N is the $n \times n$ matrix with 0's on the diagonal, 1's on the next diagonal down, and 0's elsewhere (see the example before Prop 2.8). Complete the proof sketched in the lectures that V_{2n} is an indecomposable KG-module.

7. Let S be a simple A-module, and suppose that U is an A-module such that $U/\mathrm{Rad}(U) \cong S$. Prove that U is a homomorphic image of P_S, the projective cover of S.

TURN OVER!!

8. Find the Cartan matrix for the group algebra KG in the following cases:

(a) G a p-group

(b) $G = C_n$, where $n = p^a m$ with $p \not| m$

(c) $G = S_3$, $p = 3$.

9. Let $G = S_4$ with $p = 2$.

(a) Find mutually orthogonal idempotents $e_1, \dots, e_k \in KG$ such that $1 = e_1 + \cdots + e_k$.

(b) Which KG-module $(KG)e_i$ is the projective cover of the trivial module?

10. (*Representations of direct products*) Let G_1, G_2 be finite groups.

(a) Show that the tensor product space $KG_1 \otimes_K KG_2$ becomes an algebra if we define the product by $(g_1 \otimes g_2)(g_1' \otimes g_2') = g_1 g_1' \otimes g_2 g_2'$ for $g_i, g_i' \in G_i$, extending linearly to all elements of the tensor product. Prove that as algebras, $KG_1 \otimes KG_2 \cong K(G_1 \times G_2)$.

(b) For $i = 1, 2$, let S_i be a KG_i-module, and make $S_1 \otimes S_2$ into a $K(G_1 \times G_2)$-module by defining $(g_1, g_2)(s_1 \otimes s_2) = g_1 s_1 \otimes g_2 s_2$ (for $g_i \in G_i, s_i \in S_i$). Prove that if S_1, S_2 are both simple modules, then $S_1 \otimes S_2$ is a simple $K(G_1 \times G_2)$-module. (Hint: use Q6.)

(c) Let S_i, S_i' be simple KG_i-modules for $i = 1, 2$. Show that $S_1 \otimes S_2 \cong S_1' \otimes S_2'$ iff $S_i \cong S_i'$ for $i = 1, 2$.

(d) Using Theorem 5.1 of lectures, deduce that every simple $K(G_1 \times G_2)$-module is isomorphic to one of the modules $S_1 \otimes S_2$ in part (b).

11. (*Optional: the conjugacy classes of $SL_2(p)$*) Let $G = SL_2(p)$ with p an odd prime, and for $\lambda \in \mathbb{F}_p^*$, $\mu \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ satisfying $\mu^{p+1} = 1$, define the following matrices in G:

$$t_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad s_\mu = \begin{pmatrix} 0 & 1 \\ -1 & \mu + \mu^p \end{pmatrix}.$$

(a) Show that there are $(p-3)/2$ non-conjugate matrices t_λ for $\lambda \neq \pm 1$. Work out the sizes of their conjugacy classes.

(b) Show that there are $(p-1)/2$ non-conjugate matrices s_μ, and work out the sizes of their conjugacy classes.

(c) Using the JCF theorem, show that there are exactly $2p-2$ elements of order p or $2p$ in $GL_2(p)$ (hence also in G).

(d) By adding up the numbers of elements in the classes in (a) and (b), together with those in (c) and also $\pm I$, show that all the elements of G have been accounted for.

(e) Deduce that G has exactly p conjugacy classes of p-regular elements.