M2PM2 Progress Test 3

1. (10 marks) For each of the following statements, decide whether it is true or false, and give either a proof or counterexample. The notation $M_n(F)$ means the set of $n \times n$ matrices over a field F, and A^T denotes the transpose of a matrix A.

(a) If

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 0 & 3 \\ 1 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{Z}_5),$$

then $\det(A) = 3$ $(\in \mathbb{Z}_5)$.

(b) If $A \in M_3(\mathbb{Z}_5)$ is the matrix in part (a), then there exists $B \in M_3(\mathbb{Z}_5)$ such that $B^2 = A$.

(c) If $B \in M_2(\mathbb{C})$ and $B^3 = 0$, then $B^2 = 0$.

(d) If $C, D \in M_n(F)$ are similar, then $\operatorname{trace}(CC^T) = \operatorname{trace}(DD^T)$.

(e) If $F \in M_2(\mathbb{C})$ is invertible, then there exist scalars $\lambda, \mu \in \mathbb{C}$ such that $F^{-1} = \lambda F + \mu I_2$.

2. (5 marks) Let

$$M = \begin{pmatrix} 1 & 1 & 0 \\ -3 & -2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R}),$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by $T(v) = Mv$ for all $v \in \mathbb{R}^3$. You are given that the characteristic polynomial of T is x^3.

Find a basis B of $\mathbb{R}^3$ such that the matrix $[T]_B$ is upper triangular.

3. (5 marks) Let A be an 8×8 matrix over $\mathbb{C}$ satisfying all of the following four conditions:

- the characteristic polynomial of A is x^8
- $\operatorname{rank}(A) = 5$
- $\operatorname{rank}(A^2) = 3$
- $\operatorname{rank}(A^3) = 2$.

Find all the possible Jordan Canonical Forms for A.

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1. (a) True (**1 mark**)

(b) False: if $A = B^2$ then $|B|^2 = |B^2| = |A| = 3$. But there is no element of $\mathbb{Z}_5$ that squares to 3. (**2 marks**)

(c) True: if $A^3 = 0$ then the only evalue of A is 0, so its char poly is x^2, hence $A^2 = 0$ by Cayley-Hamilton. (**2 marks**)

(d) False: eg. take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ (these both have distinct evalues 0,1 hence are similar to the diagonal matrix A). But AA^T has trace 1, whereas BB^T has trace 2. (**3 marks**)

(e) True: let the char poly of A be $x^2 + ax + b$. By Cayley-Hamilton, $A^2 + aA + bI = 0$. Multiplying through by A^{-1} gives $A + aI + bA^{-1} = 0$. Hence if $b \neq 0$ then $A^{-1} = -\frac{1}{b}(A + aI)$; and if $b = 0$ then $A = -aI$ has inverse $-a^{-1}I$ (actually b can't be 0, as $A = -aI$ has char poly $(x + a)^2$). (**2 marks**)

2. The only evalue of T is 0. First find an evector: $w_1 = e_1 - e_2 + e_3$ is one (where e_i are standard basis vectors). Now let $W = Sp(w_1)$ and work in the quotient space V/W (where $V = \mathbb{R}^3$). Get a basis of V/W – say $W + e_2, W + e_3$. Relative to this basis, the matrix of the quotient map $\bar{T} : V/W \to V/W$ is $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$. Now find an evector of this, eg. $W + e_2 + e_3$. So a final basis B such $[T]_B$ is upper triangular is

$$B = \{e_1 - e_2 + e_3, e_2 + e_3, e_3\}.$$

(Any other method getting the correct answer is OK.)

(**5 marks**)

3. As the char poly is x^8, the only evalue of A is 0. Let the JCF of A be $J_{n_1}(0) \oplus \cdots \oplus J_{n_a}(0)$, where $n_1 \geq n_2 \geq \cdots \geq n_a$.

As rank(A)=5, the geometric mult $g(0) = 3$. so the number of Jordan blocks is $a = 3$. So the JCF block sizes $n_1 n_2 n_3$ are 611, 521, 431, 422 or 332.

Now rank($J_n(0)^2$) is equal to $n - 2$, except if $n = 1$ (when it is 0). Hence rank(A^2)=3 implies that 422 and 332 are impossible. Hence the block sizes are 611, 521 or 431.

Next, rank($J_n(0)^3$) is equal to $n - 3$, except if $n = 1$ or 2 (when it is 0). Hence rank(A^3)=2 implies that 611 and 431 are impossible.

Therefore the only possible JCF is $J_5(0) \oplus J_3(0) \oplus J_1(0)$.

(**5 marks**)