M2PM2 Progress Test 4

1. (a) Let R be the ring $\mathbb{Z}[i] = \{x + yi : x, y \in \mathbb{Z}\}$, and define the subset

$$I = \{x + yi \ : \ x \equiv y \bmod 2\}.$$

(i) Show that I is an ideal of R.

(ii) Find an element $a \in R$ such that $I = aR$, the principal ideal generated by a. Justify your answer.

(b) Let S be the ring $\mathbb{Z}[\sqrt{-5}] = \{x + y\sqrt{-5} : x, y \in \mathbb{Z}\}$, and define the subset

$$J = \{x + y\sqrt{-5} \ : \ x \equiv y \bmod 2\}.$$

(i) Show that J is an ideal of S.

(ii) Prove that J is not a prinicpal ideal of S.

Solution

(a) (i) Routine to check that $(I, +)$ is a subgroup of $(R, +)$. (OK to just write this, or to do it – but it should be mentioned.) (**1 mark**)

To see that $IR \subseteq I$, let $a + bi \in I$, so $a \equiv b \bmod 2$. Then for $x + yi \in R$,

$$(a + bi)(x + yi) = (ax - by) + (ay + bx)i.$$

The sum of the coeffs of 1 and i is $ax - by + ay + bx = (a + b)x + (a - b)y$. This is even, as both $a + b$ and $a - b$ are even. Hence $IR \subseteq I$. **(4 marks)**

(ii) We claim that $I = (1 + i)R$. To see this, observe that

$$(1 + i)(x + yi) = (x - y) + (x + y)i,$$

and for any integers a, b which are congruent mod 2, there exist $x, y \in \mathbb{Z}$ such that $a = x - y$, $b = x + y$. **(5 marks)**

(b) (i) Again routine to check $(J, +)$ a subgroup (no need to mention this time). And for $a + b\sqrt{-5} \in J$ and $x + y\sqrt{-5} \in S$,

$$(a + b\sqrt{-5})(x + y\sqrt{-5}) = (ax - 5by) + (ay + bx)\sqrt{-5}.$$

The sum of the two coeffs is $(a+b)x + (a-5b)y$, which is even. Hence $JS \subseteq J$. **(3 marks)**

(ii) Suppose J is principal, so $J = zS$ where $z = x + y\sqrt{-5}$. Then since 2 and $1+\sqrt{-5}$ are in J, there exist $r, s \in S$ such that

$$2 = zr, \ 1+\sqrt{-5} = zs.$$

Taking absolute values, this gives

$$4 = |z|^2|r|^2, \ \ 6 = |z|^2|s|^2.$$

So $|z|^2$ divides 4 and 6, hence divides 2. Also $|z|^2 = x^2 + 5y^2$ which cannot be 2. Hence $|z|^2 = 1$, so $z = \pm 1$. But then $zS = S$, a contradiction (as obviously $J \neq S$). Hence J is not principal. **(7 marks)**