Proposition 14.5 *Let U, V be FG-modules with full RG-lattices U_0, V_0. Then*

(i) $\mathrm{Hom}_{RG}(U_0, V_0)$ *is a full R-lattice in* $\mathrm{Hom}_{FG}(U, V)$.

(ii) $\pi\mathrm{Hom}_{RG}(U_0, V_0) = \mathrm{Hom}_{RG}(U_0, \pi V_0)$.

(iii) *Suppose U_0 is a projective RG-module. Then*

$$\frac{\mathrm{Hom}_{RG}(U_0, V_0)}{\pi\mathrm{Hom}_{RG}(U_0, V_0)} \cong \mathrm{Hom}_{kG}(\frac{U_0}{\pi U_0}, \frac{V_0}{\pi V_0}).$$

(iv) *If U_0 is projective, then*

$$\dim_F \mathrm{Hom}_{FG}(U, V) = \dim_k \mathrm{Hom}_{kG}(\frac{U_0}{\pi U_0}, \frac{V_0}{\pi V_0}).$$

Proof (i) First we show LHS is contained in RHS. Take R- and F-bases $u_1, \ldots, u_r$ and $v_1, \ldots, v_s$ of U_0 and V_0, respectively. Any RG-homomorphism $U_0 \to V_0$ is represented by a matrix over R (wrt these bases), and this matrix also represents an FG-homomorphism $U \to V$.

Next, LHS is a lattice, as it is contained in $\mathrm{Hom}_R(U_0, V_0)$, which is a free R-module ($\cong R^{rs}$).

Finally, LHS is a full lattice: let $\phi : U \to V$ be an FG-homomorphism, and let $\phi(u_i) = \sum_j \lambda_{ji} v_j$ with $\lambda_{ji} \in F$. There exists $0 \neq a \in R$ such that $a\lambda_{ji} \in R$ for all i, j (as F is the field of fractions of R). Then $a\phi$ maps $U_0 \to V_0$, so $\phi \in F\,\mathrm{Hom}_{RG}(U_0, V_0)$. Hence $\mathrm{Hom}_{RG}(U_0, V_0)$ spans $\mathrm{Hom}_{FG}(U, V)$ over F, showing the LHS is full.

(ii) The map $V_0 \to \pi V_0$ sending $x \to \pi x$ is an RG-isomorphism, so any RG-homomorphism $U_0 \to \pi V_0$ is a composite $U_0 \to V_0 \to \pi V_0$, hence is in $\pi\mathrm{Hom}_{RG}(U_0, V_0)$.

(iii) Consider the natural map $\mathrm{Hom}_{RG}(U_0, V_0) \to \mathrm{Hom}_{RG}(U_0, V_0/\pi V_0)$ (sending a homom $\phi : U_0 \to V_0$ to its composition with the natural map $V_0 \to V_0/\pi V_0$). The kernel of this map is $\mathrm{Hom}_{RG}(U_0, \pi V_0)$, which is equal to $\pi\mathrm{Hom}_{RG}(U_0, V_0)$ by part (ii). Also the assumption that U_0 is projective implies that the map is surjective. Hence

$$\frac{\mathrm{Hom}_{RG}(U_0, V_0)}{\pi\mathrm{Hom}_{RG}(U_0, V_0)} \cong \mathrm{Hom}_{RG}(U_0, \frac{V_0}{\pi V_0}). \quad (0.1)$$

Let $\alpha \in \mathrm{Hom}_{RG}(U_0, \frac{V_0}{\pi V_0})$. Then $\pi U_0 \subseteq \mathrm{Ker}(\alpha)$, so α factors as

$$U_0 \to \frac{U_0}{\pi U_0} \to^{\beta} \frac{V_0}{\pi V_0}.$$

The map $\alpha \to \beta$ is an isomorphism

$$\mathrm{Hom}_{RG}(U_0, \frac{V_0}{\pi V_0}) \to \mathrm{Hom}_{kG}(\frac{U_0}{\pi U_0}, \frac{V_0}{\pi V_0}).$$

Combined with (0.1), this proves (iii).

(iv) By (i), the LHS is equal to the R-rank of the lattice (free R-module) $\mathrm{Hom}_{RG}(U_0, V_0)$. The RHS is the same, by (iii). □