M45P72 Modular Representation Theory Problem Sheet 1

1. (a) Let $p \geq 3$ be a prime, and let $G = \langle x \rangle \cong C_p$. Show that there are representations $\rho_1, \rho_2 : G \to GL_3(\mathbb{F}_p)$ such that

$$\rho_1(x) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_2(x) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let M_1, M_2 be the corresponding 3-dimensional $\mathbb{F}_pG$-modules. Which of M_1, M_2 are indecomposable?

(b) Let $G = \langle x, y \rangle \cong C_2 \times C_2$. Show that there are representations $\rho_1, \rho_2 : G \to GL_3(\mathbb{F}_2)$ such that

$$\rho_1(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_1(y) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho_2(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_2(y) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let M_1, M_2 be the corresponding 3-dimensional $\mathbb{F}_pG$-modules. Which of M_1, M_2 are indecomposable? Is $M_1 \cong M_2$?

(c) Let $G = \langle x, y \rangle \cong C_2 \times C_2$ and let K be an infinite field of characteristic 2. For each $\lambda \in K$, define a representation $\rho_\lambda : G \to GL_2(K)$ by

$$\rho_\lambda(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho_\lambda(y) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

Let M_λ be the corresponding KG-module.

Show that $M_\lambda \cong M_\mu$ iff $\lambda = \mu$. (Hence there are infinitely many non-isomorphic KG-modules of dimension 2.)

2. Let G be a finite group, and p a prime.

(a) Show that G has a unique largest normal p-subgroup (i.e. $\exists N \lhd G$ with $|N| = p^a$ such that N contains all other normal p-subgroups of G).

(b) Write $O_p(G)$ for the largest normal p-subgroup of G, and let K be a field of characteristic p. Show that if $\rho : G \to GL_n(K)$ is an irreducible representation, then $O_p(G) \leq \mathrm{Ker}(\rho)$.

3. Let $G = \langle x \rangle \cong C_6$, a cyclic group of order 6. Express the groups algebras $\mathbb{C}G$, $\bar{\mathbb{F}}_3G$ and $\bar{\mathbb{F}}_2G$ as direct sums of indecomposable submodules (where $\bar{\mathbb{F}}_2, \bar{\mathbb{F}}_3$ denote the algebraic closures of $\mathbb{F}_2, \mathbb{F}_3$).

4. Let $G = S_3$.

(a) Refresh your memory by writing down all the simple $\mathbb{C}G$-modules.

(b) Now find all the simple KG-modules, where $K = \bar{\mathbb{F}}_2$ or $\bar{\mathbb{F}}_3$.

TURN OVER!

5. Let A be a semisimple algebra over a field K.

(a) Prove that every A-module is semisimple.

(b) Prove that if U is an A-module such that $\mathrm{End}(U) \cong K$, then U is simple.

6. Prove that if A is an algebra, there are only finitely many non-isomorphic simple A-modules. (You may assume the Jordan-Holder theorem for A-modules.)

7. Let K be a field, and let $A = T_n(K)$ be the algebra of lower triangular $n \times n$ matrices over K.

(a) Show that there are precisely n non-isomorphic simple A-modules, all 1-dimensional.

(b) Find the radical and socle series for A_A (recall this means A regarded as an A-module).

8. Let p be a prime, K a field of characteristic p, and G a p-group. Define $I \subseteq KG$ by

$$I = \{\sum_{g \in G} \lambda_g g : \lambda_g \in K, \sum_{g \in G} \lambda_g = 0\}.$$

(a) Show that I is an ideal of the group algebra KG.

(b) Prove that the radical $\mathrm{Rad}(KG) = I$. (Use Theorem 3.2.)

9. Let K be a field and n a positive integer.

(a) Prove that the matrix algebra $M_n(K)$ is a simple algebra.

(b) Let $A = M_n(K)$, and define

$$[A, A] = \mathrm{Sp}\,(ab - ba : a, b \in A)\,.$$

Prove that $[A, A] = \{x \in A : \mathrm{trace}(x) = 0\}$.